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ON THE NON-UNIQUENESS OF CONFORMAL METRICS WITH PRESCRIBED SCALAR AND MEAN CURVATURES ON COMPACT MANIFOLDS WITH BOUNDARY

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ABSTRACT. For a compact Riemannian manifold (M^n, g) with boundary and dimension n, with $n \geq 2$, we study the existence of metrics in the conformal class of g with scalar curvature R_g and mean curvature h_g on the boundary. In this paper we find sufficient and necessary conditions for the existence of a smaller metric $\tilde{g} < g$ with curvatures $R_{\tilde{g}} = R_g$ and $h_{\tilde{g}} = h_g$. Furthermore, we establish the uniqueness of such a metric $\tilde{g}$ in the conformal class of the metric g when $R_g \geq 0$.

1. Introduction

Let (M^n, g) be a compact Riemannian manifold with boundary and dimension n, with $n \geq 2$. Let R_g denote its scalar curvature and h_g the mean curvature of its boundary, ∂M. The conformal class of the metric g, $[g]$, is the set of metrics of the form φg where φ is a smooth positive function defined in M. In recent years, there has been an increasing interest to establish to what extent the scalar curvature and the mean curvature of the boundary determine the metric within its conformal class. For instance, in the case of empty boundary ($\partial M = \varnothing$), Y. Lou in [7] established some uniqueness and non-uniqueness results. In the case of non empty boundary ($\partial M \neq \varnothing$), J. Escobar in [3] established results analogous to those obtained by Y. Lou (in the case of $\partial M = \varnothing$), when the mean curvature $h_g \leq 0$ on ∂M for $n \geq 3$ and when the geodesic curvature $k_g \leq 0$ for $n = 2$. On the other hand, G. García and J. Muñoz in [6] found sufficient conditions for the uniqueness of g, when the scalar curvature $R_g \geq 0$ for $n \geq 2$, instead of assuming that $h_g \leq 0$ for $n \geq 2$, as done by J. Escobar (see [3]). In particular, García et al's results generalized the uniqueness result of O. Montero in [8], obtained only for $R_g = 0$.

As we know, if $\tilde{g} = u^{\frac{4}{n-2}} g$ $(n \geq 3)$ for some positive function $u : \overline{M} \to \mathbb{R}$, then u satisfies the nonlinear elliptic problem,

$$\begin{cases} \Delta_g u - c(n) R_g u + c(n) R_{\tilde{g}} u^{\frac{n+2}{n-2}} = 0 & \text{in} \quad M, \\ \dfrac{\partial u}{\partial \eta} + \frac{n-2}{2} h_g u - \frac{n-2}{2} h_{\tilde{g}} u^{\frac{n}{n-2}} = 0 & \text{on} \quad \partial M, \end{cases} \tag{1}$$

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for $c(n) = \frac{n-2}{4(n-1)}$. If $\tilde{g} = e^{2u} g$ $(n = 2)$ for some positive function $u : \overline{M} \to \mathbb{R}$, then u satisfies the non linear elliptic problem

$$\begin{cases} \Delta_g u - K_g + K_{\widetilde{g}} e^{2u} = 0 & \text{in} \quad M, \\ \dfrac{\partial u}{\partial \eta_g} + k_g - k_{\widetilde{g}} e^u = 0 & \text{on} \quad \partial M, \end{cases} \tag{2}$$

where $R_g = 2K_g$ and $k_g = h_g$. Given $\widetilde{g} \in [g]$ with $R_g = R_{\widetilde{g}}$ in M, and $h_g = h_{\widetilde{g}}$ on ∂M, we see directly that $u \equiv 1$ for $n \geq 3$ and $u \equiv 0$ for $n = 2$ are solutions of the problems (1) and (2), respectively. As a consequence of these facts, we conclude that the geometrical uniqueness of the metric g is equivalent to the uniqueness of the solutions of the problems (1) for $n \geq 3$ and (2) for $n = 2$.

Before we go further, we note that if $\tilde{g} = \varphi g$ with $R_g = R_{\tilde{g}} = 0$ and $h_g = h_{\tilde{g}} = 0$ ($\tilde{g} \in [g]$), then φ must be a constant function. Thus, we will assume hereafter that (M^n, g) is an n-dimensional compact Riemannian manifold with non empty boundary such that $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$ are not identically zero, simultaneously.

As in J. Escobar in [3], and G. García and J. Muñoz in [6], we will adopt the following definition. We say that the metric $\widetilde{g}$ is smaller than the metric g ($\widetilde{g} < g$), if $\widetilde{g} = \varphi g$ for some smooth positive function φ such that $\varphi < 1$.

We will see below that the non-uniqueness of the metric g depends on the eigenvalues associated to the linear operators (L_1, B_1) defined for $n \geq 2$ by

$$\begin{cases} L_1 = \Delta_g + \frac{R_g}{n-1} & \text{in} \quad M, \\ B_1 = \frac{\partial}{\partial \eta_g} - h_g & \text{on} \quad \partial M, \end{cases} \tag{3}$$

and (L, B) defined for $n \geq 3$ by

$$\begin{cases} L = \Delta_g - c(n) R_g & \text{in} \quad M, \\ B = \frac{\partial}{\partial \eta_g} + \frac{n-2}{2} h_g & \text{on} \quad \partial M. \end{cases} \tag{4}$$

We denote the first Dirichlet eigenvalue and the first Neumann eigenvalue of this operator by $\lambda(L, B)$ and $\beta(L, B)$, respectively, and we denote the first Dirichlet eigenvalue and the first Neumann eigenvalue of the operator (L_1, B_1) by $\lambda(L_1, B_1)$ and $\beta(L_1, B_1)$, respectively. For 2-dimensional manifolds the Euler Characteristic of the manifold M, $\chi(M)$, will play the role of the eigenvalue of the conformal Laplacian (L, B).

The results in this paper are related with the work of J. Escobar in [3] on non-uniqueness of the metric g, and existence and uniqueness of small metrics, in the case of having nonpositive mean curvature. Escobar in [3] proved the following theorem

Theorem. *Let (M^n, g) be a compact manifold with boundary and mean curvature $h_g \leq 0$. If $n \geq 3$, there exists a metric $\widetilde{g} < g$ with $R_{\widetilde{g}} = R_g$ y $h_{\widetilde{g}} = h_g$ if and only if $\lambda(L, B) < 0$ and $\lambda(L_1, B_1) < 0$. If $n = 2$ there exists a metric $\widetilde{g} < g$ with $K_{\widetilde{g}} = K_g$ y $k_{\widetilde{g}} = k_g$ if and only if $\chi(M) < 0$ and $\lambda(L_1, B_1) < 0$. Furthermore, there exists at most one such a metric $\widetilde{g}$.*

In this paper we prove a stronger existence result than Escobar's theorem because we do not impose any restriction either on the scalar curvature R_g in M or on the mean curvature h_g on ∂M. Our existence theorem is

Theorem 1. *Let (M^n, g) be a compact manifold with boundary. If $n \geq 3$, $\lambda(L_1, B_1) < 0$ and $\lambda(L, B) < 0$ then there exists a metric $\widetilde{g}$ such that $\widetilde{g} < g$ with $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$. If $n = 2$, $\lambda(L_1, B_1) < 0$ and $\chi(M) < 0$ then there exists a metric $\widetilde{g}$ such that $\widetilde{g} < g$ with $K_g = K_{\widetilde{g}}$ and $k_g = k_{\widetilde{g}}$.*

We also establish a similar result to Escobar's Theorem replacing the condition about the nonpositive mean curvature $h_g \leq 0$ on ∂M by one about the nonnegative scalar curvature $R_g \geq 0$ in M.

Theorem 2. *Let (M^n, g) be a compact manifold with boundary. If $n \geq 3$ and $R_g \geq 0$ there exists a metric $\widetilde{g} < g$ with $h_g = h_{\widetilde{g}}$ and $R_g = R_{\widetilde{g}}$ if and only if $\lambda(L_1, B_1) < 0$ and $\lambda(L, B) < 0$. When $n = 2$ and $K_g \geq 0$ there exists a metric $\widetilde{g} < g$ with $K_g = k_{\widetilde{g}}$ and $k_g = k_{\widetilde{g}}$ if and only if $\chi(M) < 0$ and $\lambda(L_1, B_1) < 0$. Furthermore, there exists at most one such a metric $\widetilde{g}$.*

This paper is organized as follows: In section 2 we prove Theorem 1 and when $R_g \geq 0$ we give necessary conditions for the existence of a smaller metric. In section 3 we prove Theorem 2.

2. Existence of smaller metrics

Let (M^n, g) be a compact manifold with boundary of dimension $n \geq 2$. In this section we will give necessary and sufficient conditions for the existence of a smaller metric $\widetilde{g} < g$ with the same scalar curvature and the same mean curvature of g.

First, let us consider the linear operator $(\widehat{L}, \widehat{B})$ defined by

$$\begin{cases} \widehat{L}(\varphi) = \Delta_g \varphi - H\varphi & \text{in} \quad M, \\ \widehat{B}(\varphi) = \dfrac{\partial \varphi}{\partial \eta} + f\varphi & \text{on} \quad \partial M. \end{cases} \tag{5}$$

We will say that $\widehat{\beta}$ is a Neumann type eigenvalue of the linear operator $(\widehat{L}, \widehat{B})$ if there exists a function φ that satisfies

$$\begin{cases} \widehat{L}(\varphi) = 0 & \text{in} \quad M, \\ \widehat{B}(\varphi) = \widehat{\beta}\varphi & \text{on} \quad \partial M. \end{cases} \tag{6}$$

We also say that the function φ is an eigenfunction associated to $\widehat{\beta}$.

The study of the Neumann type eigenvalues for the linear operator $(\widehat{L}, \widehat{B})$ is associated with the functional in $H^{1,2}(M)$ given by

$$E(\varphi) = \int_M |\nabla \varphi|^2 + \int_M H\varphi^2 + \int_{\partial M} f\varphi^2. \tag{7}$$

It is not hard to see that in the case

$$\beta = \inf_{\substack{\varphi \in H^{1,2}(M) \\ \varphi \neq 0 \text{ in } \partial M}} \frac{E(\varphi)}{\int_{\partial M} \varphi^2} \tag{8}$$

is finite, then β is the first eigenvalue of problem (6), and there exists a positive function φ such that $\beta = E(\varphi)$ (see [6]). As observed by Escobar in [5], β can be $-\infty$, but in [6] García et al's showed that β is finite if and only if the first Dirichlet type eigenvalue ρ of

$$\begin{cases} \widehat{L}(\varphi) + \rho\varphi = 0 & \text{in} \quad M, \\ \varphi = 0 & \text{on} \quad \partial M \end{cases} \tag{9}$$

is non negative.

We start the discussion establishing in the coming lemmas necessary conditions for the existence of a smaller metric $\widetilde{g}$.

Lemma 1. *Let (M^n, g) be a compact manifold with boundary and $R_g \geq 0$. If there exists $\widetilde{g} < g$ with $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$, then for $n \geq 3$, $\lambda(L, B) < 0$, and for $n = 2$, $\chi(M) < 0$.*

Proof. Assume that $n \geq 3$ and let $\widetilde{g} = u^{\frac{4}{n-2}} g$ where u is a solution to problem (1). Multiplying the first equation of (1) by u and integrating by parts we get

$$\begin{aligned} E(u) &= \int_M |\nabla u|^2 + c(n) \int_M R_g u^2 + \frac{n-2}{2} \int_{\partial M} h_g u^2 \\ &= c(n) \int_M R_g u^{\frac{2n}{n-2}} + \frac{n-2}{2} \int_{\partial M} h_g u^{\frac{2(n-1)}{n-2}}. \end{aligned}$$

Using the second equation in (1) and integrating by parts we find that

$$\begin{aligned} \frac{n-2}{2} \int_{\partial M} h_g u^{\frac{2(n-1)}{n-2}} &= \int_{\partial M} \frac{\partial u}{\partial \eta} \frac{u^{\frac{2(n-1)}{n-2}}}{u^{\frac{n}{n-2}} - u} \\ &= \int_M \frac{u^{\frac{2(n-1)}{n-2}}}{u^{\frac{n}{n-2}} - u} \Delta u \\ &\qquad + \int_M \nabla u . \nabla \left(\frac{u^{\frac{2(n-1)}{n-2}}}{u^{\frac{n}{n-2}} - u} \right) \\ &= c(n) \int_M R_g \frac{u^{\frac{n}{n-2}}}{u^{\frac{2}{n-2}} - 1} \left(u - u^{\frac{n+2}{n-2}} \right) \\ &\qquad + \int_M |\nabla u|^2 \frac{u^{\frac{4}{n-2}} - \frac{n}{n-2} u^{\frac{2}{n-2}}}{\left(u^{\frac{2}{n-2}} - 1 \right)^2} \\ &= -c(n) \int_M R_g u^{\frac{2(n-1)}{n-2}} \left(1 + u^{\frac{2}{n-2}} \right) \\ &\qquad + \int_M |\nabla u|^2 \frac{u^{\frac{2}{n-2}} \left(u^{\frac{2}{n-2}} - \frac{n}{n-2} \right)}{\left(u^{\frac{2}{n-2}} - 1 \right)^2}. \end{aligned}$$

Hence

$$E(u) = -c(n)\int_M R_g u^{\frac{2(n-1)}{n-2}} + \int_M |\nabla u|^2 \frac{u^{\frac{2}{n-2}}\left(u^{\frac{2}{n-2}} - \frac{n}{n-2}\right)}{\left(u^{\frac{2}{n-2}} - 1\right)^2}.$$

Since $u < 1$ and $R_g \geq 0$ we obtain $E(u) < 0$. From the variational characterization of $\lambda(L, B)$,

$$\lambda(L, B) = \inf_{\varphi \in H^{1,2}(M),\, \varphi \neq 0 \text{ in } \partial M} \frac{E(\varphi)}{\int_M |\varphi|^2},$$

we find that $\lambda(L, B) < 0$.

Now assume that $n = 2$ and let $\widetilde{g} = e^{2u} g$. The function u satisfies problem (2). Using the boundary condition in (2) and integrating by parts we get

$$\begin{aligned} \int_{\partial M} k_g &= \int_{\partial M} \left(\frac{1}{e^u - 1}\right) \frac{\partial u}{\partial \eta} \\ &= \int_M \left(\frac{1}{e^u - 1}\right) \Delta u - \int_M |\nabla u|^2 \frac{e^u}{(e^u - 1)^2} \\ &= -\int_M K_g(1 + e^u) - \int_M |\nabla u|^2 \frac{e^u}{(e^u - 1)^2}. \end{aligned}$$

From this equation and The Gauss-Bonnet theorem we have

$$\begin{aligned} 2\pi\chi(M) &= \int_{\partial M} k_g + \int_M K_g \\ &\leq \int_{\partial M} k_g + \int_M K_g(1 + e^u) \\ &= -\int_M |\nabla u|^2 \frac{e^u}{(e^u - 1)^2}, \end{aligned}$$

and therefore $\chi(M) < 0$. □

Lemma 2. *Let (M^n, g) be a compact manifold with boundary and $R_g \geq 0$. If there exists a metric $\widetilde{g} < g$ with $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$, then $\lambda(L_1, B_1) < 0$.*

Proof. When $n \geq 3$ let $\widetilde{g} = u^{\frac{4}{n-2}} g$ and $v = u^{-\frac{2}{n-2}} - 1$. The function v satisfies the equations

$$\begin{cases} \Delta v + \dfrac{R_g}{2(n-1)} v(u^{\frac{2}{n-2}} + 1) = \dfrac{2n}{(n-2)^2} u^{-\frac{2(n-1)}{n-2}} |\nabla u|^2 & \text{in } M, \\ \dfrac{\partial v}{\partial \eta} = h_g v & \text{on } \partial M. \end{cases} \tag{10}$$

Therefore v satisfies in M the following differential inequality

$$\Delta v + \frac{R_g}{n-1} v = \frac{2n}{(n-2)^2} u^{-\frac{2(n-1)}{n-2}} |\nabla u|^2 + \frac{R_g}{2(n-1)} v(1 - u^{\frac{2}{n-2}}) > 0. \tag{11}$$

The previous inequality follows from the inequality $R_g \geq 0$ and from the fact that the function $u < 1$ is not a constant. Multiplying both sides by v, integrating and using that v is positive we find that

$$\int_M v\Delta v + \int_M \frac{R_g}{n-1} v^2 \quad > 0. \tag{12}$$

Integrating by parts we get

$$\int_M |\nabla v|^2 - \int_M \frac{R_g}{n-1} v^2 - \int_{\partial M} \frac{\partial v}{\partial \eta} v < 0. \tag{13}$$

On the other hand from equations (10) we have $\frac{\partial v}{\partial \eta} = h_g v$, and therefore

$$E(v) = \int_M |\nabla v|^2 - \int_M \frac{R_g}{n-1} v^2 - \int_{\partial M} h_g v^2 < 0. \tag{14}$$

From the variational characterization of $\lambda(L_1, B_1)$ we conclude that

$$\lambda(L_1, B_1) = \inf_{\varphi \in H^{1,2}(M), \varphi \neq 0} \frac{E(\varphi)}{\int_M |\varphi|^2} < 0.$$

When $n = 2$ we let $\widetilde{g} = e^{2u} g$ with $u < 0$. The function v defined by $v = e^{-2u} - 1$ satisfies the following equations

$$\begin{cases} \Delta v + K_g v(1 + e^u) = e^{-u} |\nabla u|^2 & \text{in} \quad M, \\ \dfrac{\partial v}{\partial \eta} = k_g v & \text{on} \quad \partial M. \end{cases} \tag{15}$$

The proof in this case follows in the same fashion as the proof in the case $n \geq 3$, keeping in mind that v satisfies equations (15) instead of the equations (10). □

Now we will prove the result on the existence of a smaller metric $\tilde{g}$.

Proof of Theorem 1. In order to show the existence of the metric $\tilde{g}$ we use the method of upper and lower solutions (see [4]). First we will find a lower solution of problem (1). If $n \geq 3$, let $\psi > 0$ be an associated positive eigenfunction to the eigenvalue $\lambda(L, B)$. That is the function ψ satisfies the problem

$$\begin{cases} L(\psi) + \lambda(L, B)\psi = 0 & \text{in} \quad M, \\ B(\psi) = 0 & \text{on} \quad \partial M. \end{cases} \tag{16}$$

Since $\lambda(L, B) < 0$ then

$$\beta(L, B) = \inf_{\varphi \in H^{1,2}(M),\, \varphi \neq 0 \text{ in } \partial M} \frac{E(\varphi)}{\int_{\partial M} |\varphi|^2}$$

is negative or $-\infty$, where

$$E(\varphi) = \int_M |\nabla \varphi|^2 + c(n) \int_M R_g \varphi^2 + \frac{n-2}{2} \int_{\partial M} h_g \varphi^2.$$

In the first case $\beta(L, B)$ is the first eigenvalue of the problem

$$\begin{cases} L(\varphi) = 0 & \text{in} \quad M, \\ B(\varphi) = \beta(L, B)\varphi & \text{on} \quad \partial M, \end{cases} \tag{17}$$

where $\varphi > 0$ is an associated positive eigenfunction to $\beta(L, B)$.

Let $\vartheta = \varphi + \psi$. Without loss of generality we assume that $\max_{x\in\bar{M}} \vartheta(x) = 1$. Define the function $\underline{u} = \epsilon\vartheta$ where $\epsilon > 0$. We will show that for ϵ small, the function $\underline{u}$ is a lower solution to problem (1) when $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$. In fact

$$\begin{aligned}
\Delta_g\underline{u} - c(n)R_g\underline{u} + c(n)R_g\underline{u}^{\frac{n+2}{n-2}} &= \epsilon\Delta\vartheta - \epsilon c(n)R_g\vartheta + c(n)R_g\epsilon^{\frac{n+2}{n-2}}\vartheta^{\frac{n+2}{n-2}} \\
&= \epsilon L(\varphi) + \epsilon L(\psi) + c(n)R_g\epsilon^{\frac{n+2}{n-2}}\vartheta^{\frac{n+2}{n-2}} \\
&= -\epsilon\lambda(L,B)\psi + c(n)R_g\epsilon^{\frac{n+2}{n-2}}\vartheta^{\frac{n+2}{n-2}} \\
&= \epsilon\left(-\lambda(L,B)\psi + c(n)R_g\epsilon^{\frac{4}{n-2}}\vartheta^{\frac{n+2}{n-2}}\right) \\
&\geq \epsilon\left(-\lambda(L,B)\min_{x\in M}\psi - c(n)\epsilon^{\frac{4}{n-2}}||R_g||_\infty\right).
\end{aligned}$$

Since $\lambda(L,B) < 0$, then taking ϵ small enough, we conclude that

$$\Delta_g\underline{u} - c(n)R_g\underline{u} + c(n)R_g\underline{u}^{\frac{n+2}{n-2}} > 0.$$

Now on ∂M we have

$$\begin{aligned}
\frac{\partial\underline{u}}{\partial\eta} + \frac{n-2}{2}h_g\underline{u} - \frac{n-2}{2}h_g\underline{u}^{\frac{n}{n-2}} &= \epsilon\frac{\partial\vartheta}{\partial\eta} + \epsilon\frac{n-2}{2}h_g\vartheta - \frac{n-2}{2}h_g(\epsilon\vartheta)^{\frac{n}{n-2}} \\
&= \epsilon\left(B(\varphi) + B(\psi)\right) - \frac{n-2}{2}h_g\epsilon^{\frac{n}{n-2}}\vartheta^{\frac{n}{n-2}} \\
&= \epsilon\beta(L,B)\varphi - \frac{n-2}{2}h_g\epsilon^{\frac{n}{n-2}}\vartheta^{\frac{n}{n-2}} \\
&\leq \epsilon\varphi\left(\beta(L,B) + \frac{n-2}{2}\epsilon^{\frac{2}{n-2}}||h_g||_\infty\right).
\end{aligned}$$

Since $\beta(L,B) < 0$, then taking ϵ small enough, we show that

$$\frac{\partial\underline{u}}{\partial\eta} + \frac{n-2}{2}h_g\underline{u} - \frac{n-2}{2}h_g\underline{u}^{\frac{n}{n-2}} < 0,$$

and consequently $\underline{u}$ is a lower solution to problem (1).

Now consider the case $\beta(L,B) = -\infty$. Then the first eigenvalue ρ of the problem

$$\left\{\begin{array}{lll} \Delta_g\phi - c(n)R_g\phi + \rho\phi = 0 & \text{in} & M, \\ \phi = 0 & \text{on} & \partial M, \end{array}\right. \tag{18}$$

is negative (see [6]).

Let ϕ be an associated eigenfunction to the eigenvalue ρ, from the variational characterization of ρ, we can take $\phi \geq 0$. From the maximum principle $\phi > 0$ in $M \setminus \partial M$. By Hopf's lemma, we have that $\dfrac{\partial\phi}{\partial\eta} < 0$. Since the boundary of ∂M is compact, there exists $\delta < 0$ such that $\dfrac{\partial\phi}{\partial\eta} \leq \delta < 0$. Now define the function $\underline{u} = \epsilon(\psi + \phi)$ with $\epsilon > 0$, where we may assume that $\max_{x\in\bar{M}}(\phi + \psi)(x) = 1$. Proceeding as in the case $\beta(L,B)$ negative, using that $\lambda(L,B)$, ρ and δ are negative and taking ϵ small enough, we find that $\underline{u}$ is a lower solution of problem (1).

Now we turn to the case $n = 2$. Let φ be a solution of the following boundary value problem:

$$\begin{aligned} \Delta\varphi &= K_g - \frac{\pi\chi(M)}{\text{vol}(M)} \qquad \text{in } M, \\ \frac{\partial\varphi}{\partial\eta} &= -k_g + \frac{\pi\chi(M)}{\text{vol}(\partial M)} \qquad \text{on } \partial M. \end{aligned} \tag{19}$$

Define the function $\underline{u} = -N + \varphi$, where $N > 0$. We claim that for N large enough the function $\underline{u}$ is a lower solution to problem (2), for $K_g = K_{\widetilde{g}}$ and $k_g = k_{\widetilde{g}}$. In fact for N large enough we have that

$$\begin{aligned} \Delta_g\underline{u} - K_g + K_g e^{2\underline{u}} &= \Delta\varphi - K_g + K_g e^{2(-N+\varphi)} \\ &= -\frac{\pi}{\text{vol}(M)}\chi(M) + K_g e^{2(-N+\varphi)} > 0 \end{aligned}$$

and

$$\begin{aligned} \frac{\partial\underline{u}}{\partial\eta} + k_g - k_g e^{2\underline{u}} &= \frac{\partial\varphi}{\partial\eta} + k_g - k_g e^{-N+\varphi} \\ &= -k_g + \frac{\pi\chi(M)}{\text{vol}(\partial M)} + k_g - k_g e^{-N+\varphi} \\ &= \frac{\pi\chi(M)}{\text{vol}(\partial M)} - k_g e^{-N+\varphi} < 0. \end{aligned}$$

Consequently $\underline{u}$ is a lower solution of problem (2).

Now we will find an upper solution of problem (1). If $n \geq 2$, let $\psi > 0$ be an associated positive eigenfunction to the eigenvalue $\lambda(L_1, B_1)$, that is ψ satisfies

$$\begin{cases} L_1(\psi) = \lambda(L_1, B_1)\psi & \text{in} \quad M, \\ B_1(\psi) = 0 & \text{on} \quad \partial M. \end{cases} \tag{20}$$

Since $\lambda(L_1, B_1) < 0$, then

$$\beta(L_1, B_1) = \inf_{\varphi \in H^{1,2}(M),\, \varphi \neq 0 \text{ } in \text{ } \partial M} \frac{E(\varphi)}{\int_{\partial M} |\varphi|^2}$$

is negative or $-\infty$, where

$$E(\varphi) = \int_M |\nabla\varphi|^2 - \int_M \frac{R_g}{n-1}\varphi^2 - \int_{\partial M} h_g\varphi^2. \tag{21}$$

In the first case, $\beta(L_1, B_1)$ is the first eigenvalue of the problem

$$\begin{cases} L_1(\varphi) = 0 & \text{in} \quad M, \\ B_1(\varphi) = \beta(L_1, B_1)\varphi & \text{on} \quad \partial M. \end{cases} \tag{22}$$

where $\varphi > 0$ is an associated positive eigenfunction to $\beta(L_1, B_1)$.

Let $\vartheta = \varphi + \psi$. Without loss of generality we assume that $\max_{x \in \bar{M}} \vartheta(x) = 1$. If $n \geq 3$ define the function $\bar{u} = 1 - \epsilon\vartheta$, for $\epsilon > 0$. We will show that for ϵ small enough, the function $\bar{u}$ is a upper solution to problem (1), for $R_g = R_{\widetilde{g}}$ and $h_g = h_{\widetilde{g}}$.

In fact

$$\begin{aligned}\Delta_g\bar{u} &- c(n)R_g\bar{u} + c(n)R_g\bar{u}^{\frac{n+2}{n-2}} = -\epsilon\Delta\vartheta - c(n)R_g(1-\epsilon\vartheta) + c(n)R_g(1-\epsilon\vartheta)^{\frac{n+2}{n-2}}\\ &= \epsilon\vartheta\left[\frac{\lambda(L_1,B_1)\psi}{\vartheta} + \frac{R_g}{n-1}\left(1 - \frac{n-2}{4}\left(\frac{1-\epsilon\vartheta}{\epsilon\vartheta}\right) + \frac{n-2}{4}\frac{(1-\epsilon\vartheta)^{\frac{n+2}{n-2}}}{\epsilon\vartheta}\right)\right]\\ &= \epsilon\vartheta\left[\frac{\lambda(L_1,B_1)\psi}{\vartheta} + \frac{R_g}{n-1}\left(1 + \frac{n-2}{4}(1-\epsilon\vartheta)\left(\frac{(1-\epsilon\vartheta)^{\frac{4}{n-2}}-1}{\epsilon\vartheta}\right)\right)\right].\end{aligned}$$

Since $\lambda(L_1,B_1)<0$, then taking ϵ small enough, we get that

$$\Delta_g\bar{u} - c(n)R_g\bar{u} + c(n)R_g\bar{u}^{\frac{n+2}{n-2}} < 0.$$

On the boundary of M, we have that

$$\begin{aligned}\frac{\partial\bar{u}}{\partial\eta} &+ \frac{n-2}{2}h_g\bar{u} - \frac{n-2}{2}h_g\bar{u}^{\frac{n}{n-2}} = -\epsilon\frac{\partial\vartheta}{\partial\eta} - \frac{n-2}{2}h_g\left((1-\epsilon\vartheta)^{\frac{n}{n-2}} - (1-\epsilon\vartheta)\right)\\ &= -\epsilon\left(\beta(L_1,B_1)\varphi + h_g\vartheta - \frac{n-2}{2}h_g\left((1-\epsilon\vartheta)^{\frac{n}{n-2}} - (1-\epsilon\vartheta)\right)\right)\\ &= \epsilon\vartheta\left(\frac{-\beta(L_1,B_1)\varphi}{\vartheta} - h_g\left[1 + \frac{n-2}{2}\frac{\left((1-\epsilon\vartheta)^{\frac{n}{n-2}} - (1-\epsilon\vartheta)\right)}{\epsilon\vartheta}\right]\right).\end{aligned}$$

Since $\beta(L_1,B_1)<0$, then taking ϵ small enough, we find that

$$\frac{\partial\bar{u}}{\partial\eta} + \frac{n-2}{2}h_g\bar{u} - \frac{n-2}{2}h_g\bar{u}^{\frac{n}{n-2}} > 0.$$

Hence $\bar{u}$ is an upper solution of problem (1).

For $n=2$, we define $\bar{u} = -\epsilon\vartheta$ and show for ϵ small that $\bar{u}$ is an lower solution to problem (2), provided that $K_g = K_{\widetilde{g}}$ y $k_g = k_{\widetilde{g}}$. In fact

$$\begin{aligned}\Delta\bar{u} - K_g + K_g e^{2\bar{u}} &= -\epsilon\Delta\vartheta - K_g + K_g e^{-2\epsilon\vartheta}\\ &= \epsilon\lambda(L_1,B_1)\psi + 2\epsilon K_g\vartheta - K_g + K_g e^{-2\epsilon\vartheta}\\ &= \epsilon\vartheta\left(\frac{\lambda(L_1,B_1)\psi}{\vartheta} + \frac{2\epsilon K_g\vartheta - K_g + K_g e^{-2\epsilon\vartheta}}{\epsilon\vartheta}\right)\\ &= \epsilon\vartheta\left(\frac{\lambda(L_1,B_1)\psi}{\vartheta} + K_g\left(\frac{2\epsilon\vartheta - 1 + e^{-2\epsilon\vartheta}}{\epsilon\vartheta}\right)\right) < 0,\end{aligned}$$

for ϵ small enough. Now, on the boundary of M we have

$$\begin{aligned}\frac{\partial\bar{u}}{\partial\eta} + k_g - k_g e^{2\bar{u}} &= -\epsilon\frac{\partial\vartheta}{\partial\eta} + k_g - k_g e^{-\epsilon\vartheta}\\ &= -\epsilon(k_g\vartheta + \beta(L_1,B_1)\varphi) + k_g - k_g e^{-\epsilon\vartheta}\\ &= \epsilon\vartheta\left(-\frac{\beta(L_1,B_1)\varphi}{\vartheta} + k_g\frac{(-\epsilon\vartheta + 1 - e^{-\epsilon\vartheta})}{\epsilon\vartheta}\right) > 0,\end{aligned}$$

for ϵ sufficiently small. Consequently $\bar{u}$ is an upper solution of problem (2).

Now consider the case $\beta(L_1, B_1) = -\infty$. Then the first eigenvalue ρ of the problem

$$\begin{cases} \Delta_g \phi + \dfrac{R_g}{n-1}\phi + \rho\phi = 0 & \text{in} \quad M, \\ \phi = 0 & \text{on} \quad \partial M, \end{cases} \tag{23}$$

is negative (see [6]).

Using the same type of arguments as above, we are able to prove for ϵ small enough that the function $\bar{u} = 1 - \epsilon(\psi + \phi)$ is an upper solution of (1) in the case $n \geq 3$, and the function $\bar{u} = -\epsilon(\psi + \phi)$ is an upper solution of problem (2) in the case $n = 2$. For $n \geq 3$, we take $\epsilon < 1/2$. Then

$$\bar{u} = 1 - \epsilon\vartheta \geq 1 - \epsilon \geq \epsilon \geq \epsilon\vartheta = \underline{u}.$$

From the result on lower solutions and upper solutions by Escobar in [4], there exists a function u satisfying (1) and such that

$$0 < \underline{u} \leq u \leq \bar{u} < 1.$$

Consequently $\widetilde{g} = u^{-\frac{4}{n-2}} g < g$. If $n = 2$, let us take $\epsilon > 0$ small enough and $N > 0$ large enough, such that $\underline{u} \leq \bar{u}$. Again using Escobar's result [4], there exists a function u satisfying (2) such that

$$\underline{u} \leq u \leq \bar{u} < 0.$$

Consequently $\widetilde{g} = e^{2u} g < g$. □

3. Uniqueness of smaller metrics

In the previous section we gave sufficient conditions for the existence of a metric $\widetilde{g} < g$ with the property of having the same scalar and mean curvature of the metric g. Assuming that $R_g \geq 0$, we are ready to establish that this metric $\widetilde{g}$ is the unique smaller metric with this property in the conformal class of g . The result follows by doing a slight modification of the proof of the uniqueness of smaller metrics due to Escobar in [3], obtained in the case $h_g \leq 0$.

Now, we will consider for $n \geq 3$ the operator $T : C(\bar{M}) \to C(\bar{M})$ defined by $T(\varphi) = \psi$, where ψ is the unique function that satisfies the following boundary value problem

$$\begin{cases} \Delta_g \psi - \gamma\psi = -\gamma\varphi + c(n) R_{\widetilde{g}} \left(\varphi - \varphi^{\frac{n+2}{n-2}}\right) & \text{in} \quad M \\ \dfrac{\partial \psi}{\partial \eta} - \rho\psi = -\rho\varphi + \frac{n-2}{2} h_g \left(\varphi^{\frac{n}{n-2}} - \varphi\right) & \text{on} \quad \partial M, \end{cases} \tag{24}$$

where $\gamma \geq \dfrac{n}{2(n-1)} ||R||_\infty$ and $\rho \leq -(n-1)||h_g||_\infty$.

For $n = 2$, we define the operator $T : C(\bar{M}) \to C(\bar{M})$ as $T(\varphi) = \psi$, where ψ is the unique function that satisfies the boundary value problem

$$\begin{cases} \Delta_g \psi - \gamma\psi = -\gamma\varphi + K_g(1 - e^{2\varphi}) & \text{in} \quad M \\ \dfrac{\partial \psi}{\partial \eta} - \rho\psi = -\rho\varphi + k_g \left(e^{\varphi} - 1\right) & \text{on} \quad \partial M, \end{cases} \tag{25}$$

where $\rho < -||k_g||_\infty$ and $\gamma > 2||K_g||_\infty$. The well definition and the compactness of the operator T follows by the general theory for elliptic operators (see [1, 2, 10]). Hereafter, we only consider the case $n \geq 3$ and we leave to the reader the details of the case $n = 2$.

Let us consider the set $A = \{u \in C(\bar{M}) |\ \underline{u} \leq u \leq \bar{u} < 1\}$ where $\underline{u}$ and $\bar{u}$ are lower and upper solutions of problem (1) obtained in last section. In particular, $\underline{u}, \bar{u} \in A$. Recall that the set A, when $n \geq 3$ depends on ϵ.

For the proof of the next lemma see ([3], Lemma 8).

Lemma 3. *The operator T satisfies $T(A) \subset int(A)$.*

As a consequence of previous lemma, we are able to define the Leray-Schauder degree of the function $I - T$ on the set A, which we will denoted by $deg(I - T, A, 0)$. Moreover, since the Leray-Schauder degree is invariant under homotopy, we conclude that $deg(I - T, A, 0) = 1$ (see [3], Lemma 9).

For a function $u \in C(\bar{M})$, we define the index of T in u as

$$i(T, u) = deg(I - T, B_\delta(u), 0),$$

where $B_\delta(u) = \left\{\varphi \in C(\bar{M}) : \|\varphi - u\|_\infty < \delta\right\}$, for δ being a small positive number.

Lemma 4. *Let $R_g = R_{\widetilde{g}} \geq 0$ and $h_g = h_{\widetilde{g}}$. If $u \in A$ is a solution of problem (1), then u is an isolated fixed point of T, and $i(T, u) = 1$.*

Proof. We start claiming that the derivative of the operator T in u, denoted by $D_u T$, does not have eigenvalues $\lambda \geq 1$. Note that $(D_u T)\varphi$ is the unique solution to the problem

$$\begin{cases} (\Delta_g - \gamma)((D_u T)\varphi) & = \left(-\gamma + c(n) R_{\widetilde{g}} \left(1 - \frac{n+2}{n-2} u^{\frac{4}{n-2}}\right)\right)\varphi \quad \text{in} \quad M, \\ \left(\dfrac{\partial}{\partial \eta} - \rho\right)((D_u T)\varphi) & = \left(-\rho + \frac{n-2}{2} h_g \left(\frac{n}{n-2} u^{\frac{2}{n-2}} - 1\right)\right)\varphi \quad \text{on} \quad \partial M. \end{cases}$$

Assume that $D_u T\varphi = \lambda\varphi$. Then φ satisfies

$$\begin{cases} \lambda \Delta_g \varphi & = \left(\gamma(\lambda - 1) + c(n) R_{\widetilde{g}} \left(1 - \frac{n+2}{n-2} u^{\frac{4}{n-2}}\right)\right)\varphi \quad \text{in} \quad M, \\ \lambda \dfrac{\partial \varphi}{\partial \eta} & = \left(\rho(\lambda - 1) + \frac{n-2}{2} h_g \left(\frac{n}{n-2} u^{\frac{2}{n-2}} - 1\right)\right)\varphi \quad \text{on} \quad \partial M. \end{cases}$$

Multiplying by φ and integrating by parts we get

$$\lambda \int_M |\nabla_g \varphi|^2 + \gamma(\lambda - 1) \int_M \varphi^2 + c(n) \int_M R_{\widetilde{g}} \left(1 - \frac{n+2}{n-2} u^{\frac{4}{n-2}}\right)\varphi^2 \\ - \rho(\lambda - 1) \int_{\partial M} \varphi^2 - \frac{n-2}{2} \int_{\partial M} h_g \left(\frac{n}{n-2} u^{\frac{2}{n-2}} - 1\right)\varphi^2 = 0.$$

Assuming that $\lambda \geq 1$, we conclude that

$$\int_M ||\nabla\varphi||^2 + c(n) \int_M R_{\widetilde{g}} \left(1 - \frac{n+2}{n-2} u^{\frac{4}{n-2}}\right)\varphi^2 - \frac{n-2}{2} \int_{\partial M} h_g \left(\frac{n}{n-2} u^{\frac{2}{n-2}} - 1\right)\varphi^2 \leq 0.$$

This fact implies that there exist an eigenfunction $\varphi_1 > 0$ and an eigenvalue $\lambda_1 \leq 0$ for the problem

$$\begin{cases} \Delta_g \varphi_1 - c(n) R_{\widetilde{g}} \left(1 - \frac{n+2}{n-2} u^{\frac{4}{n-2}}\right) \varphi_1 + \lambda_1 \varphi_1 = 0 & \text{in } M, \\ \dfrac{\partial \varphi_1}{\partial \eta} - \frac{n-2}{2} h_g \left(\frac{n}{n-2} u^{\frac{2}{n-2}} - 1\right) \varphi_1 = 0 & \text{on } \partial M. \end{cases} \tag{26}$$

Multiplying the last equation by $(u - u^{\frac{n}{n-2}}) > 0$ and integrating we obtain

$$\int_{\partial M} \frac{\partial \varphi_1}{\partial \eta}(u - u^{\frac{n}{n-2}}) - \frac{n-2}{2} \int_{\partial M} h_g \varphi_1 \left(\frac{n}{n-2} u^{\frac{2}{n-2}} - 1\right)(u - u^{\frac{n}{n-2}}) = 0.$$

Using the equation on ∂M in problem (1) we get

$$\int_{\partial M} \frac{\partial \varphi_1}{\partial \eta}(u - u^{\frac{n}{n-2}}) + \int_{\partial M} \varphi_1 \frac{\partial u}{\partial \eta} \left(\frac{n}{n-2} u^{\frac{2}{n-2}} - 1\right) = 0.$$

Integrating by parts we conclude that

$$\int_M \nabla \varphi_1 . \nabla \left(u - u^{\frac{n}{n-2}}\right) + \int_M (u - u^{\frac{n}{n-2}}) \Delta \varphi_1 + \int_M \nabla u . \nabla \left(\varphi_1 (\frac{n}{n-2} u^{\frac{2}{n-2}} - 1)\right) + \int_M (\frac{n}{n-2} u^{\frac{2}{n-2}} - 1) \varphi_1 \Delta u = 0.$$

The previous equation yields to

$$\int_M (u - u^{\frac{n}{n-2}}) \Delta \varphi_1 + \int_M (\frac{n}{n-2} u^{\frac{2}{n-2}} - 1) \varphi_1 \Delta u + \frac{2n}{(n-2)^2} \int_M \varphi_1 \left|\nabla u\right|^2 u^{-\frac{n-4}{n-2}} = 0.$$

Since u solves (1) and φ_1 solves (26)

$$\int_M c(n) R_g \varphi_1 \left[1 - \frac{n+2}{n-2} u^{\frac{4}{n-2}}\right] [u - u^{\frac{n}{n-2}}] + c(n) \int_M R_g [u - u^{\frac{n+2}{n-2}}][\frac{n}{n-2} u^{\frac{2}{n-2}} - 1] \varphi_1 + \frac{2n}{(n-2)^2} \int_M \varphi_1 \left|\nabla u\right|^2 u^{-\frac{n-4}{n-2}} = \lambda_1 \int_M \varphi_1 [u - u^{\frac{n}{n-2}}].$$

Recalling that $c(n) = \dfrac{n-2}{4(n-1)}$, from the last equality we obtain

$$\int_M \frac{R_g \varphi_1}{2(n-1)} u^{\frac{n}{n-2}} \left[1 - 2u^{\frac{2}{n-2}} + u^{\frac{4}{n-2}}\right] + \frac{2n}{(n-2)^2} \int_M \varphi_1 \left|\nabla u\right|^2 u^{-\frac{n-4}{n-2}} = \lambda_1 \int_M \varphi_1 (u - u^{\frac{n}{n-2}}).$$

Since $R_g \geq 0$ we get

$$0 \leq \int_M \frac{R_g \varphi_1}{2(n-1)} u^{\frac{n}{n-2}} \left(1 - u^{\frac{2}{n-2}}\right)^2 + \frac{2n}{(n-2)^2} \int_M \varphi_1 \left|\nabla u\right|^2 u^{-\frac{n-4}{n-2}} = \lambda_1 \int_M \varphi_1 (u - u^{\frac{n}{n-2}}) \leq 0.$$

Thus we conclude u is a constant function. Since we assumed that R_g and h_g do not vanish simultaneously and u satisfies (1) the function u must be identically equal to 1, which is a contradiction with $u \in A$ since $\bar{u} < 1$. Consequently, if λ is an eigenvalues of $D_u T$, then $\lambda < 1$, and thus we have that $i(T, u) = 1$ (see [9]). □

Proof of Theorem 2 Assume that $n \geq 3$. If there exists $\widetilde{g} \in [g]$ with $\widetilde{g} < g$ such that $h_g = h_{\widetilde{g}}$ and $R_g = R_{\widetilde{g}} \geq 0$, then Lemma 1 implies that $\lambda(L, B) < 0$, and Lemma 2 implies that $\lambda(L_1, B_1) < 0$. Conversely, if $\lambda(L, B) < 0$ and $\lambda(L_1, B_1) < 0$, then Theorem 1 implies the existence of a metric $\widetilde{g} \in [g]$ with $\widetilde{g} < g$ such that $h_g = h_{\widetilde{g}}$ and $R_g = R_{\widetilde{g}}$.

Now we want to establish the uniqueness of the smaller metric $\tilde{g}$. Let us consider the metrics $g_i = u_i^{\frac{4}{n-2}} g$ satisfying $R_g = R_{g_i} \geq 0$, $h_g = h_{g_i}$ and $0 < u_i < 1$ for $i = 1, 2$. By construction of the upper and lower solutions, we are allowed to choose $\epsilon > 0$ small enough such that $\underline{u} < u_i < \bar{u}$. From Lemma 4, any solution u of (1) in $A = A(\epsilon)$ is an isolated fixed point of T. Since A is bounded in $C(\bar{M})$ and the operator T is compact, there are $u_1, u_2, ..., u_k$ solutions to the problem (1) in A with $1 \leq k < \infty$. The additivity of the degree theory of Leray-Schauder implies that

$$1 = deg(I - T, A, 0) = \sum_{m=1}^{k} i(T, u_m) = k. \tag{27}$$

Thus $k = 1$, and so $\widetilde{g}$ is unique. The proof for $n = 2$ is similar to the case $n \geq 3$. □

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