REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 53, No. 1, 2012, 55–60

DUALISTIC STRUCTURES ON KÄHLER MANIFOLDS

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ABSTRACT. Affine connections compatible with a symplectic structure are defined and conditions for two compatible connections on a Kähler manifold to form a dualistic structure are given. The special symplectic case is detailed.

1. Introduction

Dualistic structures are closely related to statistical mathematics. They consist of pairs of affine connections on statistical manifolds, compatible with a pseudo-Riemannian metric [2]. Their importance in statistical physics was underlined by many authors: [3], [4], [5] etc.

In the present paper, we shall introduce the notion of affine connections compatible with a symplectic structure and provide conditions for two such compatible connections to form a dualistic structure on a Kähler manifold. We also prove that if two connections are compatible with a symplectic structure and one of them is symplectic, then the other is a symplectic connection, too. Recall that (M,ω) is *symplectic manifold* if M is a smooth manifold and ω a closed, nondegenerate 2-form on M.

We say that a pair of affine connections ∇ and $\nabla^{'}$ are *compatible with the symplectic structure* ω if

$$X(\omega(Y,Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla^{'}_X Z), \text{ for any } X,Y,Z \in \Gamma(TM). \tag{1.1}$$

In what follows, we shall prove some preliminary results needed to determine a dualistic structure on a Kähler manifold (see Theorem 3.3).

Proposition 1.1. *Let ∇ and ∇' be affine connections on the smooth manifold M, compatible with the symplectic structure ω. Then:*

(1) $\omega(T_\nabla(X,Y),Z) = \omega(T_{\nabla'}(X,Y),Z) + (\nabla'\omega)(X,Y,Z) - (\nabla'\omega)(Y,X,Z)$, *for any X, Y, $Z \in \Gamma(TM)$;*
(2) $\nabla\omega$ *is symmetric if and only if* $\nabla'\omega$ *is symmetric.*

2000 *Mathematics Subject Classification.* 53C05, 53D17, 70G45.
Key words and phrases. symplectic structure; covariant connection.

Proof. Using the compatibility condition, we get

$$\begin{aligned}\omega(T_\nabla(X,Y),Z) &:= \omega(\nabla_X Y,Z) - \omega(\nabla_Y X,Z) - \omega([X,Y],Z)\\ &= X(\omega(Y,Z)) - \omega(Y,\nabla'_X Z) - Y(\omega(X,Z)) + \omega(X,\nabla'_Y Z)\\ &\qquad - \omega(\nabla'_X Y - \nabla'_Y X - T_{\nabla'}(X,Y),Z)\\ &:= (\nabla'\omega)(X,Y,Z) - (\nabla'\omega)(Y,X,Z) + \omega(T_{\nabla'}(X,Y),Z),\end{aligned}$$

for any X, Y, $Z \in \Gamma(TM)$ and respectively,

$$\begin{aligned}(\nabla'\omega)(X,Y,Z) &:= X(\omega(Y,Z)) - \omega(\nabla'_X Y,Z) - \omega(Y,\nabla'_X Z)\\ &:= X(\omega(Y,Z)) - X(\omega(Y,Z)) + \omega(Y,\nabla_X Z)\\ &\qquad - X(\omega(Y,Z)) + \omega(\nabla_X Y,Z)\\ &:= -(\nabla\omega)(X,Y,Z),\end{aligned}$$

for any X, Y, $Z \in \Gamma(TM)$. □

Proposition 1.2. *Let ∇ and ∇' be affine connections on the smooth manifold M, compatible with the symplectic structure ω.*

1. *If $\nabla'\omega$ is symmetric, then $T_\nabla = T_{\nabla'}$.*
2. *If $\nabla'\omega$ is symmetric and ∇' is torsion free, then $\nabla\omega$ is symmetric and ∇ is torsion free, too. In particular, if ∇' is a symplectic connection, then ∇ is a symplectic connection, too.*

Proof. Follows from Proposition 1.1. □

Proposition 1.3. *Let ∇ and ∇' be affine connections on the smooth manifold M, compatible with the symplectic structure ω.*

1. *Then $\omega(R_\nabla(X,Y,Z),W) = -\omega(Z,R_{\nabla'}(X,Y,W))[= \omega(R_{\nabla'}(X,Y,W),Z)]$, for any X, Y, $Z \in \Gamma(TM)$.*
2. *If ∇' is flat, then ∇ is flat, too.*

Proof.

$$\begin{aligned}\omega(R_\nabla(X,Y,Z),W) &:= \omega(\nabla_X\nabla_Y Z,W) - \omega(\nabla_Y\nabla_X Z,W) - \omega(\nabla_{[X,Y]}Z,W)\\ &= X(\omega(\nabla_Y Z,W)) - \omega(\nabla_Y Z,\nabla'_X W)\\ &\qquad - Y(\omega(\nabla_X Z,W)) + \omega(\nabla_X Z,\nabla'_Y W)\\ &\qquad - [X,Y](\omega(Z,W)) + \omega(Z,\nabla'_{[X,Y]}W)\\ &= -\omega(Z,\nabla'_X\nabla'_Y W - \nabla'_Y\nabla'_X W - \nabla'_{[X,Y]}W)\\ &:= -\omega(Z,R_{\nabla'}(X,Y,W)),\end{aligned}$$

for any X, Y, Z, $W \in \Gamma(TM)$. □

Corollary 1.4. *If ∇ and ∇' are compatible with the symplectic structure ω, $\nabla'\omega$ is symmetric and ∇' is flat and torsion free, then $\nabla\omega$ is symmetric and ∇ is flat and torsion free, too.*

Proof. Follows from Propositions 1.2 and 1.3. □

2. Special symplectic case

D. V. Alekseevsky, V. Cortés and C. Devchand [1] introduced the notion of *special symplectic manifold*, generalizing Freed's definition for *special Kähler* manifold.

Definition 2.1. [1] *(M, J, ∇, ω) is called special symplectic manifold if J is a complex structure on M, ∇ is a flat symplectic connection such that $d^{\nabla} J = 0$ [where $(d^{\nabla} J)(X, Y) := (\nabla_X J)Y - (\nabla_Y J)X$, for any X, $Y \in \Gamma(TM)$].*

Lemma 2.2. *If ∇ and ∇' are compatible with the symplectic structure ω, $\nabla\omega$ is symmetric and $d^{\nabla} J = 0$, for J a complex structure, then $d^{\nabla'} J = 0$.*

Proof.

$$\begin{aligned}
\omega((d^{\nabla'} J)(X, Y), Z) &:= \omega(\nabla'_X JY - J(\nabla'_X Y), Z) - \omega(\nabla'_Y JX - J(\nabla'_Y X), Z) \\
&:= X(\omega(JY, Z)) - \omega(JY, \nabla_X Z) - Y(\omega(JX, Z)) \\
&\qquad + \omega(JX, \nabla_Y Z) - \omega(J(T_{\nabla'}(X, Y) + [X, Y]), Z) \\
&= X(\omega(JY, Z)) - \omega(JY, \nabla_X Z) - Y(\omega(JX, Z)) \\
&\qquad + \omega(JX, \nabla_Y Z) - \omega(J(\nabla_X Y - \nabla_Y X), Z) \\
&:= (\nabla\omega)(X, JY, Z) + \omega(\nabla_X JY, Z) - (\nabla\omega)(Y, JX, Z) \\
&\qquad - \omega(\nabla_Y JX, Z) - \omega(J(\nabla_X Y), Z) + \omega(J(\nabla_Y X), Z) \\
&:= (\nabla\omega)(X, JY, Z) - (\nabla\omega)(Y, JX, Z) \\
&\qquad + \omega((d^{\nabla} J)(X, Y), Z),
\end{aligned}$$

for any X, Y, $Z \in \Gamma(TM)$. □

Proposition 2.3. *If (M, J, ∇, ω) is a special symplectic manifold and ∇ and ∇' are compatible with the symplectic structure ω, then (M, J, ∇', ω) is special symplectic manifold, too.*

Proof. Follows from Corollary 1.4 and Lemma 2.2. □

3. Dualistic structures on Kähler manifolds

Let J be a complex structure on the smooth manifold M and ω a J-invariant symplectic form on M such that $g(X, Y) := \omega(X, JY)$, X, $Y \in \Gamma(TM)$ is a Riemannian structure. In this case, we say that the triple (M, J, ω) is a *Kähler manifold*. It's obvious that g is also J-invariant. In this case, $\mathcal{D} := \ker\omega$ is a J-invariant integrable distribution on M and its orthogonal complement with respect to g, $\mathcal{D}^{\perp_g}$, is also J-invariant. Indeed, the integrability of $\mathcal{D}$ follows from the fact that ω is closed and its J-invariance is equivalent to the J-invariance of $\mathcal{D}^{\perp_g}$.

Example 3.1. Let ∇ be an affine connection on the Kähler manifold (M, J, ω) and $\nabla^{(J)} := \nabla - J\nabla J$ its complex conjugate connection. In the particular case when the complex structure J is ∇-recurrent [i.e., there exists a 1-form η on M such that $\nabla_X J = \eta(X)J$, for any $X \in \Gamma(TM)$] we shall determine the condition that should be satisfied by the symplectic structure ω such that ∇ and its complex conjugate connection $\nabla^{(J)}$ to be compatible with ω. In this case,

$$\nabla_X^{(J)} Y = -J(\nabla_X JY) = \nabla_X Y + \eta(X)Y, \quad \text{for any} \quad X, Y \in \Gamma(TM)$$

and using the J-invariance of ω, the compatibility relation

$$X(\omega(Y, Z)) - \omega(\nabla_X Y, Z) - \omega(JY, \nabla_X JZ) = 0$$

can be written

$$(\nabla_X \omega)(Y, Z) - \omega(JY, (\nabla_X J)Z) = 0$$

or, equivalent,

$$(\nabla_X \omega)(Y, Z) - \eta(X)\omega(Y, Z) = 0, \quad \text{for any} \quad X, Y \in \Gamma(TM)$$

and so, $\nabla_X \omega = \eta(X)\omega$, for any $X \in \Gamma(TM)$. Remark also that from the condition $d\omega = 0$ follows $(\nabla_X \omega)(Y, Z) + (\nabla_Y \omega)(Z, X) + (\nabla_Z \omega)(X, Y) = 0$, for any X, Y, $Z \in \Gamma(TM)$ and therefore, ∇ and $\nabla^{(J)}$ are compatible with ω if and only if

$$\eta(X)\omega(Y, Z) + \eta(Y)\omega(Z, X) + \eta(Z)\omega(X, Y) = 0,$$

for any X, Y, $Z \in \Gamma(TM)$.

If ∇ and ∇' are two affine connections on the Kähler manifold (M, J, ω), compatible with ω, then their complex conjugate connections $\nabla^{(J)}$ and $\nabla'^{(J)}$ are also compatible with ω.

Some invariance and symmetry properties of affine connections on a Kähler manifold (M, J, ω) are stated in the following propositions:

Proposition 3.1. *If ∇ is a J-invariant affine connection on M [i.e. $\nabla_X JY = J(\nabla_X Y)$, for any X, $Y \in \Gamma(TM)$], ∇ and ∇' are compatible with the symplectic structure ω, then ∇' is J-invariant, too.*

Proof.

$$\begin{aligned}
\omega(Y, \nabla'_X JZ - J(\nabla'_X Z)) &= \omega(Y, \nabla'_X JZ) - \omega(Y, J(\nabla'_X Z)) \\
&= X(\omega(Y, JZ)) - \omega(\nabla_X Y, JZ) - \omega(Y, J(\nabla'_X Z)) \\
&= -X(\omega(JY, Z)) + \omega(J(\nabla_X Y), Z) + \omega(JY, \nabla'_X Z) = 0,
\end{aligned}$$

for any X, Y, $Z \in \Gamma(TM)$. □

Proposition 3.2. *Let ∇ be a J-invariant affine connection on the Kähler manifold (M, J, ω, g).*

(1) *If ∇ and ∇' are compatible with the symplectic structure ω, then ∇ and ∇' are compatible with the Riemannian structure g, too.*
(2) *If $\nabla\omega$ is symmetric, then ∇g is symmetric, too.*

Proof.

$$\begin{aligned} g(\nabla_X Y, Z) + g(Y, \nabla'_X Z) &:= \omega(\nabla_X Y, JZ) + \omega(Y, J(\nabla'_X Z)) \\ &= -\omega(J(\nabla_X Y), Z) - \omega(JY, \nabla'_X Z) \\ &= -X(\omega(JY, Z)) \\ &:= X(g(Y, Z)), \end{aligned}$$

for any X, Y, $Z \in \Gamma(TM)$ and respectively,

$$\begin{aligned} (\nabla g)(X, Y, Z) &:= X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \\ &:= X(\omega(Y, JZ)) - \omega(\nabla_X Y, JZ) - \omega(Y, J(\nabla_X Z)) \\ &:= (\nabla\omega)(X, Y, JZ), \end{aligned}$$

for any X, Y, $Z \in \Gamma(TM)$. □

Recall that (M, ∇, g) is *statistical manifold* if ∇ is a torsion free affine connection and ∇g is symmetric, for g a pseudo-Riemannian metric on M. If $\nabla^{'}$ is another affine connection on M such that ∇ and $\nabla^{'}$ are compatible with g, then $\nabla^{'}$ is also torsion free and $\nabla^{'} g$ is symmetric. In this case, $(M, \nabla^{'}, g)$ is also statistical manifold called the *dual statistical manifold of* (M, ∇, g) and we say that $(\nabla, \nabla^{'}, g)$ is a *dualistic structure on* M.

Theorem 3.3. *Let (M, J, ω, g) be a Kähler manifold, ∇' a J-invariant, torsion free affine connection on M such that $\nabla'\omega$ is symmetric and ∇ and ∇' are compatible with ω. Then (M, ∇', g) and (M, ∇, g) are statistical manifolds and (∇, ∇', g) is a dualistic structure on M.*

Proof. Follows from Propositions 1.2, 3.1 and 3.2. □

Acknowledgments

The author acknowledges the support by the research grant PN-II-ID-PCE-2011-3-0921. She also expresses the thanks to the referees for their useful remarks, which definitely improved the paper.

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Recibido: 19 de abril de 2010
Aceptado: 25 de noviembre de 2010