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INEQUALITIES FOR NORMS ON PRODUCTS OF STAR ORDERED OPERATORS

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Abstract. The aim of this paper is to relate the star order in operators in a Hilbert space with certain norm inequalities. We are showing inequalities of the type $\|BXA\|_2 \leq \|XBA\|_2$ (or $\|BXA\|_2 \geq \|XBA\|_2$), which are already known under the assumption that $A = \psi(B)$, with ψ a positive increasing (or decreasing, respectively) function defined on the spectrum of B. In this work, we will study this type of inequalities with the hypothesis that $A \leq^* B$, where $A \leq^* B$ if $A^*A = B^*A$ and $AA^* = BA^*$.

1. Introduction

A topic of interest in operator theory is the study of inequalities. Many authors have worked on this issue, among others, Corach-Porta-Recht [7], Kittaneh [10], Bourin [3] and [4].

On the other hand, many authors have studied different order relations for matrices. Among others, the star order; Hartwig and Drazin in [9], Baksalary in [2]. Since many of the usual techniques used in finite dimensional spaces (as pseudoinverses or singular value decompositions) are not available for general Hilbert spaces, in [1], the authors study the *-order on the algebra $L(\mathcal{H})$ of bounded operators on a Hilbert space $\mathcal{H}$. They introduce new techniques which allow them to show that almost all the known properties which hold for matrices can be generalized to operators acting on a Hilbert space $\mathcal{H}$.

In section 2, we expose some of the properties before raised in [1] which are referred to the star order and we study several inequalities for operator norms, previously exposed in [3], such as $\|BXA\|_2 \leq \|XBA\|_2$ (or $\|BXA\|_2 \geq \|XBA\|_2$) with $A, B, X \in \mathcal{H}$, under the assumption that $A = \psi(B)$ with ψ a positive increasing (or decreasing) function on the spectrum of B.

In section 3 we study the relationship of the star order with some norms inequalities of the type studied by Bourin [3], under the hypothesis of that $A \leq^* B$.

2. Preliminaries

Given a separable Hilbert space $\mathcal{H}, L(\mathcal{H})$ denotes the algebra of bounded linear operators on $\mathcal{H}$, and $L(\mathcal{H})^+$ the cone of positive operators for an operator $A \in \mathcal{H}$.

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We denote by $\operatorname{ran}(A)$ the range or image of $A, \ker(A)$ the nullspace of $A, \sigma(A)$ the spectrum of A, A^* the adjoint of $A, |A| = (A^*A)^{1/2}$ the modulus of A. If $\mathcal{H}$ has finite dimension, the only ideals of $L(\mathcal{H})$ are the trivial ones. If $\dim \mathcal{H} = \infty$, all proper ideals of $L(\mathcal{H})$ are included in the ideal of compacts operators.

By a unitarily invariant norm $|||.|||$, we mean a norm on an ideal $\Im$ of $L(\mathcal{H})$, making $\Im$ a Banach space, and such that $|||U.X.V||| = |||X|||$ for all X in $\Im$ and all U, V unitaries in $L(\mathcal{H})$. Examples of unitarily invariant norms are the usual operator norm $\|.\|$ and the Schatten p-norms $(1 \leq p \leq \infty)$, defined for any operator X by

$$\|X\|_p = (Tr|X|^p)^{1/p} = \left(\sum \mu_n^p(X)\right)^{1/p},$$

where $\{\mu_n(X)\}$ are the singular values of X arranged in decreasing order and repeated according to their multiplicities (even if X is not compact, there is a natural definition of μ_n for all n).

Definition 2.1. *Given $A, B \in L(\mathcal{H})$, we say that A is lower or equal than B with respect to the star order, which is denoted by $A \leq^* B$, if $A^*A = B^*A$ and $AA^* = BA^*$.*

The following results were proved in [1].

Proposition 2.2. *Let $A, B \in L(\mathcal{H})$. Then*

(1) *The following statements are equivalent*
   (a) $BA^* = AA^*$.
   (b) $A = BP$, *where P is the orthogonal projection onto* $\overline{\operatorname{ran}(A^*)}$.
   (c) $A = BP$, *where P is some orthogonal projection.*

(2) *The following statements are equivalent*
   (a) $B^*A = A^*A$.
   (b) $A = QB$, *where Q is the orthogonal projection onto* $\overline{\operatorname{ran}(A)}$.
   (c) $A = QB$, *where Q is some orthogonal projection.*

Corollary 2.3. *Let $A, B \in L(\mathcal{H})$ so that $A \leq^* B$. Then* $\operatorname{ran}(A) \subseteq \operatorname{ran}(B)$ *and* $\operatorname{ran}(A^*) \subseteq \operatorname{ran}(B^*)$.

Corollary 2.4. *Let $A, B \in L(\mathcal{H})$ so that $A \leq^* B$ and $B^2 = B$. Then $A^2 = A$.*

Theorem 2.5. *Let $A, B \in L(\mathcal{H})$ such that* $\overline{\operatorname{ran}(A)} = \overline{\operatorname{ran}(A^*)}$. *Then, $A \leq^* B$ if and only if*

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \overline{\operatorname{ran}(A)} \\ \overline{\operatorname{ran}(A)}^{\perp} \end{matrix} \quad \text{and} \quad B = \begin{pmatrix} A_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \begin{matrix} \overline{\operatorname{ran}(A)} \\ \overline{\operatorname{ran}(A)}^{\perp} \end{matrix}.$$

In such case $AB = BA$ and $\sigma(A) \subseteq \sigma(B) \cup \{0\}$.

Proposition 2.6. *Let A, B be normal operators. Then, for every continuous function $f : \sigma(A) \cup \sigma(B) \cup \{0\} \to \mathbb{C}$ satisfying that $f(0) = 0$, it holds that*

$$A \leq^* B \Rightarrow f(A) \leq^* f(B).$$

Moreover, if f is also injective, then $A \leq^* B \iff f(A) \leq^* f(B)$.

Corollary 2.7. *Let A, B be normal operators such that $A \leq^* B$. Then $|A| \leq^* |B|$ and $|A^*| \leq^* |B^*|$.*

Corollary 2.8. *Given $A \in L(\mathcal{H})$ and $B \in L(\mathcal{H})^+$, if $A \leq^* B$, then also $A \in L(\mathcal{H})^+$.*

Bourin [3] studies several inequalities for norms on matrices, in particular for the Hilbert-Schmidt and operator norms. Among others, we have:

Proposition 2.9. *Let $B \in L(\mathcal{H})^+, E$ a projection, N an self-adjoint operator and ψ a positive function defined on the spectrum of B. Then*

(1) *If ψ is increasing, $\|BE\psi(B)\| \leq \|EB\psi(B)\|$.*
(2) *If ψ is increasing, $\|BN\psi(B)\|_2 \leq \|NB\psi(B)\|_2$.*
(3) *If ψ is decreasing, $\|BN\psi(B)\|_2 \geq \|NB\psi(B)\|_2$. This inequality holds also if N is normal or if either N is in the Hilbert-Schmidt class or B is compact.*

Corollary 2.10. *Let N and B be $n \times n$ matrices with N normal and B positive and let ψ be a positive function defined on the spectrum of B. Then:*

(1) *If ψ is increasing, $\|BN\psi(B)\|_2 \leq \sqrt{n}\,\|NB\psi(B)\|_2$.*
(2) *If ψ is decreasing, $\|NB\psi(B)\|_2 \leq \sqrt{n}\,\|BN\psi(B)\|_2$.*

Definition 2.11. *An operator T defined in an Hilbert space $\mathcal{H}$ is said hyponormal if $T^*T - TT^* \geq 0$ or, equivalently, if $\|T^*x\| \geq \|Tx\|$ for all $x \in \mathcal{H}$.*

Definition 2.12. *The hyponormality index of an invertible operator X is*

$$\nu(X) = \left\|X^*X^{-1}\right\|.$$

If X is no longer invertible, we set

$$\nu(X) = lim_{\epsilon\to 0}\left\|X^*|X| + \epsilon^{-1}\right\|.$$

The hyponormality index of an operator is a number which measures the lack of normality of an operator on a finite dimensional space $\mathcal{H}$. If $\mathcal{H}$ has an infinite dimension, then this number measures the lack of hyponormality.

Theorem 2.13. *Let N be an operator, B a positive operator and ψ a positive function defined on the spectrum of B.*

(1) *If ψ is increasing and $\nu(N)$ is finite,*
$$\|BN\psi(B)\|_2 \leq \nu(N)\,\|NB\psi(B)\|_2.$$
The $\nu(N)$ constant is optimal. If N is hyponormal, the inequality holds with $\nu(N) = 1$.
(2) *If ψ is decreasing, N is normal and if either N is in the Hilbert-Schmidt class or N is self-adjoint or B is compact,*
$$\|BN\psi(B)\|_2 \geq \|NB\psi(B)\|_2.$$

Definition 2.14. *A normal operator X is said to be semi-unitary if its restriction to $\operatorname{ran}(X)$ is a unitary operator.*

Theorem 2.15. *Let B be a positive operator, E a semi-unitary operator and ψ an increasing positive function defined on the spectrum of B. Then:*

$$\|BE\psi(B)\|_2 \leq \sqrt{2}\,\|EB\psi(B)\|_2 .$$

$\sqrt{2}$ is in general the best constant possible; however, if E is a projection,

$$\|BE\psi(B)\|_2 \leq \|EB\psi(B)\|_2 .$$

Proposition 2.16. *Let H be a self-adjoint operator and X, Y, Z three positive operators such that $X^2 = YZ$. Then, for all unitarily invariant norm, we have:*

$$|||XHX||| \leq |||YHZ|||.$$

3. The relationship between the star order and some inequalities for norms

If A is invertible and $A \leq^* B$, then $A = B$. In the sequel, we assume that $A \neq B$ and, in order to restrict ourselves to the case when A is non invertible, $\dim(\ker(A)) \geq 1$.

Before relating the star order with inequalities in norms, we shall present some known results and others that arise naturally from the application of the definition of that order and properties of operators.

Lemma 3.1. *Let $A \in L(\mathcal{H})$. Then,*

(1) $\|\,|B|N|A|\,\|_2 = \|BNA^*\|_2$.
(2) $\|\,|B|N|A|\,\|_2 = \|BNA\|_2 = \|BNA^*\|_2$, *if A is normal.*

Observation 3.2. We should note that using the fact that if $A \leq^* B$ then $|A| \leq^* |B|$, and applying the definition of the star order, the following equalities result:

$$AA^* = BA^* = AB^*, A^*A = B^*A = A^*B \quad \text{and} \quad |A|^2 = |A|\,|B| = |B|\,|A|.$$

Proposition 3.3. *Let $A, B, X \in L(\mathcal{H})$, $A \leq^* B$. Then, for all unitarily invariant norm the following equalities hold:*

$$|||\,X|B|\,|A|\,||| = |||X|A|\,|B|\,||| = |||\,X\,|A|^2||| = |||XA^*A||| = |||XB^*A||| = |||XA^*B|||,$$

and if $AA^* = A^*A$ then $|||XB^*A||| = |||XBA^*|||$.

Proof. Using Observation 3.2 and applying the definition of the star order then the equalities hold.

The first clear consequence of the fact that $A \leq^* B$ implies that $|A|^2 = |A|\,|B| = |B|\,|A|$ is its application in the proposition 2.16, so the condition that they are three positive operators such that $X^2 = YZ$ can be extended to two *-ordered operators.

Proposition 3.4. *Let $A, B, H \in L(\mathcal{H})$, $A \leq^* B$. Then, for all unitarily invariant norms $|||\cdot|||$ the following inequalities hold:*

$$||| \, |B| \; H \; |A| \, ||| \geq ||| \, |A| \; H \; |A| \, ||| \quad \textit{and} \quad |||\,|A| \; H \; |B| \, ||| \geq ||| \, |A| \; H \; |A| \, |||,$$

with $H = H^*$.

Proof. The proof arises from the Proposition 2.16 and Observation 3.2.

Lemma 3.5. *Let $A, B, N \in L(\mathcal{H})$. If $A \leq^* B$, then $\|N \; |A| \; |B| \, \|_2 = \|N \; |A| \; B^* \, \|_2$.*

Proof. If $A \leq^* B$, then $|A| \leq^* |B| \Rightarrow |A| \; |B| = |B| \; |A|$ and $\|N \; |A| \; |B| \, \|_2^2 = \operatorname{tr} |B| \; |A| N N^* |A| \; |B| = \operatorname{tr} B^* B |A| N^* N |A| = \operatorname{tr} B |A| N^* N |A| B^* = \|N |A| B^*\|_2^2$.

Proposition 3.6. *Let $A, B \in L(\mathcal{H})$. If $A \leq^* B$ and $\overline{\operatorname{ran}(A)} = \overline{\operatorname{ran}(A^*)}$ then*

$$\| \, |B| - |A| \, \| = \|B - A\|.$$

Proof. To verify this, let $L(\mathcal{H}) = \overline{\operatorname{ran}(A)} \oplus \overline{\operatorname{ran}(A)}^\perp$, then from Theorem 2.5

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \overline{\operatorname{ran}(A)} \\ \overline{\operatorname{ran}(A)}^\perp \end{matrix}, \quad B = \begin{pmatrix} A_{11} & 0 \\ 0 & B_{22} \end{pmatrix} \begin{matrix} \overline{\operatorname{ran}(A)} \\ \overline{\operatorname{ran}(A)}^\perp \end{matrix}, \text{ and } \quad B - A = \begin{pmatrix} 0 & 0 \\ 0 & B_{22} \end{pmatrix}$$

hence

$$\begin{aligned} \|B - A\| &= \sup \{\|(B - A)h\| : h \in L(\mathcal{H}), \|h\| = 1\} \\ &= \sup \left\{\|Bh\| : h \in \overline{\operatorname{ran}(A)}^\perp, \|h\| = 1\right\} \\ &= \sup \left\{\| \; |B|h\| : h \in \overline{\operatorname{ran}(A)}^\perp, \|h\| = 1\right\}. \end{aligned}$$

On the other hand:

$$|A| = \begin{pmatrix} |A_{11}| & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad |B| = \begin{pmatrix} |A_{11}| & 0 \\ 0 & |B_{22}| \end{pmatrix} \quad \text{and} \quad |B| - |A| = \begin{pmatrix} 0 & 0 \\ 0 & |B_{22}| \end{pmatrix}.$$

$$\begin{aligned} \| \, |B| - |A| \, \| &= \sup \{\|(|B| - |A|)h\| : h \in L(\mathcal{H}), \|h\| = 1\} \\ &= \sup \left\{\| \; |B|h\| : h \in \overline{\operatorname{ran}(A)}^\perp, \|h\| = 1\right\} = \|B - A\|. \end{aligned}$$

To use many of the results of Bourin [3] we need operators such that either one is an increasing or a decreasing function of the other, then we wonder how to express that relationship in terms of the star order.

Definition 3.7. *Let us consider the following index:* $\gamma(A) = \inf \{\|Ah\| : \|h\| = 1\}$.

In the case of finite dimension and A positive, $\gamma(A)$ is the least eigenvalue of A.

Proposition 3.8. *Let $A, B \in L(\mathcal{H})$, $A \leq^* B$ and $\| \, |B| - |A| \, \| \leq \gamma(|A|)$. Then $|A| = \psi(|B|)$ with ψ an increasing function defined on the spectrum of $|B|$.*

Proof. $A \leq^* B$ implies $|A| \leq^* |B|$ and $|A|^* = |A|$. Then, by Theorem 2.5 $|A|$ and $|B|$ have the blocks representation in $\overline{\operatorname{ran}(|A|)} \oplus \overline{\operatorname{ran}(|A|)}^{\perp}$

$$|A| = \begin{pmatrix} M_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{matrix} \overline{\operatorname{ran}(|A|)} \\ \overline{\operatorname{ran}(|A|)}^{\perp} \end{matrix} \quad \text{and} \quad |B| = \begin{pmatrix} M_{11} & 0 \\ 0 & M_{22} \end{pmatrix} \begin{matrix} \overline{\operatorname{ran}(|A|)} \\ \overline{\operatorname{ran}(|A|)}^{\perp} \end{matrix} .$$

Since $|A|$ and $|B|$ are positive, $\sigma(|A|) \subseteq \sigma(|B|) \cup \{0\} \subseteq [0, \infty)$. If $\| \, |B| - |A| \, \| \leq \gamma(|A|)$ then $\| M_{22} \| \leq \gamma(M_{11})$. Then we can define a function ψ such that $\psi(M_{11}) = M_{11}$ and $\psi(M_{22}) = 0$. Hence, we get $|A| = \psi(|B|)$. Moreover if $x_1 \in \sigma(M_{11})$, $x_2 \in \sigma(M_{22})$, since $\| M_{22} \| \leq \gamma(M_{11})$, we obtain $x_1 \geq x_2$. Then $x_1 = \psi(x_1) \geq \psi(x_2) = 0$, and ψ is increasing.

Proposition 3.9. *Let $A, B \in L(\mathcal{H})$, $A \leq^* B$ and $\| \, |A| \, \| \leq \gamma(|B| - |A|)$. Then $|A| = \psi(|B|)$ with ψ a decreasing function defined on the spectrum of $|B|$.*

Proof. Since $A \leq^* B$ implies $|A| \leq^* |B|$ and $|A|^* = |A|$, then $|A|$ and $|B|$ have the same blocks representation in $\overline{\operatorname{ran}(|A|)} \oplus \overline{\operatorname{ran}(|A|)}^{\perp}$ stated in Proposition 3.8.

If $\gamma(|B| - |A|) \geq \| \, |A| \, \|$ then $\| M_{11} \| \leq \gamma(M_{22})$ and we can define a function ψ such that $\psi(M_{11}) = M_{11}$ and $\psi(M_{22}) = 0$ with ψ a decreasing function such that $|A| = \psi(|B|)$.

Theorem 3.10. *Let $A, B \in L(\mathcal{H})$, $A \leq^* B$ and $\gamma(|A|) \geq \| \, |B| - |A| \, \|$. Then, the following inequalities hold:*

(1) $\| \, |B| \; N \; |A| \, \|_2 \leq \nu(N) \| \; N \; |B| \; |A| \, \|_2$, *with finite $\nu(N)$.*
(2) $\| \; N \; |B| \; |A| \, \|_2 \geq \| \, |B| \; N \; |A| \, \|_2$, *for $NN^* = N^*N$.*
(3) $\sqrt{n} \|N \; |B| \; |A| \, \| \geq \| \, |B| \; N \; |A| \, \|$, *for A, B, N $n \times n$ matrices, $NN^* = N^*N$.*
(4) $\sqrt{2} \|E \; |B| \; |A| \, \| \geq \| \, |B| \; E \; |A| \, \|$, *for E semi-unitary.*
(5) $\|E \; |B| \; |A| \, \| \geq \| \, |B| \; E \; |A| \, \|$, *for a projection E.*

Proof. Using the Propositions 3.8, 2.9 and the Theorems 2.13 and 2.15 we obtain the proposition.

Observation 3.11. From the results obtained in 3.3, 3.5 and 3.10, numerous other variants of the previous inequalities arise. We cite as examples the cases not involving $|A|$ and $|B|$.

Corollary 3.12. *Let $A, B, N \in L(\mathcal{H})$, N normal, $A \leq^* B$ and $\gamma(|A|) \geq \| \, |B| - |A| \, \|$. Then*

(1) $\|NAB^*\|_2 \geq \||BNA^*|\|_2$.
(2) $\|NAB^*\|_2 \geq \|BNA\|_2$ *if $AA^* = A^*A$.*
(3) $\|NAB\|_2 \geq \|BNA^*\|_2$ *if $BB^* = B^*B$.*
(4) $\|NAB\|_2 \geq \|BNA\|_2$ *if A and B are normals.*

Proposition 3.13. *Let $A, B, N \in L(\mathcal{H})$, $A \leq^* B$ and $\gamma(|A|) \geq \| \, |B| - |A| \, \|$. Then the following inequalities hold:*

(1) $\| \operatorname{Re}(N|A| \; |B|)\|_2 \geq \| \operatorname{Re}(|B|N|A|)\|_2$.
(2) $\| \operatorname{Im}(N|A| \; |B|)\|_2 \geq \| \operatorname{Im}(|B|N|A|)\|_2$.

Proof.

(1) As $\operatorname{Re}(N|A|\,|B|) = \frac{1}{2}(N|A|\,|B| + |B|\,|A|N^*)$ we have:

$$\begin{aligned} 4\|\operatorname{Re}(N|A|\,|B|)\|_2^2 &= 2\|N|A|\,|B|\,\|_2^2 + \operatorname{tr}(N|A|\,|B|)^2 + \operatorname{tr}(|B|\,|A|N^*)^2 \\ &\geq 2\|\,|B|N\,|A|\,\|_2^2 + \operatorname{tr}(|B|N|A|)^2 + \operatorname{tr}(|B|\,|A|N^*)^2 \\ &= 4\|\operatorname{Re}(|B|N|A|)\,\|_2^2. \end{aligned}$$

(2) The proof is similar to that of the previous item.

Theorem 3.14. *Let* $A, B \in L(\mathcal{H})$, $A \leq^* B$ *and* $\gamma(|B| - |A|) \geq \|\,|A|\,\|$ *then the following inequalities hold:*

(1) $\|H|B|\,|A|\,\|_2 \leq \|\,|B|\;H\;|A|\,\|_2$, *for* $H = H^*$.
(2) $\|N|B|\,|A|\,\|_2 \leq \|\,|B|\;N\;|A|\,\|_2$, *for* $NN^* = N^*N$.
(3) $\sqrt{n}\|N|B|\,|A|\,\| \leq \|\,|B|\,|A|\,N\|$, *for* $n \times n$ *matrices* A, B, N *and* $NN^* = N^*N$.
(4) $\|\operatorname{Re}(N|A|\,|B|)\|_2 \leq \|\operatorname{Re}(|B|N|A|)\|_2$.
(5) $\|\operatorname{Im}(N|A|\,|B|)\|_2 \leq \|\operatorname{Im}(|B|N|A|)\|_2$.

Proof. Using the Propositions 2.9, 2.10, 3.9 and Theorem 2.13 we obtain the Proposition.

Observation 3.15. Using 3.1, 3.2 and 3.3 other versions of the inequalities stated in Proposition 3.14 can be obtained.

Proposition 3.16. *Let* $A, B \in L(\mathcal{H})$, $A \leq^* B$ *and* $\gamma(|A|) \geq \|\,|B| - |A|\,\|$. *Let* f *be a continuous increasing function,* $f : \sigma(|A|) \cup \sigma(|B|) \to \mathbb{C}$ *such that* $f(0) = 0$. *Then*

$$f(|A|) \leq^* f(|B|)$$

and

$$\gamma(f|A|) \geq \|\,f(|B|) - f(|A|)\,\|.$$

Proof. Using the same arguments as in Proposition 3.8, we obtain the inequalities.

Observation 3.17. If we replace at Proposition 3.4 $|A|$ and $|B|$ by $f(|A|)$ and $f(|B|)$ the inequalities also hold.

References

[1] J. Antezana, C. Cano, I. Mosconi and D. Stojanoff, A note on the star order in Hilbert spaces, Linear Multilinear Algebra. 58 (2010), no. 7-8, 1037–1051.93, 94
[2] J. Baksalary, A relationship between the star and minus orderings, Linear Algebra Appl. 82(1986), 163–167. 93
[3] J.C. Bourin, A concavity inequality for symmetric norms, Linear Algebra Appl. 413(2006), 212–217. 93, 95, 97
[4] J.C. Bourin, Some inequalities for norms on matrices and operators, Linear Algebra Appl. 292(1999), 139–154. 93
[5] C. Cano, I. Mosconi, D. Stojanoff, Some operator inequalities for unitarily invariant norms, Rev. Un. Mat. Argentina. 46 (2005), no. 1, 53–66.
[6] J. B. Conway, A Course in Functional Analysis, Springer-Verlag, New York, 1990.
[7] G. Corach, H. Porta and L. Recht, An operator inequality, Linear Algebra Appl. 142(1990), 153–158. 93

[8] M. P. Drazin, Natural structures on semigroups with involution, Bull. Amer. Math. Soc. 84(1978), 139–141.
[9] R. Hartwig and M. Drazin, Lattice properties of the ∗-order for complex matrices, J. Matrix Anal. Appl. 86(1982), 359–378. 93
[10] F. Kittaneh, On some operator inequalities, Linear Algebra Appl. 208/209(1994), 19–28. 93

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