REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 53, No. 1, 2012, 101–112

HÁJEK-RÉNYI INEQUALITY FOR DEPENDENT RANDOM VARIABLES IN HILBERT SPACE AND APPLICATIONS

YU MIAO

Abstract. In this paper, we obtain some Hájek-Rényi inequalities for sequences of Hilbert valued random variables which are associated, negatively associated and ϕ-mixing. As applications, we give some almost sure convergence theorems for these dependent sequences in Hilbert space. These results extend and improve some well-known results.

1. Introduction

In the paper [10], Hájek and Rényi established an inequality which they formulated in the following way: $X_1, X_2, \cdots$ are independent random variables and $S_n = \sum_{i=1}^n X_i, n \geq 1$. For each k, $EX_k = 0$ and $EX_k^2 < \infty$, while $\{b_k, k \geq 1\}$ is a non-increasing sequence of positive numbers. Then, for any $\varepsilon > 0$ and any positive integers n and m $(n < m)$,

$$P\left(\max_{n\leq k\leq m} b_k|S_k| \geq \varepsilon|\right) \leq \frac{1}{\varepsilon^2}\left(b_n^2\sum_{k=1}^n EX_k^2 + \sum_{k=n+1}^m b_k^2 EX_k^2\right). \tag{1.1}$$

It is well-known that Kolmogorov's inequality is the particular case $b_k = 1$, for all k and $n = 1$ in (1.1).

Afterwards this inequality was extended to real valued martingales (see [4]). Since then, this inequality has been studied by many authors. For the case of $\mathbb{R}$-valued random variables, Sung [22] obtained the Hájek-Rényi inequality for the associated sequence. Liu et al. [19] considered the negatively associated random variables. Cohn [5] studied a Hájek-Rényi inequality for Markov chain. Tómács and Líbor [23], Hu et al. [12] showed the inequality for demimartingale. For the case of Banach space, Gan [8] gave the Hájek-Rényi inequality for martingale. Furthermore, Gan and Qiu [9] studied a general version of this inequality.

In this paper, we extend the Hájek-Rényi inequality of associated, negatively associated and ϕ-mixing for random sequences to a separable real Hilbert space and derive the strong law of large numbers for these dependent sequences with values in a separable real Hilbert space.

2000 *Mathematics Subject Classification.* 60F05, 60F17, 60G10.

Key words and phrases. Hájek-Rényi inequality, associated, negatively associated, ϕ-mixing almost sure convergence.

2. Associated sequences

Lehmann [17] introduced the notion of positive quadrant dependence: Two random variables X_1 and X_2 are called positively quadrant dependent if

$$P(X_1 > x_1, X_2 > x_2) \geq P(X_1 > x_1)P(X_2 > x_2), \quad \text{for all} \ \ x_1, x_2.$$

This definition was subsequently extended to the multivariate case by Esary et al. in [6]. A finite sequence $\{X_i, 1 \leq i \leq n\}$ is said to be associated if for any componentwise nondecreasing functions f and g on $\mathbb{R}^n$

$$\mathrm{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0$$

where the covariance exists. An infinite sequence $\{X_n, n \geq 1\}$ is said to be associated if every finite subfamily is associated. It is easy to see that if $\{X_n\}$ is a sequence of associated random variables, then the covariance is nonnegative. For gaussian processes, it is well-known that positive association corresponds with positive correlation. Let us recall that the independent random variables are associated and nondecreasing functions of associated random variables are also associated.

The definition of association has found several applications in reliability theory [1]. The basic concept actually appears in [11] in the context of percolation models and it was subsequently applied to the Ising models of statistical mechanics in [7]; in the statistical mechanics literature (see, e.g., [16]), which developed independently of reliability theory, associated random variables are said to satisfy the FKG inequalities.

As in Burton et al. [3] we can give definition of association for random vectors with values in $\mathbb{R}^d$. Let $\{X_1, \ldots, X_n\}$ be a sequence of $\mathbb{R}^d$-valued random vectors. $\{X_1, \ldots, X_n\}$ is said to be associated if

$$\mathrm{Cov}(f(X_1, \ldots, X_n), g(X_1, \ldots, X_n)) \geq 0$$

for any nondecreasing functions f and g on $\mathbb{R}^{md}$, such that the covariance exists.

Let H be a separable real Hilbert space with the norm $\|\cdot\|$ generated by an inner product, $\langle \cdot, \cdot \rangle$ and let $\{e_k, k \geq 1\}$ be an orthonormal basis in H. A sequence of random variables $\{X_n, n \geq 1\}$ with values in a separable real Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is said to be associated, if for some orthonormal basis $\{e_k, k \geq 1\}$ of H and for any $d \geq 1$ the d-dimensional sequence $(\langle X_i, e_1 \rangle, \cdots, \langle X_i, e_d \rangle)$, $i \geq 1$ is associated.

Lemma 2.1. [20] *Suppose $X_1, \ldots, X_n$ are associated random variables with mean zero and finite variance. Then*

$$E\left(\max_{1 \leq k \leq n} S_k\right)^2 \leq E(S_n^2).$$

Lemma 2.2. *Let $\{X_n, n \geq 1\}$ be an associated sequence of H-valued random variables with $EX_n = 0$ and $E\|X_n\|^2 < \infty$, $n \geq 1$. Then for any $\varepsilon > 0$, we have*

$$P\left(\max_{1 \leq i \leq n} \|S_i\| \geq \varepsilon\right) \leq \frac{2E\|S_n\|^2}{\varepsilon^2}.$$

Proof. Let $\{e_k, k \geq 1\}$ be an orthonormal basis in H. Then, by Parseval's identity and Lemma 2.1, we have

$$\begin{aligned}
E \max_{1\leq k\leq n} \left\| \sum_{i=1}^{k} X_i \right\|^2 &= E \max_{1\leq k\leq n} \sum_{j=1}^{\infty} \left(\left\langle \sum_{i=1}^{k} X_i, e_j \right\rangle \right)^2 \\
&\leq \sum_{j=1}^{\infty} E \max_{1\leq k\leq n} \left(\sum_{i=1}^{k} \langle X_i, e_j \rangle \right)^2 \\
&= \sum_{j=1}^{\infty} E \max \left\{ \left(\max_{1\leq k\leq n} \sum_{i=1}^{k} \langle X_i, e_j \rangle \right)^2, \left(\max_{1\leq k\leq n} \left(- \sum_{i=1}^{k} \langle X_i, e_j \rangle \right) \right)^2 \right\} \\
&\leq \sum_{j=1}^{\infty} E \left(\max_{1\leq k\leq n} \sum_{i=1}^{k} \langle X_i, e_j \rangle \right)^2 + \sum_{j=1}^{\infty} E \left(\max_{1\leq k\leq n} \left(- \sum_{i=1}^{k} \langle X_i, e_j \rangle \right) \right)^2 \\
&\leq 2 \sum_{j=1}^{\infty} E \left(\sum_{i=1}^{n} \langle X_i, e_j \rangle \right)^2 \\
&= 2E\|S_n\|^2.
\end{aligned}$$

□

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be an associated sequence of H-valued random variables with $EX_n = 0$ and $E\|X_n\|^2 < \infty$, $n \geq 1$. Let $\{b_n, n \geq 1\}$ be a sequence of positive nondecreasing real numbers. Then for any $\varepsilon > 0$ and any $\alpha > 1$, we have*

$$P\left(\max_{1\leq k\leq n} \left\| \frac{1}{b_k} \sum_{j=1}^{k} X_j \right\| \geq \varepsilon \right) \leq \frac{2}{\varepsilon^2} \frac{\alpha^2}{1 - \frac{1}{\alpha^2}} \left\{ \sum_{j=1}^{n} \frac{E\|X_j\|^2}{b_j^2} + 2 \sum_{j=1}^{n} \frac{E\langle X_j, S_{j-1} \rangle}{b_j^2} \right\}.$$

Remark 2.1. Here the minimum in the above coefficient can be obtained, i.e.,

$$\inf_{\alpha>1} \frac{\alpha^2}{1 - \frac{1}{\alpha^2}} = 4.$$

Proof. Without loss of generality, we may assume that $b_n \geq 1$ for all $n \geq 1$. Let $\alpha > 1$. For $i \geq 0$, define

$$A_i = \left\{ 1 \leq k \leq n; \alpha^i \leq b_k < \alpha^{i+1} \right\}.$$

When $A_i \neq \emptyset$, we let $\nu(i) = \max\{k; k \in A_i\}$. Let t_n be the index of the last nonempty set A_i. If $k \in A_i$, then $\alpha^i \leq b_k \leq b_{\nu(i)} < \alpha^{i+1}$. By Lemma 2.2, we have

$$\begin{aligned}
P\left(\max_{1\leq k\leq n}\left\|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right\|\geq\varepsilon\right) &= P\left(\max_{0\leq i\leq t_n, A_i\neq\emptyset}\max_{k\in A_i}\left\|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right\|\geq\varepsilon\right)\\
&\leq \sum_{i=0,A_i\neq\emptyset}^{t_n} P\left(\frac{1}{\alpha^i}\max_{1\leq k\leq\nu(i)}\left\|\sum_{j=1}^{k}X_j\right\|\geq\varepsilon\right)\\
&\leq \frac{2}{\varepsilon^2}\sum_{i=0,A_i\neq\emptyset}^{t_n}\frac{1}{\alpha^{2i}}E\left(\left\|\sum_{j=1}^{\nu(i)}X_j\right\|^2\right)\\
&= \frac{2}{\varepsilon^2}\sum_{i=0,A_i\neq\emptyset}^{t_n}\frac{1}{\alpha^{2i}}\sum_{l=1}^{\infty}E\left\langle\sum_{j=1}^{\nu(i)}X_j, e_l\right\rangle^2\\
&= \frac{2}{\varepsilon^2}\sum_{i=0,A_i\neq\emptyset}^{t_n}\frac{1}{\alpha^{2i}}\sum_{j=1}^{\nu(i)}\left(E\|X_j\|^2+2E\langle X_j,S_{j-1}\rangle\right)\\
&= \frac{2}{\varepsilon^2}\sum_{j=1}^{n}\left(E\|X_j\|^2+2E\langle X_j,S_{j-1}\rangle\right)\sum_{i=0,A_i\neq\emptyset,\nu(i)\geq j}^{t_n}\frac{1}{\alpha^{2i}}.
\end{aligned}$$

Let $i_0 = \min\{i; A_i \neq \emptyset, \nu(i) \geq j\}$, then it is easy to check $b_j \leq b_{\nu(i_0)} < \alpha^{i_0+1}$. It follows that

$$\sum_{i=0,A_i\neq\emptyset,\nu(i)\geq j}^{t_n}\frac{1}{\alpha^{2i}} < \frac{1}{1-\frac{1}{\alpha^2}}\frac{1}{\alpha^{2i_0}} < \frac{\alpha^2}{1-\frac{1}{\alpha^2}}\frac{1}{b_j^2}.$$

From the above discussions, the proof of the theorem can be completed. □

By Theorem 2.3, we can obtain the following Hájek-Rényi inequality for associated random variables.

Theorem 2.4. *Let $\{X_n, n \geq 1\}$ be an associated sequence of H-valued random variables with $EX_n = 0$ and $E\|X_n\|^2 < \infty$, $n \geq 1$. Let $\{b_n, n \geq 1\}$ be a sequence of positive nondecreasing real numbers. Then for any $\varepsilon > 0$, any $\alpha > 1$ and any $m < n$, we have*

$$\begin{aligned}
P\left(\max_{m\leq k\leq n}\left\|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right\|\geq\varepsilon\right) &\leq \frac{4}{\varepsilon^2 b_m^2}E\|S_m\|^2 + \frac{4}{\varepsilon^2}\frac{\alpha^2}{1-\frac{1}{\alpha^2}}\sum_{j=m+1}^{n}\frac{E\langle X_j,S_j\rangle}{b_j^2}\\
&\leq \frac{8}{\varepsilon^2 b_m^2}\sum_{i=1}^{m}E\langle X_i,S_i\rangle + \frac{16}{\varepsilon^2}\frac{\alpha^2}{1-\frac{1}{\alpha^2}}\sum_{j=m+1}^{n}\frac{E\langle X_j,S_j\rangle}{b_j^2}.
\end{aligned}$$

Proof. Observe that

$$P\left(\max_{m\le k\le n}\left\|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right\|\ge\varepsilon\right)\le P\left(\max_{m\le k\le n}\frac{\|S_m\|}{b_k}\ge\frac{\varepsilon}{2}\right)+P\left(\max_{m\le k\le n}\frac{\|S_k-S_m\|}{b_k}\ge\frac{\varepsilon}{2}\right)$$
$$=: I_n+II_n.$$

For the term I_n, it is easy to see that

$$I_n\le\frac{4}{\varepsilon^2b_m^2}E\|S_m\|^2=\frac{4}{\varepsilon^2b_m^2}E\left(\sum_{i=1}^{m}\|X_i\|^2+2\sum_{i<j}\langle X_i,X_j\rangle\right)\le\frac{8}{\varepsilon^2b_m^2}\sum_{i=1}^{m}E\langle X_i,S_i\rangle. \tag{2.1}$$

For the term II_n, noting that

$$\max_{m\le k\le n}\frac{\|S_k-S_m\|}{b_k}=\max_{1\le k\le n-m}\frac{\left\|\sum_{j=1}^{k}X_{m+j}\right\|}{b_{m+k}},$$

then, by Theorem 2.3, we obtain

$$\begin{aligned}II_n&\le\frac{8}{\varepsilon^2}\frac{\alpha^2}{1-\frac{1}{\alpha^2}}\left\{\sum_{j=1}^{n-m}\frac{E\|X_{m+j}\|^2}{b_{m+j}^2}+2\sum_{j=1}^{n-m}\frac{E\langle X_{m+j},S_{m+j-1}-S_m\rangle}{b_{m+j}^2}\right\}\\&\le\frac{16}{\varepsilon^2}\frac{\alpha^2}{1-\frac{1}{\alpha^2}}\sum_{j=1}^{n-m}\frac{E\langle X_{m+j},S_{m+j}-S_m\rangle}{b_{m+j}^2}\\&=\frac{16}{\varepsilon^2}\frac{\alpha^2}{1-\frac{1}{\alpha^2}}\sum_{j=m+1}^{n}\frac{E\langle X_j,S_j-S_m\rangle}{b_j^2}\\&\le\frac{16}{\varepsilon^2}\frac{\alpha^2}{1-\frac{1}{\alpha^2}}\sum_{j=m+1}^{n}\frac{E\langle X_j,S_j\rangle}{b_j^2}.\end{aligned}\tag{2.2}$$

Here we use the definition of association. Thus the desired result can be obtained by (2.1) and (2.2). □

Corollary 2.5. *Let $\{b_n,n\ge1\}$ be a sequence of positive nondecreasing real numbers. Let $\{X_n,n\ge1\}$ be an associated sequence of H-valued random variables with $EX_n=0$ and satisfying $\sum_{j=1}^{\infty}E\langle X_j,S_j\rangle/b_j^2<\infty$. If $0<r<2$, then*

$$E\left(\sup_{n\ge1}(\|S_n\|/b_n)^r\right)<\infty.$$

Proof. By Theorem 2.3, we get

$$
\begin{aligned}
E\left(\sup_{n\geq 1}(\|S_n\|/b_n)^r\right) &= \int_0^\infty P\left(\sup_{n\geq 1}(\|S_n\|/b_n)^r > t\right)dt \\
&\leq 1+\int_1^\infty P\left(\sup_{n\geq 1}(\|S_n\|/b_n)^r > t\right)dt \\
&\leq 1+\int_1^\infty \frac{2}{t^{2/r}}\frac{\alpha^2}{1-\frac{1}{\alpha^2}}\left\{\sum_{j=1}^\infty \frac{E\|X_j\|^2}{b_j^2}+2\sum_{j=1}^\infty \frac{E\langle X_j, S_{j-1}\rangle}{b_j^2}\right\}dt \\
&\leq 1+\frac{4\alpha^2}{1-\frac{1}{\alpha^2}}\left\{\sum_{j=1}^\infty \frac{E\langle X_j, S_j\rangle}{b_j^2}\right\}\int_1^\infty \frac{1}{t^{2/r}}dt < \infty.
\end{aligned}
$$

□

Corollary 2.6. *Let $\{b_n, n \geq 1\}$ be a nondecreasing unbounded sequence of positive real numbers. Let $\{X_n, n \geq 1\}$ be an associated sequence of H-valued random variables with $EX_n = 0$ and satisfying $\sum_{j=1}^\infty E\langle X_j, S_j\rangle/b_j^2 < \infty$. Then*

$$S_n/b_n \to 0, \quad a.s.$$

Proof. For any $m < N$, by Theorem 2.4, it follows that

$$P\left(\max_{m\leq n\leq N}\frac{\|S_n\|}{b_n} > \varepsilon\right) \leq \frac{8}{\varepsilon^2 b_m^2}\sum_{i=1}^m E\langle X_i, S_i\rangle + \frac{4}{\varepsilon^2}\frac{\alpha^2}{1-\frac{1}{\alpha^2}}\sum_{j=m+1}^N \frac{E\langle X_j, S_j\rangle}{b_j^2},$$

then, by Kronecker's lemma, we can obtain

$$
\begin{aligned}
\lim_{m\to\infty} P\left(\bigcup_{n=m}^\infty\left\{\frac{\|S_n\|}{b_n} > \varepsilon\right\}\right) &= \lim_{m\to\infty} P\left(\bigcup_{N=m}^\infty\left\{\max_{m\leq n\leq N}\frac{\|S_n\|}{b_n} > \varepsilon\right\}\right) \\
&= \lim_{m\to\infty}\lim_{N\to\infty} P\left(\left\{\max_{m\leq n\leq N}\frac{\|S_n\|}{b_n} > \varepsilon\right\}\right) \\
&\to 0
\end{aligned}
$$

which implies the desired result. □

Remark 2.2. In particular, taking $b_n = n$, the result of Ko et al. in [15, Theorem 2.4] is a simple corollary.

Corollary 2.7. *Let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers and $b_n = \sum_{i=1}^n a_i$, $n \geq 1$. Let $\{X_n, n \geq 1\}$ be an associated sequence of H-valued random variables with $EX_n = 0$ and satisfying*

$$\sum_{j=1}^\infty \frac{1}{b_j^2}\sum_{i=1}^j a_i a_j E\langle X_j, X_i\rangle < \infty.$$

Then

$$\sum_{i=1}^n \frac{a_i X_i}{b_n} \to 0, \quad a.s.$$

Proof. Noting that $\{a_nX_n, n \geq 1\}$ is a sequence of associated random variables and by Corollary 2.6, we can complete the proof of the desired result. □

Remark 2.3. All the above results extend the works of Sung in [22] from $\mathbb{R}$-valued random variables to Hilbert space.

3. Negatively associated sequences

A finite sequence $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (abbreviated to NA) if for any disjoint subsets $A, B \subset \{1, 2, \dots, n\}$ and any coordinatewise nondecreasing functions f on $\mathbb{R}^{|A|}$ and g on $\mathbb{R}^{|B|}$

$$\mathrm{Cov}(f(X_i, i \in A), g(X_i, i \in B)) \leq 0$$

where the covariance exists. An infinite sequence $\{X_n, n \geq 1\}$ is said to be NA if every finite subfamily is NA. This concept was introduced by Joag-Dev and Proschan in [14]. A number of well-known multivariate distributions posses the NA property, such as multinomial distribution, multivariate hypergeometric distribution, negatively correlated normal distribution, permutation distribution and joint distribution of ranks.

As the definition of associated random variables, we can define the NA sequences in Hilbert space. A sequence of random variables $\{X_n, n \geq 1\}$ with values in a separable real Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is said to be as NA, if for some orthonormal basis $\{e_k, k \geq 1\}$ of H and for any $d \geq 1$ the d-dimensional sequence $(\langle X_i, e_1\rangle, \cdots, \langle X_i, e_d\rangle)$, $i \geq 1$ is NA.

Lemma 3.1. [21] *Let $\{X_n, n \geq 1\}$ be a sequence of NA random variables with finite second moments and zero means. Then we have*

$$E \max_{1\leq k\leq n} \left(\sum_{i=1}^{k} X_i \right)^2 \leq 2 \sum_{i=1}^{n} EX_i^2.$$

Theorem 3.2. *Let $\{X_n, n \geq 1\}$ be a NA sequence of H-valued random variables with $EX_n = 0$ and $E\|X_n\|^2 < \infty$, $n \geq 1$. Let $\{b_n, n \geq 1\}$ be a sequence of positive nondecreasing real numbers. Then for any $\varepsilon > 0$, we have*

$$P\left(\max_{1\leq k\leq n} \left\| \frac{1}{b_k} \sum_{i=1}^{k} X_i \right\| \geq \varepsilon \right) \leq 8 \sum_{i=1}^{n} \frac{E\|X_i\|^2}{\varepsilon^2 b_i^2}.$$

Proof. From Lemma 3.1, we obtain

$$\begin{aligned} E\max_{1\le k\le n}\left\|\sum_{i=1}^{k}X_i\right\|^2 &= E\max_{1\le k\le n}\sum_{j=1}^{\infty}\left(\left\langle\sum_{i=1}^{k}X_i,e_j\right\rangle\right)^2\\ &\le \sum_{j=1}^{\infty}E\max_{1\le k\le n}\left(\left\langle\sum_{i=1}^{k}X_i,e_j\right\rangle\right)^2\\ &= \sum_{j=1}^{\infty}E\max_{1\le k\le n}\left(\sum_{i=1}^{k}\langle X_i,e_j\rangle\right)^2\\ &\le 2\sum_{j=1}^{\infty}\sum_{i=1}^{n}E\left(\langle X_i,e_j\rangle\right)^2 = 2\sum_{i=1}^{n}E\|X_i\|^2. \end{aligned} \tag{3.1}$$

Let $S_n=\sum_{j=1}^{n}X_j$. Without loss of generality, setting $b_0=0$, we have

$$\begin{aligned} S_k &= \sum_{j=1}^{k}b_j\frac{X_j}{b_j} = \sum_{j=1}^{k}\left(\sum_{i=1}^{j}(b_i-b_{i-1})\frac{X_j}{b_j}\right)\\ &= \sum_{i=1}^{k}(b_i-b_{i-1})\sum_{i\le j\le k}\frac{X_j}{b_j}. \end{aligned}$$

Since $(1/b_k)\sum_{j=1}^{k}(b_j-b_{j-1})=1$, then we have

$$\begin{aligned} \max_{1\le k\le n}\frac{1}{b_k}\|S_k\| &\le \max_{1\le k\le n}\max_{1\le i\le k}\left\|\sum_{i\le j\le k}\frac{X_j}{b_j}\right\|\\ &\le \max_{1\le i\le k\le n}\left\|\sum_{j\le k}\frac{X_j}{b_j}-\sum_{j<i}\frac{X_j}{b_j}\right\| \le 2\max_{1\le i\le n}\left\|\sum_{j=1}^{i}\frac{X_j}{b_j}\right\|. \end{aligned}$$

Since $\{X_j/b_j, j\ge 1\}$ is still a sequence of NA random variables, then by (3.1), we have

$$P\left(\max_{1\le k\le n}\frac{1}{b_k}\|S_k\|\ge\varepsilon\right)\le P\left(2\max_{1\le i\le n}\left\|\sum_{j=1}^{i}\frac{X_j}{b_j}\right\|\ge\varepsilon\right)\le 8\sum_{i=1}^{n}\frac{E\|X_i\|^2}{\varepsilon^2 b_i^2}.$$

□

Theorem 3.3. *Let $\{X_n, n\ge 1\}$ be a NA sequence of H-valued random variables with $EX_n=0$ and $E\|X_n\|^2<\infty$, $n\ge 1$. Let $\{b_n, n\ge 1\}$ be a sequence of positive nondecreasing real numbers. Then for any $\varepsilon>0$ and any $m<n$, we have*

$$P\left(\max_{m\le k\le n}\left\|\frac{1}{b_k}\sum_{i=1}^{k}X_i\right\|\ge\varepsilon\right)\le\frac{32}{\varepsilon^2}\left(\frac{1}{b_m^2}\sum_{i=1}^{m}E\|X_i\|^2+\sum_{i=m+1}^{n}\frac{E\|X_i\|^2}{b_i^2}\right).$$

Proof. Observe that

$$P\left(\max_{m\le k\le n}\left\|\frac{1}{b_k}\sum_{j=1}^{k}X_j\right\|\ge\varepsilon\right)\le P\left(\max_{m\le k\le n}\frac{\|S_m\|}{b_k}\ge\frac{\varepsilon}{2}\right)+P\left(\max_{m\le k\le n}\frac{\|S_k-S_m\|}{b_k}\ge\frac{\varepsilon}{2}\right)=:I_n+II_n.$$

For the term I_n, by the similar proof of Theorem 3.2, it is easy to check that

$$I_n\le\frac{32}{\varepsilon^2 b_m^2}\sum_{i=1}^{m}E\|X_i\|^2. \tag{3.2}$$

For the term II_n, noting that

$$\max_{m\le k\le n}\frac{\|S_k-S_m\|}{b_k}=\max_{1\le k\le n-m}\frac{\left\|\sum_{j=1}^{k}X_{m+j}\right\|}{b_{m+k}},$$

then, by Theorem 3.2, we obtain

$$II_n\le\frac{32}{\varepsilon^2}\sum_{i=1}^{n-m}\frac{E\|X_{m+i}\|^2}{b_{m+i}^2}=\frac{32}{\varepsilon^2}\sum_{i=m+1}^{n}\frac{E\|X_i\|^2}{b_i^2}. \tag{3.3}$$

Here we use the definition of association. Thus the desired result can be obtained by (3.2) and (3.3). □

Remark 3.1. Let $\{X_n, n\ge 1\}$ be a NA sequence of $\mathbb{R}$-valued random variables with $EX_n=0$ and $E|X_n|^2<\infty$, $n\ge 1$. Let $\{b_n, n\ge 1\}$ be a sequence of positive nondecreasing real numbers. Then for any $\varepsilon>0$ and any $m<n$, Liu et al. [19] obtained

$$P\left(\max_{m\le k\le n}\left|\frac{1}{b_k}\sum_{i=1}^{k}X_i\right|\ge\varepsilon\right)\le\frac{128}{\varepsilon^2}\left(\frac{1}{b_m^2}\sum_{i=1}^{m}EX_i^2+\sum_{i=m+1}^{n}\frac{EX_i^2}{b_i^2}\right). \tag{3.4}$$

Hence Theorem 3.3 improves and extends the result of Liu et al. in [19].

As the proofs of Corollary 2.5 and 2.6, we have following

Corollary 3.4. *Let $\{b_n, n\ge 1\}$ be a sequence of positive nondecreasing real numbers. Let $\{X_n, n\ge 1\}$ be a NA sequence of H-valued random variables with $EX_n=0$ and satisfying $\sum_{j=1}^{\infty}E\|X_j\|/b_j^2<\infty$. If $0<r<2$, then*

$$E\left(\sup_{n\ge 1}(\|S_n\|/b_n)^r\right)<\infty.$$

Corollary 3.5. *Let $\{b_n, n\ge 1\}$ be a nondecreasing unbounded sequence of positive real numbers. Let $\{X_n, n\ge 1\}$ be a NA sequence of H-valued random variables with $EX_n=0$ and satisfying $\sum_{j=1}^{\infty}E\|X_j\|/b_j^2<\infty$. Then*

$$S_n/b_n\to 0,\quad a.s.$$

Remark 3.2. In particular, taking $b_n = n$, the result of Ko et al. in [15, Corollary 3.5] is a simple corollary.

Corollary 3.6. *Let $\{X_n, n \geq 1\}$ be a NA sequence of H-valued random variables with $EX_n = 0$. Then for any $0 < t < 2$ and $\varepsilon > 0$, we have*

$$P\left(\sup_{j\geq m}\frac{\|S_j\|}{j^{1/t}} \geq \varepsilon\right) \leq 8\varepsilon^{-2}\frac{2}{2-t}m^{(t-2)/t}\sup_n E\|X_n\|^2.$$

Corollary 3.7. *Let $\{X_n, n \geq 1\}$ be a NA sequence of H-valued random variables with $EX_n = 0$ and $\sup_n E\|X_n\|^2 < \infty$. Then for any $0 < t < 2$, we have*

$$E\sup_n\left(\frac{\|S_n\|}{n^{1/t}}\right)^r < \infty, \quad \textit{for all} \ \ 0 < r < 2$$

and

$$\frac{S_n}{n^{1/t}} \to 0, \quad a.s.$$

4. ϕ-mixing sequences

Let $\{X_i; i \geq 1\}$ be a sequence of random variables and for any $1 \leq i \leq j \leq \infty$ denote $\mathcal{M}_i^j$ as the σ-field generated by $\{X_k; i \leq k \leq j\}$. A sequence of random variables $\{X_i; i \geq 1\}$ is said to be ϕ-mixing, if for any $A \in \mathcal{M}_1^k$ and $B \in \mathcal{M}_{k+j}^\infty$,

$$|P(B|A) - P(B)| \leq \phi(j), \quad \phi(j) \geq 0,$$

where $1 \geq \phi(1) \geq \phi(2) \geq \cdots$, and $\lim_{j\to\infty}\phi(j) = 0$. For more information of ϕ-mixing (see Billingsley [2]). Intuitively, $\{X_1, X_2, \cdots, X_n\}$ is ϕ-mixing if X_i and X_{i+j} become virtually independent as j becomes large. For example, the waiting time W_i of an M/M/1 delay-in-queue is ϕ-mixing, because W_i and W_{i+j} become virtually independent as j becomes large. In addition, m-dependent sequence implies ϕ-mixing, while for gaussian processes, ϕ-mixing corresponds to m-dependence (see [13]).

A sequence of random variables $\{X_n, n \geq 1\}$ with values in a separable real Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is said to be as ϕ-mixing, if for some orthonormal basis $\{e_k, k \geq 1\}$ of H and for any $d \geq 1$ the d-dimensional sequence $(\langle X_i, e_1\rangle, \cdots, \langle X_i, e_d\rangle)$, $i \geq 1$ is ϕ-mixing.

Lemma 4.1. [18] *Let $\{X_n, n \geq 1\}$ be a sequence of ϕ-mixing random variables with finite second moments, zero means and $\sum_n \phi^{1/2}(n) < \infty$. Then there exists a positive constant C, such that*

$$E\max_{1\leq k\leq n}\left(\sum_{i=1}^k X_i\right)^2 \leq C\sum_{i=1}^n EX_i^2.$$

Based on Lemma 4.1 and by using the same proofs as NA sequence we can also obtain some estimates and limit behaviors for ϕ-mixing sequence, which are similar to the results in Section 3.

Acknowledgements

The author is very grateful to the editor and referees for their critical and valuable comments, which improved the presentation of this article. This work is supported by NSFC (No. 11001077), NCET (No. NCET-11-0945) and HASTIT (No. 2011HASTIT011).

References

[1] R. E. Barlow and F. Proschan, Statistical theory of reliability and life testing. Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1975. 102

[2] P. Billingsley, *Convergence of probability measures. Second edition.* Wiley Series in Probability and Statistics: Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999. 110

[3] R. M. Burton, A. R. Dabrowski and H. Dehling, *An invariance principle for weakly associated random vectors.* Stochastic Process. Appl. 23 (1986), no. 2, 301–306. 102

[4] Y. S. Chow and H. Teicher, *Probability theory. Independence, interchangeability, martingales. Third edition.* Springer Texts in Statistics. Springer-Verlag, New York, 1997. 101

[5] H. Cohn, *Limit theorems for sums of random variables defined on finite inhomogeneous Markov chains.* Ann. Math. Statist. 43 (1972), 1283–1292. 101

[6] J. D. Esary, F. Proschan and D. W. Walkup, *Association of random variables, with applications.* Ann. Math. Statist. 38 (1967) 1466–1474. 102

[7] C. M. Fortuin, P. W. Kasteleyn and J. Ginibre, Correlation inequalities on some partially ordered sets. Comm. Math. Phys. 22 (1971), 89-103. 102

[8] S. X. Gan, *The Hájek-Rényi inequality for Banach space valued martingales and the p smoothness of Banach spaces.* Statist. Probab. Lett. 32 (1997), no. 3, 245–248. 101

[9] S. X. Gan and D. H. Qiu, *On the Hájek-Rényi inequality.* Wuhan Univ. J. Nat. Sci. 12 (2007), no. 6, 971–974. 101

[10] J. Hájek and A. Rényi, *Generalization of an inequality of Kolmogorov.* Acta Math. Acad. Sci. Hungar. 6 (1955), 281–283. 101

[11] T. E. Harris, A lower bound for the critical probability in a certain percolation process. Proc. Cambridge Philos. Soc. 56 (1960), 13-20. 102

[12] S. H. Hu, X. J. Wang, W. Z. Yang, and T. Zhao, *The Hájek-Rényi-type inequality for associated random variables.* Statist. Probab. Lett. 79 (2009), no. 7, 884–888. 101

[13] I. A. Ibragimov and Y. V. Linnik, Independent and stationary sequences of random variables. Wolters-Noordhoff Publishing, Groningen, 1971. 110

[14] K. Joag-Dev and F. Proschan, *Negative association of random variables, with applications.* Ann. Statist. 11 (1983), no. 1, 286–295. 107

[15] M. H. Ko, T. S. Kim and K. H. Han, *A note on the almost sure convergence for dependent random variables in a Hilbert space.* J. Theoret. Probab. 22 (2009), no. 2, 506–513. 106, 110

[16] J. L. Lebowitz, Bounds on the correlations and analyticity properties of ferromagnetic Ising spin systems. Comm. Math. Phys. 28 (1972), 313-321. 102

[17] E. L. Lehmann, *Some concepts of dependence.* Ann. Math. Statist. 37 (1966), 1137–1153. 102

[18] J. J. Liu, P. Y. Chen and S. X. Gan, *The laws of large numbers for ϕ-mixing sequences.* (Chinese) J. Math. (Wuhan) 18 (1998), no. 1, 91–95. 110

[19] J. J. Liu, S. X. Gan and P. Y. Chen, *The Hájek-Rényi inequality for the NA random variables and its application.* Statist. Probab. Lett. 43 (1999), 99–105. 101, 109

[20] C. M. Newman and A. L. Wright, *An invariance principle for certain dependent sequences.* Ann. Probab. 9 (1981), no. 4, 671–675. 102

[21] Q. M. Shao, *A comparison theorem on moment inequalities between negatively associated and independent random variables.* J. Theoret. Probab. 13 (2000), no. 2, 343–356. 107

[22] S. H. Sung, *A note on the Hájek-Rényi inequality for associated random variables.* Statist. Probab. Lett. 78 (2008), no. 7, 885–889. 101, 107

[23] T. Tómács and Z. Líbor, A Hájek-Rényi type inequality and its applications. Ann. Math. Inform. 33 (2006), 141–149. 101

Yu Miao
College of Mathematics and Information Science,
Henan Normal University,
Henan Province, 453007, China
yumiao728@yahoo.com.cn

Recibido: 23 de julio de 2010
Aceptado: 23 de abril de 2012