REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 53, No. 2, 2012, 1–12

ON BARYCENTRIC CONSTANTS

FLORIAN LUCA, OSCAR ORDAZ, AND MARÍA TERESA VARELA

Abstract. Let G be an abelian group with n elements. Let S be a sequence of elements of G, where the repetition of elements is allowed. Let $|S|$ be the cardinality, or the length of S. A sequence $S \subseteq G$ with $|S| \geq 2$ is barycentric or has a barycentric-sum if it contains one element a_j such that $\sum_{a_i \in S} a_i = |S|a_j$. This paper is a survey on the following three barycentric constants: the k-barycentric Olson constant $BO(k, G)$, which is the minimum positive integer $t \geq k \geq 3$ such that any subset of t elements of G contains a barycentric subset with k elements, provided such an integer exists; the k-barycentric Davenport constant $BD(k, G)$, which is the minimum positive integer t such that any subsequence of t elements of G contains a barycentric subsequence with k terms; the barycentric Davenport constant $BD(G)$, which is the minimum positive integer $t \geq 3$ such that any subset of t elements of G contains a barycentric subset. New values and bounds on the above barycentric constants when $G = \mathbb{Z}_n$ is the group of integers modulo n are also given.

1. Introduction

Let G be an abelian group with n elements. The study of barycentric sequences in G was started in [7] and [8]. Recall that a sequence of length ≥ 2 in G is called barycentric if it contains one element which is the "average" of its terms. Formally, the barycentric sequences are defined as follows:

Definition 1 ([8]). Let $a_1, a_2, \ldots, a_k$ be k not necessarily distinct elements of G. The above sequence is *k-barycentric* if there exists j such that $a_1 + a_2 + \ldots + a_j + \ldots + a_k = ka_j$. The element a_j is called a barycenter. When $\{a_1, a_2, \ldots, a_k\}$ is a set, we refer to it as a *k-barycentric set.*

For example, the set $\{0, 1, 2, 3, 4\} \subseteq \mathbb{Z}_8$ is 5-barycentric with barycenter 2, whereas the set $\{0, 2, 3, 4, 5\} \subseteq \mathbb{Z}_8$ is not 5-barycentric. The barycentric-sum problem consists in finding the smallest integer $t \geq 2$ such that any sequence of length t contains a k-barycentric subsequence for some k, which might be specified beforehand or not. The barycentric-sum problem was born in 1995 in [7] and [8], and was inspired by a theorem of Hamidoune from [21] asserting that every sequence of length $n + k - 1$ contains a k-barycentric subsequence.

2000 *Mathematics Subject Classification.* 11B50, 11P70, 11B75.
Key words and phrases. k-barycentric sequences; k-barycentric set; k-barycentric Olson constant; k-barycentric Davenport constant; barycentric Davenport constant; Olson constant.
Corresponding author. E-mail address: oscarordaz55@gmail.com (O. Ordaz).

Notice that a k-barycentric sequence in G with k a multiple of n is a sequence with zero-sum. The problem of the zero-sum asks to find the smallest positive integer t such that any sequence of t elements of G contains a subsequence of length k of terms that sum to 0 for some $k \geq 2$, where again k might be specified beforehand or not. The zero-sum problem started in 1961 with the celebrated Erdős, Ginzburg and Ziv theorem [10], which asserts that in $G = \mathbb{Z}_n$ every sequence of $2n-1$ elements contains a subsequence of length n having a zero-sum. The Davenport constant $D(G)$ was first introduced in 1966, as being the minimum integer t such that every sequence of length t contains some subsequence with zero-sum. For the state of the art on results, problems and conjectures on the zero-sum theory the reader is referred to the nicely structured surveys of Caro [3] and of Gao and Geroldinger [13]. More information on zero-sum problems can also be found in [9, 12, 18, 21, 22, 23, 26, 25, 32, 35, 40]. Both the zero-sum problem and the barycentric-sum problem are central questions in the field of combinatorial number theory.

The following notion was introduced in [37].

Definition 2. The *k-barycentric Olson constant* $BO(k, G)$ is the minimal positive integer $t \geq 3$ such that every subset of $t \geq k \geq 3$ elements of G contains a k-barycentric set, provided such an integer exists.

Conditions for the existence of $BO(k, \mathbb{Z}_n)$ as well as some values for this constant for specific k and/or n are given in [37, 39].

The following notion was introduced in [8].

Definition 3. The *barycentric Davenport constant* $BD(G)$ is the minimal positive integer $t \geq 3$ such that every subset with t elements of G contains a barycentric set.

It is easy to see that $3 \leq BD(\mathbb{Z}_n) \leq n$.

The following notion was introduced in [7].

Definition 4. The *k-barycentric Davenport constant* $BD(k, G)$ is the minimal positive integer t such that every sequence with t terms in G contains a k-barycentric sequence.

Let $|S|$ be the cardinality, or the length, of a set, or a sequence S in G, respectively. In [22], Hamidoune gives the following condition.

The Hamidoune Condition. Let S be a sequence of elements of G with $|S| \geq n + k - 1$. Then there exists a k-barycentric subsequence of S. Moreover, in the case when $k \geq n$, the above lower bound can be replaced by $|S| \geq k + D(G) - 1$, where $D(G)$ is the Davenport constant, and the conclusion still holds.

Thus, by the Hamidoune condition, we have that $BD(k, \mathbb{Z}_n) \leq n + k - 1$.

The existence of $BD(k, G)$ is assured by the Hamidoune condition. In [8], it was shown that $BD(G) \leq O(G) + 1$, where $O(G)$ is the Olson constant. Recall that $O(G)$ is the minimum integer t such that every subset S with t elements of G contains a subset with a zero-sum. It is known (see [34]) that $O(G) \leq 3\sqrt{|G|}$. In general, the existence of $BO(k,G)$ is an open problem. This problem is equivalent to the existence of a subset $\{a_1, a_2, \ldots, a_k\}$ of G such that $a_1 + a_2 + \cdots + (1-k)a_j + \cdots + a_k = 0$ for some j. Thus, one needs to understand zero-sums with weights. A nice survey on zero-sums with weights is [13]. Not covered by these surveys are the recent results of Griffiths [17], Hamidoune [24], and Luca [29]. The main result up to now in this area is a weighted version of the Erdős, Ginzburg and Ziv theorem obtained by Grynkiewicz [19].

The main goal of our paper is to survey the state of the art on the barycentric integer constants $BO(k,\mathbb{Z}_n)$, $BD(k,\mathbb{Z}_n)$, $BD(\mathbb{Z}_n)$, as well as on the constant $O(\mathbb{Z}_n)$. We also give some new values and bounds on these constants. In Section 2, we describe the theory of orbits which are used to calculate the barycentric constants. Sections 3, 4 and 5 deal with $BO(k,\mathbb{Z}_n)$, $BD(k,\mathbb{Z}_n)$, and $BD(\mathbb{Z}_n)$, respectively. We use the letter p to denote a prime number.

2. Orbits

In this section, we present the orbits as well as some of their properties. More information about them is available in [11]. They were used in [7, 8, 37] to obtain some exact values as well as some existence theorems for $BD(\mathbb{Z}_n)$, $BO(k,\mathbb{Z}_n)$ and $BD(k,\mathbb{Z}_n)$. Analogous investigations for non-cyclic groups where so far not undertaken. They were also used by Chapman, Freeze and Smith in their investigations on the minimal zero-sequences and the strong Davenport constant [4].

Let $\mathbb{Z}_{n,k}$ be the subset of $\mathbb{Z}_n^k$ consisting of all the subsets with k elements of $\mathbb{Z}_n$. In what follows, we shall introduce an equivalence relation on $\mathbb{Z}_{n,k}$ which partitions this set in orbits with certain properties.

Let

$$G_n = \{f_{a,b} : \mathbb{Z}_n \to \mathbb{Z}_n, f_{a,b}(x) = ax + b,\ a,\ b \in \mathbb{Z}_n, \gcd(a,n) = 1\}.$$

Observe that G_n is a group of order $n\phi(n)$, where $\phi(n)$ is the Euler function of n. The group G_n acts linearly on $\mathbb{Z}_{n,k}$ in the obvious way

$$f_{a,b}(\{x_1, \ldots, x_k\}) = \{f_{a,b}(x_1), \ldots, f_{a,b}(x_k)\}.$$

The above action defines an equivalence relation on $\mathbb{Z}_{n,k}$ in the usual way in which two sets with k elements are equivalent if there is some map $f_{a,b}$ in G_n which maps one of the sets onto the other. These equivalence classes are called *orbits*. Their most important property for us is that they preserve the barycentric property. Namely, if some element $\mathbf{x}$ in $\mathbb{Z}_{n,k}$ is either barycentric or contains a barycentric subset with t elements, then all elements in the orbit of $\mathbf{x}$ will also have the same property. Thus, it makes sense to make the following definition.

Definition 5 ([37]). If $\mathbf{x}$ is a barycentric set with k elements, then its orbit $\theta(\mathbf{x})$ is called barycentric. If $\mathbf{x}$ contains a t-barycentric set, then its orbit $\theta(\mathbf{x})$ is called t-barycentric. An orbit is called barycentric-free if it is not t-barycentric for any $t \geq 3$. Finally, an orbit is not barycentric-free if it is t-barycentric for some $t \geq 3$.

It is easy to see that $\theta(\{0\})$ is the only orbit of $\mathbb{Z}_{n,1}$. It is also easy to see that the orbits of $\mathbb{Z}_{n,2}$ are all of the form $\theta(\{0,z\})$. The next result is a finer characterization of the orbits of $\mathbb{Z}_{n,2}$.

Lemma 6. *The orbits in $\mathbb{Z}_{n,2}$ are of the form $\theta(\{0,d\})$ with $d \mid n$ and $d < n$. Furthermore, there are as many orbits in $\mathbb{Z}_{n,2}$ as there are proper divisors d of n.*

Proof. Let $\mathbf{x} = \{x_1, x_2\}$ be some point in $\mathbb{Z}_{n,2}$. Observe that $f_{1,-x_1}(\mathbf{x}) = \{0, y\}$, where $y = x_2 - x_1$ is nonzero modulo n. Let $d = \gcd(y, n)$ and write $y = dy_1$. Then $d < n$ and y_1 is coprime to n/d, so it is invertible modulo n/d. Let z_1 be the inverse of y_1 modulo n/d and extend z_1 to some congruence class modulo n. Then $f_{z_1,0}(\{0,y\}) = \{0, z_1 y\} = \{0, d\}$, which is what we wanted. To see that distinct divisors d of n give rise to distinct orbits, assume that this is not so, and let $1 \le d_1 < d_2 < n$ be two divisors of n such that $f_{a,b}(\{0,d_1\}) = \{0,d_2\}$. If $f_{a,b}$ maps 0 to 0, then $b = 0$ and $ad_1 \equiv d_2 \pmod{n}$. Since d_2 divides n, we get that $d_2 \mid d_1 a$, and since $d_1 < d_2$, it must be that a and d_2 have a greatest common divisor larger than 1, which is false. If $f_{a,b}$ maps 0 to d_2, then $b = d_2$ and $0 = f_{a,b}(d_1) = ad_1 + d_2$. Thus, we get $ad_1 \equiv -d_2 \pmod{n}$, and the preceding argument applies and leads to a contradiction. This completes the proof of the lemma. □

Remark 7 ([37])**.** Let us observe that the orbits of $\mathbb{Z}_{n,k}$ can be obtained from the orbits of $\mathbb{Z}_{n,k-1}$ in the following way: for each orbit $\theta(\{x_1,\ldots,x_{k-1}\})$ of $\mathbb{Z}_{n,k-1}$, we get an orbit $\theta(\{x_1,\ldots,x_{k-1},x_k\})$ for some $x_k \in \mathbb{Z}_n \setminus \{x_1,\ldots,x_{k-1}\}$. All possible orbits of $\mathbb{Z}_{n,k}$ are obtained in this way, but of course they are not all distinct since the elements of group G_n might induce some further identifications among these orbits.

We have the following obvious lemma.

Lemma 8 ([37])**.** *The number of orbits of $\mathbb{Z}_{n,k}$ is equal to the number of orbits of $\mathbb{Z}_{n,n-k}$.*

In [20], the following inductive algorithm was used to obtain all the orbits of $\mathbb{Z}_{n,k}$. All these were calculated for all $3 \le k \le n \le 11$.

Process 1. [20] Set $\mathbb{Z}_{n,k} = \{\{x_1, x_2, \ldots, x_k\} : x_i \in \mathbb{Z}_n\}$ with $1 \le k \le n-1$. When $k = 1$, we have that $\mathbb{Z}_{n,1} = \theta(\{0\})$, while Lemma 6 gives all the orbits of $\mathbb{Z}_{n,2}$. Finally, Remark 7 explains how to get all the orbits of $\mathbb{Z}_{n,k}$ for $k \ge 3$ from the orbits of $\mathbb{Z}_{n,k-1}$.

We have also the following nice result:

Theorem 1. If $p > 3$ is prime, then $\mathbb{Z}_{p,3}$ has $\lceil p/6 \rceil$ orbits.

Proof. We start with the only orbit $\theta(\{0,1\})$ of $\mathbb{Z}_{p,2}$ and analyze the number of ways in which we can extend it to an orbit with three elements. Let $\{0,1,x\}$ be a set with three elements modulo p and assume that it is equivalent to $\{0,1,y\}$. Since $f_{a,b}$ is determined on the images of two points, it follows that there are only 6 such possible maps, namely the ones sending the pair $(0,1)$ to the pairs $(0,1), (1,0), (0,y), (y,0), (1,y), (y,1)$, respectively. Solving for each one of them the resulting equations for a and b and using also the fact that x must go to the only element of $\{0,1,y\}\backslash\{f_{a,b}(0), f_{a,b}(1)\}$ via $f_{a,b}$ in each of the six cases, we get

$$y = x, \quad 1-x, \quad \frac{1}{x}, \quad \frac{1}{1-x}, \quad \frac{x-1}{x}, \quad \frac{x}{x-1}, \tag{1}$$

respectively. Indeed, let us do one of these *calculations*. Assume that $f_{a,b}$ sends $(0,1)$ to $(1,y)$. Then $1 = f_{a,b}(0) = a\cdot 0 + b = b$ and $y = f_{a,b}(1) = a \cdot 1 + b = a + b$, from where we get $(a,b) = (y-1,1)$. Now $0 = f_{a,b}(x) = a\cdot x + b = (y-1)x+1$ and solving for y we get that $y = (x-1)/x$, as claimed. Let us now look at instances when the six residue classes appearing in (1) are not distinct. Say $x \equiv 1/x \pmod p$. Then $x^2 \equiv 1 \pmod p$, leading to $x \equiv -1 \pmod p$. This leads to

$$y \in \{-1, 2, 1/2\}.$$

If $x \notin \{-1,2,1/2\}$, then all possible identifications among the six residue classes modulo p showing up at (1) all lead to $x^2 - x + 1 \equiv 0 \pmod p$, which is also equivalent to $(2x-1)^2 \equiv -3 \pmod p$. For $p > 3$, by Quadratic Reciprocity, this last congruence has no solutions when $p \equiv 2 \pmod 3$ and has precisely two distinct solutions x_1 and x_2 modulo p when $p \equiv 1 \pmod 3$. So, when $p \equiv 2 \pmod 3$, we get that if $x \notin \{0,1,-1,2,1/2\}$, then all residue classes appearing in (1) are distinct and therefore $\mathbb{Z}_{p,3}$ has precisely

$$1 + \frac{p-5}{6} = \frac{p+1}{6} = \left\lceil \frac{p}{6} \right\rceil$$

orbits in this case. When $p \equiv 1 \pmod 3$ and $x \in \{x_1, x_2\}$, then among the six residue appearing in (1) one checks that there are only two distinct ones, namely x_1 and x_2, and so if $x \notin \{0,1,-1,2,1/2,x_1,x_2\}$, then again all residue classes appearing in (1) are distinct. Thus, the number of orbits in $\mathbb{Z}_{p,3}$ is now

$$1 + 1 + \frac{p-7}{6} = \frac{p+5}{6} = \left\lceil \frac{p}{6} \right\rceil$$

as well, which completes the proof of the theorem.

□

3. The k-barycentric Olson constant

In this section, we deal with the existence of $BO(k, \mathbb{Z}_n)$ and give some of its values. Note first that if n is odd, then $BO(n,\mathbb{Z}_n) = n$, whereas if n is even then $BO(n,\mathbb{Z}_n)$ does not exist. Moreover, if n is odd, then $BO(n-1,\mathbb{Z}_n)$ does not exist, and when n is even we have $BO(n-1,\mathbb{Z}_n) = n-1$. From now on, we deal with the case $k \le n-2$.

We have the following easy lemma.

Lemma 9.

i. $\theta(\{0,1,\ldots,k-1\})$ *is a barycentric orbit of* $\mathbb{Z}_{n,k}$ *for* k *odd and* $n \ge k$.
ii. $\theta(\{0,1,2,5\})$ *is a barycentric orbit of* $\mathbb{Z}_{n,4}$ *for all* $n \ge 6$.
iii. $\theta(\{0,1,2,\ldots,k/2,k/2+2,k/2+3,\ldots,k-1,k+1\})$ *is a barycentric orbit of* $\mathbb{Z}_{n,k}$ *for all* $n > k+1 \ge 7$ *with* k *even.*

Lemma 9 implies easily the following theorem.

Theorem 10 ([37]). *$BO(k,\mathbb{Z}_n)\le n-1$ whenever $3\le k\le n-2$.*

Proof. By Lemma 8, $\mathbb{Z}_{n,n-1}$ and $\mathbb{Z}_{n,1}$ have the same number of orbits, namely one, and this orbit is k-barycentric by Lemma 9. □

Lemma 9 has also the following easy consequence.

Lemma 11. *The inequality $BO(k,\mathbb{Z}_p)\le p-2$ holds for all $3\le k\le p-2$ with p prime.*

Process 2. [20] The k-barycentric Olson constant $BO(k,\mathbb{Z}_n)$, whenever it exists, is the minimal positive integer t such that all the orbits of $\mathbb{Z}_{n,t}$ are k-barycentric.

Using Process 2, the exact values of $BO(k,\mathbb{Z}_n)$ with $3\le k\le n\le 12$ were calculated in [20].

The following theorem was proved by Dias da Silva and Hamidoune in [6].

Theorem 12 ([6]). *Let H be a subset of $\mathbb{Z}_p$. Let $d\in[2,|H|]$ be an integer. Set*

$$\wedge^d H=\{\sum_{x\in S}x : S\subset H \text{ with } |S|=d\}.$$

Then $|\wedge^d H|\ge\min\{p,d(|H|-d)+1\}$.

Using Theorem 12, the following two results were proved in [37].

Theorem 13 ([37]). *The inequality $BO(3,\mathbb{Z}_p)\le\lceil p/3\rceil+1$ holds for all $p\ge 5$.*

Theorem 14 ([37]). *The inequality*

$$BO\left(\left\lceil\frac{p-1}{d}\right\rceil+1,\mathbb{Z}_p\right)\le\left\lceil\frac{p-1}{d}\right\rceil+1+d$$

holds whenever $d\ge 2$ and $p\ge 2d+1$. Furthermore, the inequality

$$BO\left(\left\lceil\frac{p-1}{2}\right\rceil,\mathbb{Z}_p\right)\le\frac{p+3}{2}$$

holds for all $p\ge 7$.

The following theorem shows that $BO(k,\mathbb{Z}_p)=o(p)$ as $p\to\infty$.

Theorem 15 ([41]). *Let k be a positive integer and $c>0$. Set $A\subseteq\{1,2,\cdots,N\}$ with $|A|>cN$. Then for N sufficiently large, A contains an arithmetic progression T of length k.*

4. The k-barycentric Davenport constant

Recall that by the Hamidoune condition, we have $BD(k,\mathbb{Z}_n)\le n+k-1$. Furthermore, when $BO(k,\mathbb{Z}_n)$ exists, then every subsequence with $BO(k,\mathbb{Z}_n)$ different elements contain a k-barycentric subsequence.

The following theorem is derived from the Dias da Silva-Hamidoune Theorem 12.

Theorem 16 ([8])**.** *The inequality*

$$BD(3,\mathbb{Z}_p) \leq 2\left\lceil \frac{p}{3} \right\rceil + 1$$

holds for all $p \geq 5$*.*

Furthermore, from the theorems of Vosper and Cauchy-Davenport one can deduce the following result.

Theorem 17 ([7])**.** *The inequality*

$$BD(k,\mathbb{Z}_p) \leq p + k - 2$$

holds for all $4 \leq k \leq p-1$*.*

Remark 18.

(i) $BD(n,\mathbb{Z}_n) = 2n-1$: by Hamidoune condition or by Erdős, Ginzburg and Ziv theorem, we have the inequality $BD(n,\mathbb{Z}_n) \leq 2n-1$. In the reverse direction, the sequence $0^{n-1}1^{n-1}$ does not contain any n-barycentric sequence.

(ii) $2n-2 \leq BD(n-1,\mathbb{Z}_n) \leq 2n-3$: by a similar reasoning because the sequence $0^{n-2}1^{n-2}$ does not contain any $(n-1)$-barycentric sequence.

Theorem 19 ([7])**.** *Assume that* $k > p \geq 5$ *and that the remainder of the division of* k *by* p *is in* $\{4,\ldots,p-1\}$*. Then* $BD(k,\mathbb{Z}_p) \leq p+k-2$*. When the remainder is* $p-1$*, we actually have* $BD(k,\mathbb{Z}_p) = p+k-2$*.*

Process 3 ([20])**.** Here, we discuss a process producing a sequence of length $BD(k,\mathbb{Z}_n)$. **When** $BO(k,\mathbb{Z}_n)$ **exists**, the process starts with sequences of length $BO(k,\mathbb{Z}_n)$ and with $2 \leq s \leq BO(k,\mathbb{Z}_n)-1$ distinct elements from the non-k-barycentric orbits of $\mathbb{Z}_{n,s}$. **When** $BO(k,\mathbb{Z}_n)$ **does not exist**, the process initiates with sequences of length $n+1$ and with $2 \leq s \leq n$ different elements from the non-k-barycentric orbits of $\mathbb{Z}_{n,s}$. The length of the sequence being constructed increases when a larger sequence without k-barycentric subsequences is found. We repeat this construction as many times as possible by changing the number of initial elements when we reach a sequence containing a k-barycentric subsequence.

Using Process 3, the exact values of $BD(k,\mathbb{Z}_n)$ with $3 \leq k \leq n \leq 12$ and $3 \leq k \leq n$ were calculated in [20].

5. The Barycentric Davenport constant

The following algorithm is obvious.

Process 4 ([20])**.** The barycentric Davenport constant $BD(\mathbb{Z}_n)$ is the minimal positive integer $t \geq 3$ such that all orbits of $\mathbb{Z}_{n,t}$ are non-barycentric-free.

Using Process 4, the exact values of $BD(\mathbb{Z}_n)$ for all $3 \leq n \leq 13$ were calculated in [20].

Theorem 20 ([8])**.** *The inequality* $BD(G) \leq O(G)+1$ *holds for all finite abelian groups* G*.*

From Theorem 20, we can see that there is a relation between constants $BD(G)$ and $O(G)$. Some results about $O(G)$ are given in what follows. The reader is invited to look at paper [38], where last advances on $O(G)$ are found; they can be inspiring to obtain results of $BD(G)$ from $O(G)$.

Let $ZFS_s(G)$ be the set of zero-free subsets of G. It is clear that

$$O(G) = 1 + \max\{|S| : S \in ZFS_s(G)\}.$$

In [32], Olson showed that $O(\mathbb{Z}_p) \leq 2\sqrt{p}$. This is better than the bound $O(G) \leq 3\sqrt{|G|}$ from [34] valid for all finite abelian groups G mentioned in the introduction. Some values of this constant appear in Table 1 in [8, 40]. Hamidoune and Zemor [27] improved Olson's bound by showing that $O(\mathbb{Z}_p) \leq \sqrt{2p} + 5\ln(p) + 1$. This was further improved by Nguyen, Szemerédi and Vu in [31] where they eliminated the logarithmic term. In particular, one gets immediately that $O(\mathbb{Z}_3) = 2$, $O(\mathbb{Z}_5) = 3$ and $O(\mathbb{Z}_7) = 4$. More numerical examples are presented in [8, 40].

We next present a theorem due to Gao and Geroldinger [12].

Theorem 21 ([12]). *Let*

$$G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r} \oplus \mathbb{Z}_n^{s+1}$$

where $r \geq 0$, $s \geq 0$, $1 < n_1 | \cdots | n_r | n$ *and* $n_r \neq n$. *If* G *is a* p*-group and* $r + s/2 \geq n$, *then*

$$O(G) = \sum_{i=1}^{r}(n_i - 1) + (s+1)(n-1) + 1.$$

Corollary 22. $O(\mathbb{Z}_p^s) = s(p-1) + 1$ *for* $s \geq 2p + 1$.

From the above corollary, we obtain immediately the following values.

Theorem 23 ([8]).

i. $O(\mathbb{Z}_2^s) = s + 1$ *for all* $s \geq 1$.
ii. $O(\mathbb{Z}_3^s) = 2s + 1$ *for all* $s \geq 3$.

We have the following theorem.

Theorem 24 ([30]). $O(\mathbb{Z}_p \oplus \mathbb{Z}_p) \leq 2p - 2$ *for* $p \geq 3$.

From the above theorem, one can easily deduce that $O(\mathbb{Z}_3^2) = 4$. Next we turn our attention to lower bounds for $O(G)$.

Theorem 25 ([8]). *Set* $f(n) = \left\lfloor \sqrt{2n} - 1/2 \right\rfloor$. *We then have the following inequalities:*

i. $O(G \oplus H) \geq O(G) + O(H) - 1$.
ii. $O(\mathbb{Z}_n) \geq f(n) + 1$.
iii. $O(\mathbb{Z}_a \oplus \mathbb{Z}_{ab}) \geq O(\mathbb{Z}_a) + af(b)$.

The following two lemmas can be used to inductively build zero-free sets.

Lemma 26 ([40]). *Let* S *be a zero-free set in* $\mathbb{Z}_3^s$ *with* $s \geq 3$ *and* $|S| = 2s$. *Then* $S \cup \{e_{s+1}, e_1 + e_{s+1}\}$ *is a zero-free set in* $\mathbb{Z}_3^{s+1}$.

Lemma 27 ([40]). *Let S be a zero-free set in $\mathbb{Z}_3^s$ with $s \geq 3$ and $|S| = 2s$. Suppose that each vector in S has two coordinates equal to 1 and all other coordinates equal to 0. Then $S \cup \{e_1 + e_{s+1}, e_2 + e_{s+1}\}$ is a zero-free set in $\mathbb{Z}_3^{s+1}$.*

We have the following theorem.

Theorem 28 ([15]). *If $p > 4.67 \times 10^{34}$, then $O(\mathbb{Z}_p^2) = p + O(\mathbb{Z}_p) - 1$.*

The next theorem improves Theorem 28:

Theorem 29 ([1]). *If $p > 6000$, then $O(\mathbb{Z}_p^2) = p + O(\mathbb{Z}_p) - 1$.*

By Corollary 22, we have the equality $O(\mathbb{Z}_p^s) = D(\mathbb{Z}_p^s) = s(p-1)+1$ provided that $s \geq 2p+1$.

For $S \subseteq G$ let us put

$$\sum(S) = \{\sum_{x\in A} x : A \subseteq S\}.$$

For any natural number k let us define

$$F(G,k) = \min\{|\sum(S)| : S \in ZFS_s(G) \text{ with } |S| = k\}.$$

Recently Gao, Li, Peng and Sun [14] obtained the following result.

Theorem 30. *Assume that k is a natural number, $c > 0$ is a real number, and that $F(G,k) \geq 1 + c^{-2}k^2$. Then $O(G) < c\sqrt{|G|-1} + 1$.*

The following conjecture from [35, 40] is still open.

Conjecture 31 ([40]). *$O(G) \leq O(\mathbb{Z}_n)$.*

Using Theorem 20 together with the Hamidoune-Zemor bound on $O(\mathbb{Z}_p)$, one calculates easily that $BD(\mathbb{Z}_3) = 3$, $BD(\mathbb{Z}_5) = 3$, $BD(\mathbb{Z}_7) = 4$, and $BD(\mathbb{Z}_{11}) = 5$. More examples are given in [8].

Theorem 32 ([8]). *$BD(\mathbb{Z}_2^s) = s + 2$ for all $s \geq 1$.*

The following two conjectures are equivalent.

Conjecture 33. *Let $s \geq 2$ and $S \subseteq \mathbb{Z}_3^s$ be such that $|S| = 2s+1$. Then there exists $a \in S$ such that $S - a \setminus \{0\}$ contains a subset with zero-sum.*

Conjecture 34. *$BD(\mathbb{Z}_3^s) = 2s + 1$ for $s \geq 1$.*

Next we show that Conjectures 33 and 34 are true for small values of s.

Theorem 35. *Conjectures 33 and 34 are true for $s = 2, 3, 4, 5$.*

Proof. We start with some general considerations. It is clear that $BD(\mathbb{Z}_3) = 3$. Assume now that $s \geq 2$. Let $\{e_i : 1 \leq i \leq s\}$ be the canonical basis of the vector space $\mathbb{Z}_3^s$. Set

$$A = \{0\} \cup \{e_i : 1 \leq i \leq s\} \cup \{e_i + e_{i+1} : 1 \leq i \leq s-1\}.$$

It is clear that $A \setminus \{0\}$ is a zero-free subset. It is also easy to see that A is a barycentric free set, from where we get that $2s+1 \le BD(\mathbb{Z}_3^s)$. Since $BD(G) \le O(G)+1$ and $O(\mathbb{Z}_3^s) = 2s+1$ for $s \ge 3$, we have that $BD(\mathbb{Z}_3^s) \le O(\mathbb{Z}_3^s)+1 = 2s+2$.

Now let $T \subseteq \mathbb{Z}_3^s$ have cardinality $2s+1$. Assume that T does not contain barycentric subsets. Then we can write $T = S \cup \{0\}$, where S is a zero-free subset. Since $D(\mathbb{Z}_3^s) = 2s+1$, it follows that $\mathbb{Z}_3^s = \langle S \rangle$ (see [12] for this assertion). In particular, S contains a basis of $\mathbb{Z}_3^s$ and without loss of generality we can suppose that it is the canonical basis. Let $S^1 = S \setminus \{e_1, e_2, \dots, e_s\}$. We will see that it is not possible that $|S^1| = s$. Let x be in S^1. Suppose, up to a reordering of its coordinates, that its coordinates are

$$x = (2, \dots, 2, 1, \dots, 1, 0, \dots, 0).$$

Say, x has its first a_2 coordinates equal to 2, the next a_1 coordinates equal to 1, and the last $a_0 = s - a_2 - a_1$ coordinates equal to 0. If $a_2 = 0$, then we take the barycentric-free set $\{x, e_{a_2+1}, \dots, e_{a_2+a_1}\}$. This set has sum $2x$. Hence, $a_1 + 1 \not\equiv 2 \pmod 3$. Further, by looking at $\{0, x, e_{a_2+1}, \dots, e_{a_2+a_1}\}$, we see that $a_1 + 2 \not\equiv 2 \pmod 3$. To summarize, if $a_2 = 0$, then $a_1 \equiv 2 \pmod 3$. If $a_1 = 0$, then we take the barycentric-free set $\{0, x, e_1, \dots, e_{a_2}\}$. This set has sum 0, a contradiction. Thus, $a_1 > 0$. If $a_1 = 1$, we then take the barycentric-free sets $\{x, e_1, \dots, e_{a_2+a_1}\}$ and $\{0, x, e_1, \dots, e_{a_2+a_1}\}$. Both have sum $2e_{a_2+a_1}$, so $a_2 + a_1 + 1 \not\equiv 2 \pmod 3$, and $a_2 + a_1 + 2 \not\equiv 2 \pmod 3$. Thus, if $a_1 = 1$, then $a_2 \equiv 1 \pmod 3$.

Now let us analyze the small values of s.

- $s = 2$ The only possible elements in S^1 are of the form $(1, 1)$ or $(2, 1)$. Since $(1, 0) + (1, 1) + (1, 2) = (0, 0)$, we conclude that S^1 cannot have 2 elements.
- $s = 3$ The only possible elements in S^1 are of the form $(1, 1, 0)$ or $(2, 1, 0)$ or $(2, 1, 1)$. It is clear that each pair of such elements together with an appropriate element of the basis yields a barycentric set in $S \cup \{0\}$. Thus, it is not possible to find a set S^1 with 3 elements.
- $s = 4$ The only possible elements in S^1 are $(1, 1, 0, 0)$, $(2, 1, 0, 0)$, $(2, 1, 1, 0)$, $(2, 1, 1, 1)$, $(2, 2, 1, 1)$. By simple inspection, we can discard barycentric subsets in $S \cup \{0\}$ until reaching the conclusion that it is not possible to find such S^1 with 4 elements.
- $s = 5$ The only possible elements in S^1 are of the form: $(1, 1, 1, 1, 1)$, $(1, 1, 0, 0, 0)$, $(2, 1, 1, 1, 1)$, $(2, 1, 1, 1, 0)$, $(2, 1, 1, 0, 0)$, $(2, 1, 0, 0, 0)$, $(2, 2, 1, 1, 1)$, $(2, 2, 1, 1, 0)$, $(2, 2, 2, 1, 1)$, $(2, 2, 2, 2, 1)$. Again, we can derive from these elements as many barycentric sets in $S \cup \{0\}$ as to conclude that we cannot have such an S^1 with 5 elements. However, the computations are becoming cumbersome.

□

References

[1] G. Bhowmik and J. Schlage-Puchta, An improvement on Olson constant for $\mathbb{Z}_p \oplus \mathbb{Z}_p$, *Acta Arith.*, **141** (2010), 311–319. 9

[2] Y. Caro, On zero-sum subsequences in abelian non-cyclic groups, *Israel J. Math.* **92** (1995), 221–233.

[3] Y. Caro, Zero-sum problems: a survey, *Discrete Math.* **152** (1996), 93–113. 2
[4] S. T. Chapman, M. Freeze and W. W. Smith, Minimal zero-sequences and the strong Davenport constant, *Discrete Math.* **203** (1999), 271–277. 3
[5] L. Childs, Introduction to Higher Algebra, Springer-Verlag (1979).
[6] J. A. Dias da Silva and Y. O. Hamidoune, Cyclic spaces for Grassmann derivatives and additive theory, *Bull. London Math. Soc.* **26** (1994), 140–146. 6
[7] C. Delorme, S. González, O. Ordaz and M.T. Varela, Barycentric sequences and barycentric Ramsey numbers stars, *Discrete Math.* **277** (2004), 45–56. 1, 2, 3, 7
[8] C. Delorme, I. Márquez, O. Ordaz and A. Ortuño, Existence condition for barycentric sequences, *Discrete Math.* **281** (2004), 163–172. 1, 2, 3, 7, 8, 9
[9] C. Delorme, O. Ordaz and D. Quiroz, Some remarks on the Davenport constant, *Discrete Math.* **237** (2001), 119–128. 2
[10] P. Erdős, A. Ginzburg and A. Ziv, Theorem in the additive number theory, *Bull. Res. Council Israel* **10** (1961), 41–43. 2
[11] J. B. Fraleigh, A first course in Abstract Algebra, Addison Wesley Longman 2000. 3
[12] W. Gao and A. Geroldinger, On long minimal zero sequences in finite abelian groups, *Per. Math. Hungarica* **38** (1999), 179–211. 2, 8, 10
[13] W. Gao and A. Geroldinger, Zero-sum problems in finite abelian groups: A survey, *Exp. Math.* **24** (2006), 337–369. 2, 3
[14] W. Gao, Y. Li, J. Peng and F. Sun, Subsums of a zero-sum free subset of an abelian group, *Elec. J. Combin.* **15** (2008), 1–35. 9
[15] W. Gao, I. Ruzsa and R. Thangadurai, Olson's constant for the group $\mathbb{Z}_p \oplus \mathbb{Z}_p$, *J. Combin. Theory Ser. A* **107** (2004), 49–67. 9
[16] R.L. Graham, B. L. Rothschild and J. H. Spencer, Ramsey theory, Second edition, John Wiley and Sons N. Y., 1990.
[17] S. Griffiths, The Erdős-Ginzburg-Ziv theorem with units, *Discrete Math.* **308** (2008), 5473–5484. 3
[18] D. J. Grynkiewicz, O. Ordaz, M.T. Varela and F. Villarroel, On the Erdős-Ginzburg-Ziv Inverse Theorems, *Acta Arith.* **129** (2007), 307–318. 2
[19] D. J. Grynkiewicz, A weighted Erdős-Ginzburg-Ziv theorem, *Combinatorica* **26** (2006), 445–453. 3
[20] C. Guia, F. Losavio, O. Ordaz, M. T. Varela and F. Villarroel, Barycentric Davenport constants, to appear in *Divulg. Mat.* 4, 6, 7
[21] Y. O. Hamidoune, On weighted sequences sums, *Combinatorics, Probability and Computing* **4** (1995), 363–367. 1, 2
[22] Y. O. Hamidoune, On weighted sums in abelian groups, *Discrete Math.* **162** (1996), 127–132. 2
[23] Y. O. Hamidoune, Subsequence sums, *Combinatorics, Probability and Computing* **12** (2003), 413–425. 2
[24] Y. O. Hamidoune, A weighted generalization of Gao's $n + D - 1$, *Combinatorics, Probability and Computing* **17** (2007), 1–6. 3
[25] Y. O. Hamidoune, O. Ordaz and A. Ortuño, On a combinatorial theorem of Erdős-Ginzburg-Ziv, *Combinatorics, Probability and Computing* **7** (1998), 403–412. 2
[26] Y. O. Hamidoune and D. Quiroz, On subsequence weigted products, *Combinatorics, Probability and Computing* **14** (2005), 485–489. 2
[27] Y. O. Hamidoune and G. Zemor, On zero-free subset sums, *Acta Arith.* **78** (1996), 143-152. 8
[28] S. Lang, Algebra, Addison Wesley Publishing Company, 1965.
[29] F. Luca, A generalization of a classical zero-sum problem, *Discrete Math.* **307** (2007), 1672–1678. 3
[30] H. B. Mann and J. E. Olson, Sums of sets in the elementary abelian group of type $(p, p)^*$, *J. Combin. Theory* **2** (1967), 275–284. 8

[31] H. H. Nguyen, E. Szmerédi and V. H. Vu, Subset sums modulo a prime, *Acta Arith.*, 131 (2008), 303–316. 8
[32] J. E. Olson, An addition theorem modulo p, *J. Combin. Theory* **5** (1968), 45–52. 2, 8
[33] J. E. Olson, A combinatorial problem on finite abelian groups I, *J. Number theory* **1** (1969), 195–199.
[34] J. E. Olson, Sums of sets of group elements, *Acta Arith.* **28** (1975), 147–156. 3, 8
[35] O. Ordaz and D. Quiroz, On zero free sets, *Divulg. Mat.* **14** (2006), 1–10. 2, 9
[36] O. Ordaz and D. Quiroz, Barycentric-sum problem: a survey, *Divulg. Mat.* **15** (2007), 193–206.
[37] O. Ordaz, M.T. Varela and F. Villarroel, k-barycentric Olson constant, *Math. Rep.* **11 (61)**(2009), 33–45. 2, 3, 4, 6
[38] O. Ordaz, A. Philipp, I Santos and W. Schmid, On the Olson and the strong Davenport constants, *J. de Théor. Nombres Bordeaux* **23** (2011), no. 3, 715–750. 8
[39] O. Ordaz, A. Plagne and W. Schmid, Some remarks on barycentric-sum problems over cyclic groups, to appear in *The European Journal of Combinatorics.* 2
[40] J. Subocz, Some values of Olson's constant, *Divulg. Mat.* **8** (2000), 121–128. 2, 8, 9
[41] E. Szemerédi, On sets of integers containing no k elements in arithmetic progression, *Acta Arith.* **27** (1975), 199–245. 6
[42] F. Villarroel, La constante de Olson k-baricéntrica y un teorema inverso de Erdős-Ginzburg-Ziv, Tesis Doctoral en Matemática, Facultad de Ciencias, Universidad Central de Venezuela, 2008.

Florian Luca
Centro de Ciencias Matemáticas,
Universidad Nacional Autónoma de México,
Ap. Postal 61–3 (Xangari),
CP 58089, Morelia, México.
fluca@matmor.unam.mx

Oscar Ordaz
Escuela de Matemáticas y Laboratorio MoST,
Centro ISYS, Facultad de Ciencias,
Universidad Central de Venezuela
Ap. 47567, Caracas, Venezuela.
oscarordaz55@gmail.com

María Teresa Varela
Departamento de Matemáticas Puras y Aplicadas,
Universidad Simón Bolívar.
Ap. 89000, Caracas 1080-A, Venezuela.
mtvarela@usb.ve

Recibido: 23 de julio de 2010
Aceptado: 2 de abril de 2012