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A REMARK ON PRIME REPUNITS

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Abstract. A formula for the generating function of prime repunits is given in terms of a Lambert series using S. Golomb's formula.

1. Introduction and main result

Identities can be sometimes used to prove that certain sequences of numbers are infinite. Recall the following known example attributed to J. Hacks in Dickson's *History of the theory of numbers*. From the well-known formula $\prod_p(\frac{1}{1-1/p^2}) = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ (Euler) and the fact that π^2 is irrational (Legendre) one obtains that the number of primes p is infinite (if the number of primes were finite then the left hand side would be a rational number). Of course, this is not a simple proof, see [1].

The aim of this note is, using Solomon Golomb's formula (2.1) (see [2]), to give a formula which involves the generating function of prime repunits and to make a remark with the above idea. We need some notation first.

A *repunit* is a natural number whose decimal expansion contains only the digit one: $R_n := \underbrace{1\cdots1}_{n} = \frac{10^n-1}{9}$. It is known that R_n is a prime repunit for $n = 2, 19, 23, 317, 1031$. An open question is to know whether the number of prime repunits is infinite.

For a natural number m_0 we write $m_0 = p_1^{r_1}\cdots p_\ell^{r_\ell}$ where p_i are distinct primes and $r_i \geq 1$ (we shall always use p to denote a prime number). We write lcm for the least common multiple, gcd for the greatest common divisor and μ to denote the Möbius function. We denote by $\nu(m_0)$ an additive function i.e. a function defined at positive integer numbers so that $\nu(a)+\nu(b) = \nu(ab)$ if $\gcd(a,b) = 1$. Also, we write as usual $\omega(p_1^{r_1}\cdots p_\ell^{r_\ell}) = \ell$ and $\Omega(n)$ the (completely) additive function which counts the number of prime divisors of n with multiplicity.

Define the function

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$$S_\nu(z) := \sum_{\substack{n\geq 5\\ R_n \text{ prime}}} z^n \nu(R_n).$$

Observe that $S_\nu(z)$ is the generating function of the prime repunits greater than 1111.

For $d > 1$, $\gcd(d, 10) = 1$ we define $m = m(d)$ the multiplicative order of 10 mod$d$ i.e. m is the smallest positive integer such that $10^m = 1 \bmod d$. Define $F_d(z)$ as:

$$F_d(z) := \begin{cases} \frac{z^m}{1-z^{6m}} + \frac{z^{5m}}{1-z^{6m}}, & \text{if } m = 1, 5 \bmod 6, \ \textit{and } 5 \leq m, \\ 0, & \text{otherwise.} \end{cases} \tag{1.1}$$

We prove the following theorem.

Theorem. *Let ν be any additive function such that $\nu(d) = O(d^k)$ for some positive k. Then in a neighborhood of zero one has*

$$2\nu(3)S_\nu(z) = -\nu(3)^2\Big(\frac{z^7}{1-z^6} + \frac{z^5}{1-z^6}\Big) + \sum_{\substack{d\geq 7\\ \gcd(d,10)=1}} \mu(d)\nu(d)^2 F_d(z). \tag{1.2}$$

Remarks: The right hand side of (1.2) converges in some neighborhood of zero. Indeed one has $m \geq \log d$ (log is the logarithm in base 10). Therefore for, say, $|z| < \frac{1}{2}$,

$$\sum_{\substack{d\geq 7\\ \gcd(d,10)=1}} |\mu(d)\nu(d)^2 F_d(z)| \leq O\Big(\sum_{d=1}^{\infty} d^{2k}|z|^m\Big) \leq O\Big(\sum_{d=1}^{\infty} d^{2k}|z|^{[\log d]}\Big),$$

the last series being convergent in a suitable neighborhood of zero, where $[\cdot]$ is the nearest integer function.

In the spirit of the beginning of this note we observe the following immediate corollary of (1.2) (taking $\nu(p) = 1$): *assume that there exists a natural number $q \geq 11$, such that the (absolutely convergent) series*

$$\sum_{\substack{d\geq 7\\ \gcd(d,10)=1}} \mu(d)\omega(d)^2 F_d\Big(\frac{1}{q}\Big) = \sum_{\substack{d\geq 7\\ \gcd(d,10)=1}} \mu(d)\Omega(d)^2 F_d\Big(\frac{1}{q}\Big)$$

is an irrational number. Then the number of prime repunits is infinite. (Note: both series are equal due to the factor $\mu(d)$.)

Of course, these series are difficult to analyze and they bear some similarity with the Lambert series $\sum_1^\infty \frac{1}{2^n-1}$ which have been proved irrational by Erdös [8] (see also [7]). The difficulty arises due to the extra arithmetical elements present of μ, ω (or Ω) and the dependence on m and d in F_d.

As an easy exercise one can prove that if the number of prime repunits is infinite then the above series is irrational for $q = 100$. Hint: Use (1.2) and the fact that the number $S_\omega(\frac{1}{100})$, whose decimal expansion contains only the digits $0, 1$ is an irrational number (this number has the digit 1 in place $2n$ iff $1111 < R_n$ is prime). To see this notice that if R_n is prime then n must be prime (see below Lemma 2.1 i)).

Finally observe that for an odd square-free number $1 < d = p_1 \cdots p_\ell$ (distinct primes) with $\gcd(d, 10) = 1$ the number $m(d)$ can be obtained as follows: if $1 \le n_i$ is the smallest integer such that $10^{n_i} = 1 \bmod p_i$ then $m(d) = \operatorname{lcm}\{n_1, \ldots, n_\ell\}$. To see this notice that $m | \operatorname{lcm}\{n_1, \ldots, n_\ell\}$ for $10^{\operatorname{lcm}\{n_1,\ldots,n_\ell\}} = 1 \bmod p_i$ and therefore $10^{\operatorname{lcm}\{n_1,\ldots,n_\ell\}} = 1 \bmod d$. On the other hand $10^m = 1 \bmod d$ and thus $10^m = 1 \bmod p_i$; therefore $n_i | m$ and then $\operatorname{lcm}\{n_1, \ldots, n_\ell\} | m$.

2. Proof

For the proof of the theorem we need the following auxiliary lemma.

Lemma 2.1.

i) If R_n is prime then n must be prime.

ii) If $10^n - 1$ has at most two distinct prime factors, then either $n = 1, 2, 3$, or $n \ge 5$ and R_n is prime.

Proof: i) If $n = ab$ then $\frac{10^{ab}-1}{3^2} = \frac{10^{ab}-1}{10^b-1}\,\frac{10^b-1}{3^2}$.

ii) Assume that $10^n - 1$ has exactly one prime divisor. But $10^n - 1 = 3^r$, with $n, r > 1$ has no solution because this is a special case of Catalan's equation (see [3]).

Therefore, assume that $10^n - 1$ has exactly two prime divisors and n is coprime to 3. We write $10^n - 1 = 3^{2+a} p^r$ and then $R_n = 3^a p^r$. If $a > 0$ then we must have $3|n$ (the sum of the digits of a number must be divisible by 3 if the number is divisible by 3; the sum of the digits of R_n is n). This is absurd and therefore $a = 0$. Bugeaud and Mignotte [6], who completed a theorem of Shorey and Tijdeman ([5], Theorem 5 i)), showed that R_n is not a perfect power if $1 < n$. Thus R_n is prime if n is coprime to 3.

Now if $3|n$ then

$$10^n - 1 = (10^{n/3} - 1)(10^{2n/3} + 10^{n/3} + 1),$$

and the second factor is 3 mod 9, so 3 divides n but 9 does not. So, the second factor must have some other prime factor $p > 3$, therefore the first factor is a power of 3, again false for $n > 3$ by results on Catalan's equation. Thus $n = 3$. ■

We recall S. Golomb's formula (see [2])

$$\sum_{d | m' = p_1^{r_1} \cdots p_\ell^{r_\ell}} \mu(d)\ \nu(d)^2 = \begin{cases} -\nu(p_1)^2, & \text{if } \ell = 1, \\ 2\nu(p_1)\nu(p_2), & \text{if } \ell = 2, \\ 0, & \text{if } \ell > 2. \end{cases} \tag{2.1}$$

This could be proved by grouping d as having one divisor, two divisors, three divisors etc., as

$$-\binom{\ell-1}{0}\sum_{i=1}^{\ell}\nu(p_i)^2+\Big\{\binom{\ell-1}{1}\sum_{i=1}^{\ell}\nu(p_i)^2+2\binom{\ell-2}{0}\sum_{i<j}\nu(p_i)\nu(p_j)\Big\}$$
$$-\Big\{\binom{\ell-1}{2}\sum_{i=1}^{\ell}\nu(p_i)^2+2\binom{\ell-2}{1}\sum_{i<j}\nu(p_i)\nu(p_j)\Big\}+\dots$$

which gives the desired formula (2.1) after grouping terms.

We have, using (2.1) and the above lemma, that

$$2\nu(3)S_\nu(z)=\sum_{\substack{n\geq 5\\ n=1,5 \bmod 6}} z^n\Big\{\sum_{d|10^n-1}\mu(d)\nu(d)^2\Big\}.$$

Indeed this last formula follows from (2.1) which gives zero in the case that $m'=10^n-1$ has three or more prime divisors and from the fact that if $R_n>1111$ is a prime repunit then n must be prime and therefore $n=1$ or $n=5, \bmod 6$, $n\geq 5$.

We continue our proof. We have

$$\sum_{\substack{n\geq 5\\ n=1,5 \bmod 6}} z^n\Big\{\sum_{d|10^n-1}\mu(d)\nu(d)^2\Big\}=\sum_{\substack{d=3\\ \gcd(d,10)=1}}^{\infty}\mu(d)\nu(d)^2\Big\{\sum_{\substack{n\geq 5\\ n=1,5 \bmod 6\\ 10^n=1 \bmod d}} z^n\Big\}.$$

Notice that, for fixed d, the positive solutions of $10^n=1 \bmod d$ are given by the set $\{m,2m,3m,4m,\dots\}$.

Assume $d\geq 3$, $\gcd(d,10)=1$ and d is a square-free number. Then $\sum_{10^n=1 \bmod d} z^n$ $=z^m+z^{2m}+z^{3m}+z^{4m}+\dots$ and therefore the sum

$$\sum_{\substack{n=1,5 \bmod 6\\ 10^n=1 \bmod d}} z^n=(z^m+z^{7m}+\dots)+(z^{5m}+z^{11m}+\dots)$$

if $m=1,5 \bmod 6$, and is zero otherwise. Thus

$$\sum_{\substack{n\geq 5\\ n=1,5 \bmod 6\\ 10^n=1 \bmod d}} z^n=\begin{cases}\frac{z^m}{1-z^{6m}}+\frac{z^{5m}}{1-z^{6m}}, & \text{if } m=1,5 \bmod 6; 5\leq m,\\ \frac{z^7}{1-z^6}+\frac{z^5}{1-z^6}, & \text{if } m=1,\\ 0, & \text{otherwise.}\end{cases} \tag{2.2}$$

But if $m=1$ then one must have $d=3$. So

$$\sum_{\substack{d=3\\ \gcd(d,10)=1}}^{\infty}\mu(d)\nu(d)^2\Big\{\sum_{\substack{n\geq 5\\ n=1,5 \bmod 6\\ 10^n=1 \bmod d}} z^n\Big\}=$$

$$-\nu(3)^2\Big(\frac{z^7}{1-z^6}+\frac{z^5}{1-z^6}\Big)+\sum_{\substack{d=7\\ \gcd(d,10)=1}}^{\infty}\mu(d)\nu(d)^2\Big\{\sum_{\substack{n\geq 5\\ n=1,5 \bmod 6\\ 10^n=1 \bmod d}} z^n\Big\}=$$

$$-\nu(3)^2\Big(\frac{z^7}{1-z^6}+\frac{z^5}{1-z^6}\Big)+\sum_{\substack{d=7\\ \gcd(d,10)=1}}^{\infty}\mu(d)\nu(d)^2F_d(z),$$

where (1.1) follows from (2.2).

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