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ON $\mathcal{I}$-NULL LIE ALGEBRAS

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ABSTRACT. We consider the class of complex Lie algebras for which the Koszul 3-form is zero, and prove that it contains all quotients of Borel subalgebras, or of their nilradicals, of finite dimensional complex semisimple Lie algebras. A list of Kac-Moody types for indecomposable nilpotent complex Lie algebras of dimension ≤ 7 is given.

1. Introduction

Leibniz algebras are non-antisymmetric versions $\mathfrak{g}$ of Lie algebras: the commutator is not required to be antisymmetric, and the right adjoint operations $[.,Z]$ are required to be derivations for any $Z \in \mathfrak{g}$ ([10]). In the presence of antisymmetry, that is equivalent to the Jacobi identity. Leibniz algebras have a cohomology of their own, the Leibniz cohomology $HL^\bullet(\mathfrak{g},\mathfrak{g})$, associated to the complex $CL^\bullet(\mathfrak{g},\mathfrak{g}) = \text{Hom }(\mathfrak{g}^{\otimes\bullet},\mathfrak{g}) = \mathfrak{g}\otimes(\mathfrak{g}^*)^{\otimes\bullet}$ and the Leibniz coboundary δ defined for $\psi \in CL^n(\mathfrak{g},\mathfrak{g})$ by

$$(\delta\psi)(X_1,X_2,\dots,X_{n+1}) =$$
$$[X_1,\psi(X_2,\dots,X_{n+1})] + \sum_{i=2}^{n+1}(-1)^i[\psi(X_1,\dots,\hat{X}_i,\dots,X_{n+1}),X_i]$$
$$+\sum_{1\leqslant i<j\leqslant n+1}(-1)^{j+1}\,\psi(X_1,\dots,X_{i-1},[X_i,X_j],X_{i+1},\dots,\hat{X}_j,\dots,X_{n+1})$$

(If $\mathfrak{g}$ is a Lie algebra, δ coincides with the usual coboundary d on $C^\bullet(\mathfrak{g},\mathfrak{g}) = \mathfrak{g}\otimes\bigwedge^\bullet\mathfrak{g}^*$). Since Lie algebras are Leibniz algebras, a natural question is, given some fixed Lie algebra, whether or not it has more infinitesimal Leibniz deformations (i.e. deformations as a Leibniz algebra) than infinitesimal deformations as a Lie algebra. That amounts to the comparison of the adjoint Leibniz 2-cohomology group $HL^2(\mathfrak{g},\mathfrak{g})$ and the ordinary one $H^2(\mathfrak{g},\mathfrak{g})$, and was addressed by elementary methods in [5]. There we proved that

$$HL^2(\mathfrak{g},\mathfrak{g}) = H^2(\mathfrak{g},\mathfrak{g})\oplus ZL_0^2(\mathfrak{g},\mathfrak{g})\oplus\mathcal{C},$$

where $ZL_0^2(\mathfrak{g},\mathfrak{g})$ is the space of symmetric Leibniz 2-cocycles and $\mathcal{C}$ is a space consisting of *coupled* Leibniz 2-cocycles, i.e. the nonzero elements have the property that their symmetric and antisymmetric parts are not cocycles. The Lie algebra $\mathfrak{g}$

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is said to be (adjoint) ZL^2-uncoupling if $\mathcal{C} = \{0\}$. That is best understood in terms of the Koszul map $\mathcal{I}$ which associates to any invariant bilinear form B on the Lie algebra $\mathfrak{g}$ the Koszul form $(X,Y,Z) \mapsto I_B(X,Y,Z) = B([X,Y],Z)$ $(X,Y,Z \in \mathfrak{g})$. Then $ZL^2_0(\mathfrak{g},\mathfrak{g}) = \mathfrak{c} \otimes \ker \mathcal{I}$ ($\mathfrak{c}$ the center of $\mathfrak{g}$) and $\mathcal{C} \cong (\mathfrak{c} \otimes \operatorname{Im} \mathcal{I}) \cap B^3(\mathfrak{g},\mathfrak{g})$. Hence $\mathfrak{g}$ is ZL^2-uncoupling if and only if $(\mathfrak{c} \otimes \operatorname{Im} \mathcal{I}) \cap B^3(\mathfrak{g},\mathfrak{g}) = \{0\}$. The class of (adjoint) ZL^2-uncoupling Lie algebras is rather extensive since it contains, beside the class of zero center Lie algebras, the class of Lie algebras having zero Koszul form, which we call $\mathcal{I}$-null Lie algebras.

In the present paper, we examine some properties of the class of $\mathcal{I}$-null Lie algebras. First, after proving basic properties of $\mathcal{I}$-null Lie algebras, we state in Proposition 2.6 a result for Lie algebras having a codimension 1 ideal, connecting $\mathcal{I}$-nullity of the ideal and $\mathcal{I}$-nullity or $\mathcal{I}$-exactedness (i.e. the Koszul form is a coboundary) of the Lie algebra itself. Several corollaries are given, and fundamental examples are treated in detail. We also give a table (Table 1) for all non $\mathcal{I}$-null complex Lie algebras of dimension $\leqslant 7$. This table is a new result. Then comes the main result of the paper, Theorem 3.1, which states that any nilradical of a Borel subalgebra of a finite-dimensional semi-simple Lie algebra is $\mathcal{I}$-null.

We also give a list of Kac-Moody types for indecomposable nilpotent Lie algebras of dimension ≤ 7 (Table 2). Again, that result is new.

Throughout the paper, the base field is $\mathbb{C}$.

2. The Koszul map and $\mathcal{I}$-null Lie algebras

Let $\mathfrak{g}$ be any finite dimensional complex Lie algebra. Recall that a symmetric bilinear form $B \in S^2\mathfrak{g}^*$ is said to be invariant (see [9]), i.e. $B \in \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$ if and only if $B([Z,X],Y) = -B(X,[Z,Y])$ $\forall X,Y,Z \in \mathfrak{g}$. The Koszul map $\mathcal{I}$: $\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} \to \left(\bigwedge^3 \mathfrak{g}^*\right)^{\mathfrak{g}} \subset Z^3(\mathfrak{g},\mathbb{C})$ is defined by $\mathcal{I}(B) = I_B$, with $I_B(X,Y,Z) = B([X,Y],Z)$ $\forall X,Y,Z \in \mathfrak{g}$.

Lemma 2.1. *Denote $\mathcal{C}^2\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$. The projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathcal{C}^2\mathfrak{g}$ induces an isomorphism*

$$\varpi \ : \ \ker \mathcal{I} \to S^2 \left(\mathfrak{g}/\mathcal{C}^2\mathfrak{g}\right)^* .$$

Proof. For $B \in \ker \mathcal{I}$, define $\varpi(B) \in S^2 \left(\mathfrak{g}/\mathcal{C}^2\mathfrak{g}\right)^*$ by

$$\varpi(B)(\pi(X),\pi(Y)) = B(X,Y), \ \forall X,Y \in \mathfrak{g}.$$

$\varpi(B)$ is well-defined since for $X,Y,U,V \in \mathfrak{g}$

$$\begin{aligned} B(X+[U,V],Y) &= B(X,Y) + B([U,V],Y) \\ &= B(X,Y) + I_B(U,V,Y) \\ &= B(X,Y) \text{ (as } I_B = 0). \end{aligned}$$

The map ϖ is injective since $\varpi(B) = 0$ implies $B(X,Y) = 0$ $\forall X,Y \in \mathfrak{g}$. To prove that it is onto, let $\bar{B} \in S^2 \left(\mathfrak{g}/\mathcal{C}^2\mathfrak{g}\right)^*$, and let $B_\pi \in S^2\mathfrak{g}^*$ defined by $B_\pi(X,Y) = \bar{B}(\pi(X),\pi(Y))$. Then $B_\pi([X,Y],Z) = \bar{B}(\pi([X,Y]),\pi(Z)) = \bar{B}(0,\pi(Z)) = 0$ $\forall X,Y,Z \in \mathfrak{g}$, hence $B_\pi \in \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$ and $B_\pi \in \ker \mathcal{I}$. Now, $\varpi(B_\pi) = \bar{B}$. □

From Lemma 2.1, $\dim\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} = \frac{\ell(\ell+1)}{2} + \dim \operatorname{Im}\mathcal{I}$, where $\ell = \dim H^1(\mathfrak{g},\mathbb{C}) = \dim\left(\mathfrak{g}/\mathcal{C}^2\mathfrak{g}\right)$. For reductive $\mathfrak{g}$, $\dim\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} = \dim H^3(\mathfrak{g},\mathbb{C})$ ([9]).

Definition 2.2. $\mathfrak{g}$ is said to be $\mathcal{I}$-null (resp. $\mathcal{I}$-exact) if $\mathcal{I} = 0$ (resp. $\operatorname{Im}\mathcal{I} \subset B^3(\mathfrak{g},\mathbb{C})$).

$\mathfrak{g}$ is $\mathcal{I}$-null if and only if $\mathcal{C}^2\mathfrak{g} \subset \ker B\ \forall B \in \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$. It is standard that for any $B \in \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$, there exists $B_1 \in \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$ such that $\ker(B+B_1) \subset \mathcal{C}^2\mathfrak{g}$. Hence $\bigcap_{B\in(S^2\mathfrak{g}^*)^{\mathfrak{g}}} \ker B \subset \mathcal{C}^2\mathfrak{g}$, and $\mathfrak{g}$ is $\mathcal{I}$-null if and only $\bigcap_{B\in(S^2\mathfrak{g}^*)^{\mathfrak{g}}} \ker B = \mathcal{C}^2\mathfrak{g}$.

Lemma 2.3. *(i) Any quotient of a (not necessarily finite dimensional) $\mathcal{I}$-null Lie algebra is $\mathcal{I}$-null;*
(ii) Any finite direct product of $\mathcal{I}$-null Lie algebras is $\mathcal{I}$-null.

Proof. (i) Let $\mathfrak{g}$ be any $\mathcal{I}$-null Lie algebra, $\mathfrak{h}$ an ideal of $\mathfrak{g}$, $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{h}$, $\pi : \mathfrak{g} \to \bar{\mathfrak{g}}$ the projection, and $\bar{B} \in \left(S^2\bar{\mathfrak{g}}^*\right)^{\bar{\mathfrak{g}}}$. Define $B_\pi \in S^2\mathfrak{g}^*$ by $B_\pi(X,Y) = \bar{B}(\pi(X),\pi(Y))$, $X,Y \in \mathfrak{g}$. Then $B_\pi([X,Y],Z) = \bar{B}(\pi([X,Y]),\pi(Z)) = \bar{B}([\pi(X),\pi(Y)],\pi(Z)) = \bar{B}(\pi(X),[\pi(Y),\pi(Z)]) = \bar{B}(\pi(X),\pi([Y,Z])) = B_\pi(X,[Y,Z])\ \forall X,Y,Z \in \mathfrak{g}$, hence $B_\pi \in \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$ and $I_{\bar{B}} \circ (\pi\times\pi\times\pi) = I_{B_\pi} = 0$ since $\mathfrak{g}$ is $\mathcal{I}$-null. Hence $I_{\bar{B}} = 0$.
(ii) Let $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ ($\mathfrak{g}_1, \mathfrak{g}_2$ $\mathcal{I}$-null) and $B \in \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$. As $B(X_1,[Y_2,Z_2]) = B([X_1,Y_2],Z_2) = B(0,Z_2) = 0\ \forall X_1 \in \mathfrak{g}_1, Y_2, Z_2 \in \mathfrak{g}_2$, B vanishes on $\mathfrak{g}_1 \times \mathcal{C}^2\mathfrak{g}_2$ and on $\mathcal{C}^2\mathfrak{g}_1 \times \mathfrak{g}_2$ as well, hence $I_B = 0$. □

Lemma 2.4. *Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra, with Cartan subalgebra $\mathfrak{h}$, simple root system S, positive roots Δ_+, and root subspaces $\mathfrak{g}^\alpha$. Let $\mathfrak{k} \neq \{0\}$ be any subspace of $\mathfrak{h}$, and $\Gamma \subset \Delta_+$ such that $\alpha+\beta \in \Gamma$ for $\alpha,\beta \in \Gamma$, $\alpha+\beta \in \Delta_+$. Consider $\mathfrak{u} = \mathfrak{k} \oplus \bigoplus_{\alpha\in\Gamma}\mathfrak{g}^\alpha$.*
(i) Suppose that $\alpha_{|\mathfrak{k}} \neq 0\ \forall\alpha \in \Gamma$. Then $\mathfrak{u}$ is $\mathcal{I}$-null;
(ii) Suppose that $\alpha_{|\mathfrak{k}} = 0\ \forall\alpha \in \Gamma\cap S$, and $\alpha_{|\mathfrak{k}} \neq 0\ \forall\alpha \in \Gamma\setminus S$. Then $\mathfrak{u}$ is $\mathcal{I}$-null.

Proof. (i) Let $\mathfrak{u}_+ = \bigoplus_{\alpha\in\Gamma}\mathfrak{g}^\alpha$, and X_α a root vector in $\mathfrak{g}^\alpha : \mathfrak{g}^\alpha = \mathbb{C}X_\alpha\ \forall\alpha \in \Gamma$. Let $B \in \left(S^2\mathfrak{u}^*\right)^{\mathfrak{u}}$. First, $B(H,X) = 0\ \forall H \in \mathfrak{k}, X \in \mathfrak{u}_+$. In fact, for any $\alpha \in \Gamma$, since there exists $H_\alpha \in \mathfrak{k}$ such that $\alpha(H_\alpha) \neq 0$, $B(H,X_\alpha) = \frac{1}{\alpha(H_\alpha)} B(H,[H_\alpha,X_\alpha]) = \frac{1}{\alpha(H_\alpha)} B([H,H_\alpha],X_\alpha]) = \frac{1}{\alpha(H_\alpha)} B(0,X_\alpha) = 0$. Second, that entails that the restriction of B to $\mathfrak{u}_+ \times \mathfrak{u}_+$ is zero, since for any $\alpha,\beta \in \Gamma$,

$$B(X_\alpha,X_\beta) = \frac{1}{\alpha(H_\alpha)}B([H_\alpha,X_\alpha],X_\beta) = \frac{1}{\alpha(H_\alpha)}B(H_\alpha,[X_\alpha,X_\beta]) = 0$$

as $[X_\alpha,X_\beta] \in \mathfrak{u}_+$. Then $\mathfrak{u}$ is $\mathcal{I}$-null.
(ii) In that case, $X_\alpha \notin \mathcal{C}^2\mathfrak{u}\ \forall\alpha \in \Gamma\cap S$, and $\dim\left(\mathfrak{u}/\mathcal{C}^2\mathfrak{u}\right) = \dim\mathfrak{k} + \#(\Gamma\cap S)$. For $\mathfrak{u}$ to be $\mathcal{I}$-null, one has to prove that, for any $B \in \left(S^2\mathfrak{u}^*\right)^{\mathfrak{u}}$:

$$B(H,X_\beta) = 0,\ \forall H \in \mathfrak{k},\ \beta \in \Gamma\setminus S; \tag{2.1}$$

$$B(X_\alpha,X_\beta) = 0,\ \forall\alpha \in \Gamma\cap S,\ \beta \in \Gamma\setminus S; \tag{2.2}$$

$$B(X_\beta,X_\gamma) = 0,\ \forall\beta,\gamma \in \Gamma\setminus S. \tag{2.3}$$

(2.1) is proved as in case (i). To prove (2.2), let $H_\beta \in \mathfrak{k}$ such that $\beta(H_\beta) \neq 0$. Then

$$B(X_\alpha, X_\beta) = \frac{1}{\beta(H_\beta)} B(X_\alpha, [H_\beta, X_\beta]) = \frac{1}{\beta(H_\beta)} B([X_\alpha, H_\beta], X_\beta) = -\frac{1}{\beta(H_\beta)} B(\alpha(H_\beta)X_\alpha, X_\beta) = -\frac{1}{\beta(H_\beta)} B(0, X_\beta) = 0.$$

As to (2.3),

$$B(X_\beta, X_\gamma) = \frac{1}{\beta(H_\beta)} B([H_\beta, X_\beta], X_\gamma) = \frac{1}{\beta(H_\beta)} B(H_\beta, [X_\beta, X_\gamma]) = 0$$

from (2.1).

□

Example 2.5. Any Borel subalgebra is $\mathcal{I}$-null.

Proposition 2.6. *Let $\mathfrak{g}_2$ be a codimension 1 ideal of the Lie algebra $\mathfrak{g}$, $(x_1, \dots, x_N)$ a basis of $\mathfrak{g}$ with $x_1 \notin \mathfrak{g}_2$, $x_2, \dots, x_N \in \mathfrak{g}_2$, π_2 the corresponding projection onto $\mathfrak{g}_2$, and $(\omega^1, \dots, \omega^N)$ denote the dual basis for $\mathfrak{g}^*$. Let $B \in \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$, and denote $B_2 \in \left(S^2\mathfrak{g}_2^*\right)^{\mathfrak{g}_2}$ the restriction of B to $\mathfrak{g}_2 \times \mathfrak{g}_2$. Then:*
(i)

$$I_B = d(\omega^1 \wedge f) + I_{B_2} \circ (\pi_2 \times \pi_2 \times \pi_2). \tag{2.4}$$

where $f = B(\cdot, x_1) \in \mathfrak{g}^*$;
(ii) Let $\gamma \in \bigwedge^2 \mathfrak{g}_2^* \subset \bigwedge^2 \mathfrak{g}^*$, and denote $d_{\mathfrak{g}_2}$ the coboundary operator of $\mathfrak{g}_2$. Then

$$d\gamma = \omega^1 \wedge \theta_{x_1}(\gamma) + d_{\mathfrak{g}_2}\gamma \circ (\pi_2 \times \pi_2 \times \pi_2) \tag{2.5}$$

where θ_{x_1} stands for the coadjoint action of x_1 on the cohomology of $\mathfrak{g}$;
(iii) Suppose $I_{B_2} \in B^3(\mathfrak{g}_2, \mathbb{C})$, and let $\gamma \in \bigwedge^2 \mathfrak{g}_2^* \subset \bigwedge^2 \mathfrak{g}^*$ such that $I_{B_2} = d_{\mathfrak{g}_2}\gamma$. Then $I_B \in B^3(\mathfrak{g}, \mathbb{C})$ if and only if $\omega^1 \wedge \theta_{x_1}(\gamma) \in B^3(\mathfrak{g}, \mathbb{C})$. In particular, the condition

$$\theta_{x_1}(\gamma) = df \tag{2.6}$$

implies $I_B = d\gamma$.

Proof. (i) For $X, Y, Z \in \mathfrak{g}$ one has

$$\begin{aligned} B([X,Y],Z) &= B([\omega^1(X)x_1 + \pi_2(X), \omega^1(Y)x_1 + \pi_2(Y)], \omega^1(Z)x_1 + \pi_2(Z)) \\ &= B\big(\omega^1(X)[x_1, \pi_2(Y)] - \omega^1(Y)[x_1, \pi_2(X)] + [\pi_2(X), \pi_2(Y)], \\ &\qquad \omega^1(Z)x_1 + \pi_2(Z)\big) \\ &= \omega^1(X)\omega^1(Z)B([x_1, \pi_2(Y)], x_1) - \omega^1(Y)\omega^1(Z)B([x_1, \pi_2(X)], x_1) \\ &\qquad + \beta(X,Y,Z) + B\left([\pi_2(X), \pi_2(Y)], \pi_2(Z)\right) \\ &= \beta(X,Y,Z) + B\left([\pi_2(X), \pi_2(Y)], \pi_2(Z)\right) \end{aligned} \tag{2.7}$$

where

$$\begin{aligned}\beta(X,Y,Z) &= \omega^1(Z)B([\pi_2(X),\pi_2(Y)],x_1)+\omega^1(X)B([x_1,\pi_2(Y)],\pi_2(Z))\\ &\qquad -\omega^1(Y)B([x_1,\pi_2(X)],\pi_2(Z))\\ &= \omega^1(Z)B([\pi_2(X),\pi_2(Y)],x_1)+\omega^1(X)B(x_1,[\pi_2(Y),\pi_2(Z)])\\ &\qquad -\omega^1(Y)B(x_1,[\pi_2(X),\pi_2(Z)]).\end{aligned}$$

Now

$$\begin{aligned}df(X,Y) &= -B([X,Y],x_1)\\ &= -B([\omega^1(X)x_1+\pi_2(X),\omega^1(Y)x_1+\pi_2(Y)],x_1)\\ &= -B(\omega^1(X)[x_1,\pi_2(Y)]-\omega^1(Y)[x_1,\pi_2(X)]+[\pi_2(X),\pi_2(Y)],x_1)\\ &= -B([\pi_2(X),\pi_2(Y)],x_1),\end{aligned}$$

hence

$$\begin{aligned}\beta(X,Y,Z) &= -(\omega^1(Z)df(X,Y)+\omega^1(X)df(Y,Z)-\omega^1(Y)df(X,Z))\\ &= -(\omega^1\wedge df)(X,Y,Z).\end{aligned}$$

Since $d\omega^1=0$, (2.7) then reads

$$I_B = d(\omega^1\wedge f)+I_{B_2}\circ(\pi_2\times\pi_2\times\pi_2). \tag{2.8}$$

(ii) One has for any $X,Y,Z\in\mathfrak{g}$

$$\begin{aligned}d\gamma(X,Y,Z) &= d\gamma(\pi_2(X),\pi_2(Y),\pi_2(Z))+\omega^1(X)d\gamma(x_1,\pi_2(Y),\pi_2(Z))\\ &\qquad +\omega^1(Y)d\gamma(\pi_2(X),x_1,\pi_2(Z))+\omega^1(Z)d\gamma(\pi_2(X),\pi_2(Y),x_1).\end{aligned}$$

Now, since γ vanishes if one of its arguments is x_1,

$$\begin{aligned}d\gamma(x_1,\pi_2(Y),\pi_2(Z)) &= -\gamma([x_1,\pi_2(Y)],\pi_2(Z))+\gamma([x_1,\pi_2(Z)],\pi_2(Y))\\ d\gamma(\pi_2(X),x_1,\pi_2(Z)) &= -\gamma([\pi_2(X),x_1],\pi_2(Z))-\gamma([x_1,\pi_2(Z)],\pi_2(X))\\ d\gamma(\pi_2(X),\pi_2(Y),x_1) &= \gamma([\pi_2(X),x_1],\pi_2(Y))-\gamma([\pi_2(Y),x_1],\pi_2(X)),\end{aligned}$$

hence

$$\begin{aligned}d\gamma(X,Y,Z) &= d\gamma(\pi_2(X),\pi_2(Y),\pi_2(Z))+\omega^1(X)\theta_{x_1}\gamma(\pi_2(Y),\pi_2(Z))\\ &\qquad -\omega^1(Y)\theta_{x_1}\gamma(\pi_2(X),\pi_2(Z))+\omega^1(Z)\theta_{x_1}\gamma(\pi_2(X),\pi_2(Y))\\ &= d\gamma(\pi_2(X),\pi_2(Y),\pi_2(Z))+\left(\omega^1\wedge\theta_{x_1}\gamma\right)(X,Y,Z)\end{aligned}$$

since $\theta_{x_1}\gamma(\pi_2(U),\pi_2(V))=\theta_{x_1}\gamma(U,V)$ for all $U,V\in\mathfrak{g}$.
(iii) Results immediately from (i) and (ii). □

Corollary 2.7. *Under the hypotheses of Proposition 2.6, suppose that x_1 commutes with every x_i $(2\leqslant i\leqslant N)$ except for $x_{i_1},\dots,x_{i_r}$ and that $x_{i_1},\dots,x_{i_r}$ commute to one another. Then, if $\mathfrak{g}_2$ is $\mathcal{I}$-null, $\mathfrak{g}$ is $\mathcal{I}$-null.*

Proof. From Equation 2.4, one has to prove that for any invariant bilinear symmetric form B on $\mathfrak{g}$, $f = B(\cdot, x_1) \in \mathfrak{g}^*$ verifies $df = 0$, i.e. for any $2 \leqslant i, j \leqslant N$, $B(x_1, [x_i, x_j]) = 0$. For $i \neq i_1, \dots, i_r$, and any $j \geqslant 2$, $B(x_1, [x_i, x_j]) = B([x_1, x_i], x_j) = B(0, x_j) = 0$. For $i, j \in \{i_1, \dots, i_r\}$, $B(x_1, [x_i, x_j]) = B(x_1, 0) = 0$. □

Definition 2.8. The n-dimensional standard filiform Lie algebra is the Lie algebra with basis $\{x_1, \dots, x_n\}$ and commutation relations $[x_1, x_i] = x_{i+1}$ $(1 \leqslant i < n)$.

Corollary 2.9. *Any standard filiform Lie algebra or any Heisenberg Lie algebra is $\mathcal{I}$-null.*

Corollary 2.10. *Any Lie algebra containing some $\mathcal{I}$-null codimension 1 ideal is $\mathcal{I}$-exact.*

Corollary 2.11. *Suppose that the Lie algebra $\mathfrak{g}$ is such that* $\dim Im\mathcal{I} = 0$ *or* 1. *Let $\tau \in Der\,\mathfrak{g}$ such that $\tau x_k \in \mathcal{C}^2\mathfrak{g}$ $\forall k \geqslant 2$ where $(x_1, \dots, x_N)$ is some basis of $\mathfrak{g}$. Denote $\tilde{\mathfrak{g}}_\tau = \mathbb{C}\tau \oplus \mathfrak{g}$ the Lie algebra obtained by adjoining the derivation τ to $\mathfrak{g}$, and by $\tilde{\mathcal{I}}$ the Koszul map of $\tilde{\mathfrak{g}}_\tau$. Then* $\dim Im\tilde{\mathcal{I}} = 0$ *if* $\dim Im\mathcal{I} = 0$, *and* $\dim Im\tilde{\mathcal{I}} = 0$ *or* 1 *if* $\dim Im\mathcal{I} = 1$.

Proof. Let $B \in \left(S^2\tilde{\mathfrak{g}}_\tau{}^*\right)^{\tilde{\mathfrak{g}}_\tau}$. One has

$$I_B = \omega^\tau \wedge df_\tau + I_{B_2} \circ (\pi_2 \times \pi_2 \times \pi_2) \tag{2.9}$$

where $(\tau, x_1, \dots, x_N)$ is the basis of $\tilde{\mathfrak{g}}_\tau$, $(\omega^\tau, \omega^1, \dots, \omega^N)$ the dual basis, B_2 the restriction of B to $\mathfrak{g}$, $f_\tau = B(\tau, \cdot\,)$ and π_2 the projection on $\mathfrak{g}$. We will also use the projection π_3 on $\mathrm{vect}(x_2, \dots, x_N)$. For $\tilde{X}, \tilde{Y} \in \tilde{\mathfrak{g}}_\tau$, $\tilde{X} = \omega^\tau(\tilde{X})\tau + X$, $\tilde{Y} = \omega^\tau(\tilde{Y})\tau + Y$, $X = \pi_2(\tilde{X}), Y = \pi_2(\tilde{Y})$, so that $df_\tau(\tilde{X}, \tilde{Y}) = -B(\tau, [\tilde{X}, \tilde{Y}]) = -B(\tau, [X, Y]) = -B(\tau, [\omega^1(X)x_1 + \pi_3(X), \omega^1(Y)x_1 + \pi_3(Y)]) = -\omega^1(X)B(\tau, [x_1, \pi_3(Y)]) +\omega^1(Y)B(\tau, [x_1, \pi_3(X)]) - B(\tau, [\pi_3(X), \pi_3(Y)])$, hence

$$df_\tau(\tilde{X}, \tilde{Y}) = \omega^1(X)B_2(\tau\pi_3(Y), x_1) - \omega^1(Y)B_2(\tau\pi_3(X), x_1) - B_2(\tau\pi_3(X), \pi_3(Y)). \tag{2.10}$$

Note that $\tau\pi_3(X), \tau\pi_3(Y) \in \mathcal{C}^2\mathfrak{g}$ by the hypotheses. Suppose first that $\mathfrak{g}$ is $\mathcal{I}$-null. Then $B_2(\tau\pi_3(Y), x_1), B_2(\tau\pi_3(X), x_1), B_2(\tau\pi_3(X), \pi_3(Y))$ all vanish. From Equations (2.9), (2.10), $\tilde{\mathfrak{g}}_\tau$ is $\tilde{\mathcal{I}}$-null. Suppose now that $\mathfrak{g}$ verifies $\dim \mathrm{Im}\,\mathcal{I} = 1$ and let $C \in \left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$ with $I_C \neq 0$. If $\tilde{\mathfrak{g}}_\tau$ is not $\tilde{\mathcal{I}}$-null we may suppose that $I_B \neq 0$. There exists $\lambda \in \mathbb{C}$ such that $I_{B_2} = \lambda I_C$. Then $B_2(\tau\pi_3(Y), x_1) = \lambda C(\tau\pi_3(Y), x_1)$, $B_2(\tau\pi_3(X), x_1) = \lambda C(\tau\pi_3(X), x_1)$, $B_2(\tau\pi_3(X), \pi_3(Y)) = \lambda C(\tau\pi_3(X), \pi_3(Y))$. It follows from Equations (2.9), (2.10), that $\dim \mathrm{Im}\,\tilde{\mathcal{I}} = 1$. □

Definition 2.12. A Lie algebra $\mathfrak{g}$ is said to be quadratic if there exists a nondegenerate invariant bilinear form on $\mathfrak{g}$.

Clearly, quadratic nonabelian Lie algebras are not $\mathcal{I}$-null.

Example 2.13. This example is an illustration to Corollary 2.11. The nilpotent Lie algebra $\mathfrak{g}_{7,2.4}$ has commutation relations $[x_1, x_2] = x_3$, $[x_1, x_3] = x_4$, $[x_1, x_4] =$

x_5, $[x_1,x_5]=x_6$, $[x_2,x_5]=-x_7$, $[x_3,x_4]=x_7$. $\mathfrak{g}_{7,2.4}$ is quadratic and $\dim \operatorname{Im}\mathcal{I}=1$. The elements of Der $\mathfrak{g}_{7,2.4}$(mod ad$\mathfrak{g}_{7,2.4}$) are

$$\tau=\begin{pmatrix} \xi_1^1 & 0 & 0 & 0 & 0 & 0 & 0\\ \xi_1^2 & \xi_2^2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \xi_1^1+\xi_2^2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 2\xi_1^1+\xi_2^2 & 0 & 0 & 0\\ 0 & \xi_2^5 & 0 & 0 & 3\xi_1^1+\xi_2^2 & 0 & 0\\ \xi_1^6 & \xi_2^6 & \xi_2^5 & 0 & 0 & 4\xi_1^1+\xi_2^2 & 0\\ \xi_1^7 & 0 & 0 & 0 & 0 & -\xi_2^1 & 3\xi_1^1+2\xi_2^2 \end{pmatrix} \tag{2.11}$$

τ is nilpotent if $\xi_1^1=\xi_2^2=0$. Denote the nilpotent τ by $(\xi_1^2;\xi_2^5;\xi_1^6,\xi_2^6;\xi_1^7)$. Now, projectively equivalent derivations τ,τ' (see [12]) give isomorphic $\tilde{\mathfrak{g}}_\tau$, $\tilde{\mathfrak{g}}_{\tau'}$. By reduction using projective equivalence, we are reduced to the following cases: Case 1. $\xi_1^2\neq 0$: $(1;\varepsilon;0,\eta;0)$; Case 2. $\xi_1^2=0$: $(0;\varepsilon;0,\eta;\lambda)$; where $\varepsilon,\eta,\lambda=0,1$. In both cases $\tilde{\mathfrak{g}}_\tau$ is $\mathcal{I}$-null, except when $\tau=0$ in case 2 where $\tilde{\mathfrak{g}}_\tau$ is the direct product $\mathbb{C}\times\mathfrak{g}_{7,2.4}$ which is quadratic. Hence any indecomposable 8-dimensional nilpotent Lie algebra containing a subalgebra isomorphic to $\mathfrak{g}_{7,2.4}$ is $\mathcal{I}$-null, though $\mathfrak{g}_{7,2.4}$ is quadratic. That is in line with the fact that, from the double extension method of [16], [15], any indecomposable quadratic solvable Lie algebra is a *double extension* of a quadratic solvable Lie algebra by $\mathbb{C}$.

Example 2.14. Among the 170 (non isomorphic) nilpotent complex Lie algebras of dimension $\leqslant 7$, only a few are not $\mathcal{I}$-null. Those are listed in Table 1 in the classification of [11], [13] (they are all $\mathcal{I}$-exact). Table 1 gives for each of them $\dim\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$, a basis for $\left(\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} / \ker\mathcal{I}\right)$ (which in those cases is one-dimensional), and the corresponding I_Bs. The results in Table 1 are new and have been obtained, first by explicit computation of all invariant bilinear forms on each one of the 170 Lie algebras with the computer algebra system *Reduce* and a program similar to those in [12],[13], and second by hand calculation of I_B for non $\mathcal{I}$-null Lie algebras. $\sharp$ denotes quadratic Lie algebras; for $\omega,\pi\in\mathfrak{g}^*$, $\odot$ stands for the symmetric product $\omega\odot\pi=\omega\otimes\pi+\pi\otimes\omega$; $\omega^{i,j,k}$ stands for $\omega^i\wedge\omega^j\wedge\omega^k$.

Remark 2.15. There are nilpotent Lie algebras of higher dimension with $\dim\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} / \ker\mathcal{I}>1$. For example, in the case of the 10 dimensional Lie algebra $\mathfrak{g}$ with commutation relations $[x_1,x_2]=x_5,[x_1,x_3]=x_6,[x_1,x_4]=x_7,[x_2,x_3]=x_8,[x_2,x_4]=x_9,[x_3,x_4]=x_{10}$, $\dim\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} / \ker\mathcal{I}=4$, and in the analogous case of the 15 dimensional nilpotent Lie algebra with 5 generators one has $\dim\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} / \ker\mathcal{I}=10$. Those algebras are $\mathcal{I}$-exact and not quadratic.

Example 2.16. The quadratic 5-dimensional nilpotent Lie algebra $\mathfrak{g}_{5,4}$ has commutation relations $[x_1,x_2]=x_3$, $[x_1,x_3]=x_4$, $[x_2,x_3]=x_5$. Consider the 10-dimensional direct product $\mathfrak{g}_{5,4}\times\mathfrak{g}_{5,4}$, with the commutation relations: $[x_1,x_2]=x_5$, $[x_1,x_5]=x_6$, $[x_2,x_5]=x_7$, $[x_3,x_4]=x_8$, $[x_3,x_8]=x_9$, $[x_4,x_8]=x_{10}$. The only 11-dimensional nilpotent Lie algebra with an invariant bilinear form which reduces to $B_1=\omega^1\odot\omega^7-\omega^2\odot\omega^6+\omega^5\otimes\omega^5$, $B_2=\omega^3\odot\omega^{10}-\omega^4\odot\omega^9+\omega^8\otimes\omega^8$, on respectively the first and second factor is the direct product $\mathbb{C}\times\mathfrak{g}_{5,4}\times\mathfrak{g}_{5,4}$,

TABLE 1. Non $\mathcal{I}$-null nilpotent complex Lie algebras of dimension $\leqslant 7$.

algebra	$\dim\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}}$	basis for $\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} / \ker\mathcal{I}$	I_B.
$\mathfrak{g}_{5,4}$ ♯	4	$\omega^1\odot\omega^5-\omega^2\odot\omega^4+\omega^3\otimes\omega^3$	$\omega^{1,2,3}=d\omega^{1,5}$
$\mathfrak{g}_{6,3}$ ♯	7	$\omega^1\odot\omega^6-\omega^2\odot\omega^5+\omega^3\odot\omega^4$	$\omega^{1,2,3}=d\omega^{1,6}$
$\mathfrak{g}_{6,14}$	4	$\omega^1\odot\omega^6-\omega^2\odot\omega^4+\omega^3\otimes\omega^3$	$\omega^{1,2,3}=-d\omega^{1,4}$
$\mathfrak{g}_{5,4}\times\mathbb{C}$ ♯	7	$\omega^1\odot\omega^5-\omega^2\odot\omega^4+\omega^3\otimes\omega^3$	$\omega^{1,2,3}=d\omega^{1,5}$
$\mathfrak{g}_{5,4}\times\mathbb{C}^2$ ♯	11	$\omega^1\odot\omega^5-\omega^2\odot\omega^4+\omega^3\otimes\omega^3$	$\omega^{1,2,3}=d\omega^{1,5}$
$\mathfrak{g}_{6,3}\times\mathbb{C}$ ♯	11	$\omega^1\odot\omega^6-\omega^2\odot\omega^5+\omega^3\odot\omega^4$	$\omega^{1,2,3}=d\omega^{1,6}$
$\mathfrak{g}_{7,0.4(\lambda)}$, $\mathfrak{g}_{7,0.5}$, $\mathfrak{g}_{7,0.6}$, $\mathfrak{g}_{7,1.02}$, $\mathfrak{g}_{7,1.10}$, $\mathfrak{g}_{7,1.13}$, $\mathfrak{g}_{7,1.14}$, $\mathfrak{g}_{7,1.17}$	4	$\omega^1\odot\omega^5-\omega^2\odot\omega^4+\omega^3\otimes\omega^3$	$\omega^{1,2,3}=d\omega^{1,5}$
$\mathfrak{g}_{7,1.03}$	4	$\omega^1\odot\omega^6-\omega^2\odot\omega^4+\omega^3\otimes\omega^3$	$\omega^{1,2,3}=d\omega^{1,6}$
$\mathfrak{g}_{7,2.2}$	7	$\omega^1\odot\omega^4-\omega^2\odot\omega^6+\omega^3\odot\omega^5$	$\omega^{1,2,3}=d\omega^{1,4}$
$\mathfrak{g}_{7,2.4}$ ♯	4	$\omega^1\odot\omega^7+\omega^2\odot\omega^6-\omega^3\odot\omega^5+\omega^4\otimes\omega^4$	$\omega^{1,3,4}-\omega^{1,2,5}=d\omega^{1,7}$
$\mathfrak{g}_{7,2.5}$, $\mathfrak{g}_{7,2.6}$, $\mathfrak{g}_{7,2.7}$, $\mathfrak{g}_{7,2.8}$, $\mathfrak{g}_{7,2.9}$,	4	$\omega^1\odot\omega^5-\omega^2\odot\omega^4+\omega^3\otimes\omega^3$	$\omega^{1,2,3}=d\omega^{1,5}$
$\mathfrak{g}_{7,2.18}$	7	$\omega^1\odot\omega^6-\omega^2\odot\omega^5+\omega^4\otimes\omega^4$	$\omega^{1,2,4}=d\omega^{1,6}$
$\mathfrak{g}_{7,2.44}$, $\mathfrak{g}_{7,3.6}$	7	$\omega^1\odot\omega^6-\omega^2\odot\omega^5+\omega^3\odot\omega^4$	$\omega^{1,2,3}=d\omega^{1,6}$
$\mathfrak{g}_{7,3.23}$	7	$\omega^1\odot\omega^6-\omega^2\odot\omega^5+\omega^3\otimes\omega^3$	$\omega^{1,2,3}=d\omega^{1,6}$

Example 2.17. The 4-dimensional solvable "diamond" Lie algebra $\mathfrak{g}$ with basis (x_1,x_2,x_3,x_4) and commutation relations $[x_1,x_2]=x_3,[x_1,x_3]=-x_2,[x_2,x_3]=x_4$ cannot be obtained as in Lemma 2.4. Here $\dim\left(\left(S^2\mathfrak{g}^*\right)^{\mathfrak{g}} / \ker\mathcal{I}\right)=1$, with basis element $B=\omega^1\odot\omega^4+\omega^2\otimes\omega^2+\omega^3\otimes\omega^3$. $I_B=\omega^{1,2,3}=d\omega^{1,4}$; $\mathfrak{g}$ is quadratic and $\mathcal{I}$-exact. In fact, one verifies that all other solvable 4-dimensional Lie algebras are $\mathcal{I}$-null (for a list, see e.g. [17]). For a complete description of Leibniz and Lie deformations of the diamond Lie algebra (and a study of the case of $\mathfrak{g}_{5,4}$), see [5].

3. CASE OF A NILRADICAL

We now state and prove our main result. The proof is by case analysis over the simple complex finite dimensional Lie algebras. In the classical cases, the point consists in an inductive use of Corollary 2.7. In the exceptional cases, we either utilize directly the commutation relations (G_2,F_4), or make use of a certain property of the pattern of positive roots, which we call property $(\mathcal{P})$ (E_6,E_7,E_8).

Theorem 3.1. *Any nilradical $\mathfrak{g}$ of a Borel subalgebra of a finite-dimensional semi-simple Lie algebra is $\mathcal{I}$-null.*

Proof. It is enough to consider the case of a simple Lie algebra, hence of one of the 4 classical types plus the 5 exceptional ones.

Case A_n. Denote $E_{i,j}, 1\leqslant i,j\leqslant n+1$ the canonical basis of $\mathfrak{gl}(n+1,\mathbb{C})$. One may suppose that the Borel subalgebra of $A_n=\mathfrak{sl}(n+1)$ is comprised of the upper triangular matrices with zero trace, and the Cartan subalgebra $\mathfrak{h}$ is $\bigoplus_{i=1}^{i=n}\mathbb{C}H_i$ with $H_i=E_{i,i}-E_{i+1,i+1}$. The nilradical is $\mathfrak{g}=A_n^+=\bigoplus_{1\leqslant i<j\leqslant n+1}\mathbb{C}E_{i,j}$. For

$n = 1$, $\mathfrak{g} = \mathbb{C}$ is $\mathcal{I}$-null. Suppose the result holds for the nilradical of the Borel subalgebra of $A_{n-1} = \mathfrak{sl}(n)$. One has $\mathfrak{g} = \mathbb{C}E_{1,2} \oplus \cdots \oplus \mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2'$ with $\mathfrak{g}_2' = \bigoplus_{2\leqslant i<j\leqslant n+1} \mathbb{C}E_{i,j}$ being the nilradical of the Borel subalgebra of A_{n-1}, hence $\mathcal{I}$-null. $E_{1,n+1}$ commutes with $\mathfrak{g}_2'$, hence $\mathfrak{g}_2'$ is a codimension 1 ideal of $\mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2'$, and, from Corollary 2.7, $\mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2'$ is $\mathcal{I}$-null. Now $E_{1,n}$ commutes with all members of the basis of $\mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2'$, except for $E_{n,n+1}$, and $[E_{1,n}, E_{n,n+1}] = E_{1,n+1}$. Then $\mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2'$ is a codimension 1 ideal of $\mathbb{C}E_{1,n} \oplus (\mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2')$, and from Corollary 2.7, $\mathbb{C}E_{1,n} \oplus (\mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2')$ is $\mathcal{I}$-null. Consider $\mathbb{C}E_{1,n-1} \oplus (\mathbb{C}E_{1,n} \oplus \mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2')$. $E_{1,n-1}$ commutes with all members of the basis of $\mathbb{C}E_{1,n} \oplus \mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2'$ except for $E_{n-1,n}, E_{n-1,n+1}$, and yields respectively $E_{1,n}, E_{1,n+1}$. Then $\mathbb{C}E_{1,n} \oplus \mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2'$ is a codimension 1 ideal of $\mathbb{C}E_{1,n-1} \oplus (\mathbb{C}E_{1,n} \oplus \mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2')$, and since $E_{n-1,n}, E_{n-1,n+1}$ commute, we get from Corollary 2.7 that $\mathbb{C}E_{1,n-1} \oplus (\mathbb{C}E_{1,n} \oplus \mathbb{C}E_{1,n+1} \oplus \mathfrak{g}_2')$ is $\mathcal{I}$-null. The result then follows by induction.

Case D_n. We may take D_n as the Lie algebra of matrices

$$\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -{}^tZ_1 \end{pmatrix} \tag{3.1}$$

with $Z_i \in \mathfrak{gl}(n, \mathbb{C}), Z_2, Z_3$ skew symmetric (see [7], p. 193). Denote $\tilde{E}_{i,j} = \begin{pmatrix} E_{i,j} & 0 \\ 0 & -E_{j,i} \end{pmatrix}$, $\tilde{F}_{i,j} = \begin{pmatrix} 0 & E_{i,j}-E_{j,i} \\ 0 & 0 \end{pmatrix}$ ($E_{i,j}, 1 \leqslant i, j \leqslant n$ the canonical basis of $\mathfrak{gl}(n, \mathbb{C})$). The Cartan subalgebra $\mathfrak{h}$ is $\bigoplus_{i=1}^{i=n} \mathbb{C}H_i$ with $H_i = \tilde{E}_{i,i}$ and the nilradical of the Borel subalgebra is

$$D_n^+ = \bigoplus_{1\leqslant i<j\leqslant n} \mathbb{C}\tilde{E}_{i,j} \oplus \bigoplus_{1\leqslant i<j\leqslant n} \mathbb{C}\tilde{F}_{i,j}. \tag{3.2}$$

All $\tilde{F}_{i,j}$'s commute to one another, and one has:

$$[\tilde{E}_{i,j}, \tilde{F}_{k,l}] = \delta_{j,k}\tilde{F}_{i,l} - \delta_{j,l}\tilde{F}_{i,k}. \tag{3.3}$$

We identify D_{n-1} to a subalgebra of D_n by simply taking the first row and first column of each block to be zero in (3.1). For $n = 2$, $D_2^+ = \mathbb{C}^2$ is $\mathcal{I}$-null. Suppose the result holds true for D_{n-1}^+. One has

$$D_n^+ = \mathbb{C}\tilde{E}_{1,2} \oplus \mathbb{C}\tilde{E}_{1,3} \oplus \ldots \mathbb{C}\tilde{E}_{1,n} \oplus \mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+.$$

Start with $\mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+$. From (3.3), $\tilde{F}_{1,2}$ commutes with all $\tilde{E}_{i,j}$ ($2 \leqslant i < j \leqslant n$) hence with D_{n-1}^+. Then D_{n-1}^+ is a codimension 1 ideal of $\mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+$ and $\mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+$ is $\mathcal{I}$-null from Corollary 2.7. Consider now $\mathbb{C}\tilde{F}_{1,3} \oplus (\mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$. Again from (3.3), $\tilde{F}_{1,3}$ commutes with all elements of the basis of D_{n-1}^+ except $\tilde{E}_{2,3}$ and $[\tilde{E}_{2,3}, \tilde{F}_{1,3}] = \tilde{F}_{1,2}$. Then $\mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+$ is a codimension 1 ideal of $\mathbb{C}\tilde{F}_{1,3} \oplus (\mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$, and the latter is $\mathcal{I}$-null. Suppose that $\mathbb{C}\tilde{F}_{1,s-1} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+$ is a codimension 1 ideal of $\mathbb{C}\tilde{F}_{1,s} \oplus (\mathbb{C}\tilde{F}_{1,s-1} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$, and that the latter is $\mathcal{I}$-null. Consider $\mathbb{C}\tilde{F}_{1,s+1} \oplus (\mathbb{C}\tilde{F}_{1,s} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$. From (3.3), for $2 \leqslant i < j \leqslant n$, $[\tilde{E}_{i,j}, \tilde{F}_{1,s+1}] = \delta_{j,s+1}\tilde{F}_{1,i}$ is nonzero only for $i = 2, \ldots, s$, and $j = s+1$, and it is then equal to $\tilde{F}_{1,i}$. Then first $\mathbb{C}\tilde{F}_{1,s} \oplus (\mathbb{C}\tilde{F}_{1,s-1} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$ is a codimension 1 ideal of $\mathbb{C}\tilde{F}_{1,s+1} \oplus (\mathbb{C}\tilde{F}_{1,s} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D_{n-1}^+)$. Second, the latter is $\mathcal{I}$-null from

Corollary 2.7. By induction the above property holds for $s = n$. Consider now $\mathbb{C}\tilde{E}_{1,n} \oplus (\mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D^+_{n-1})$. One has for $2 \leqslant i < j \leqslant n$, $[\tilde{E}_{1,n}, \tilde{E}_{i,j}] = 0$, $[\tilde{E}_{1,n}, \tilde{F}_{i,j}] = -\delta_{n,j}\tilde{F}_{1,i}$, $[\tilde{E}_{1,n}, \tilde{F}_{1,j}] = 0$. Hence $\mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D^+_{n-1}$ is an ideal of $\mathbb{C}\tilde{E}_{1,n} \oplus (\mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D^+_{n-1})$ and the latter is $\mathcal{I}$-null. For $2 \leqslant i < j \leqslant n$, $1 \leqslant k \leqslant n-2$,

$$\begin{array}{rcl} [\tilde{E}_{1,n-k}, \tilde{E}_{i,j}] &=& \delta_{n-k,i}\tilde{E}_{1,j}, \\ [\tilde{E}_{1,n-k}, \tilde{F}_{i,j}] &=& \delta_{n-k,i}\tilde{F}_{1,j} - \delta_{n-k,j}\tilde{F}_{1,i}, \\ [\tilde{E}_{1,n-k}, \tilde{E}_{1,n}] &=& \delta_{n-k,1}\tilde{E}_{1,n} = 0, \\ [\tilde{E}_{1,n-k}, \tilde{F}_{1,j}] &=& \delta_{n-k,1}\tilde{F}_{1,j} = 0. \end{array}$$

$[\tilde{E}_{1,n-1}, \tilde{E}_{i,j}]$ is nonzero only for $(i = n-1, j = n)$ and then yields $\tilde{E}_{1,n}$; $[\tilde{E}_{1,n-1}, \tilde{F}_{i,j}]$ is nonzero only for $(i = n-1, j = n)$ or for $(i < j = n-1)$ and yields respectively $\tilde{F}_{1,n}$, or $-\tilde{F}_{1,i}$. $[\tilde{E}_{1,n-1}, \tilde{E}_{1,n}]$ and $[\tilde{E}_{1,n-1}, \tilde{F}_{1,j}]$ are zero for $n \geqslant 3$. Hence, first $\mathbb{C}\tilde{E}_{1,n} \oplus \mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D^+_{n-1}$ is a codimension 1 ideal of $\mathbb{C}\tilde{E}_{1,n-1} \oplus (\mathbb{C}\tilde{E}_{1,n} \oplus \mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D^+_{n-1})$, and second the latter is $\mathcal{I}$-null, since $\tilde{E}_{n-1,n}$ commutes with $\tilde{F}_{n-1,n}$, $\tilde{F}_{i,n-1}$. Suppose that $\mathbb{C}\tilde{E}_{1,n-k+1} \oplus \cdots \oplus \mathbb{C}\tilde{E}_{1,n} \oplus \mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D^+_{n-1}$ is a codimension 1 ideal of... $\mathbb{C}\tilde{E}_{1,n-k} \oplus (\mathbb{C}\tilde{E}_{1,n-k+1} \oplus \cdots \oplus \mathbb{C}\tilde{E}_{1,n} \oplus \mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D^+_{n-1})$ and that the latter is $\mathcal{I}$-null. Consider $\mathbb{C}\tilde{E}_{1,n-k-1} \oplus (\mathbb{C}\tilde{E}_{1,n-k} \oplus \cdots \oplus \mathbb{C}\tilde{E}_{1,n} \oplus \mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D^+_{n-1})$. $[\tilde{E}_{1,n-k-1}, \tilde{E}_{i,j}]$ is nonzero only for $i = n-k-1$ and yields then $\tilde{E}_{1,j}$; $[\tilde{E}_{1,n-k-1}, \tilde{F}_{i,j}] = \delta_{n-k-1,i}\tilde{F}_{1,j} - \delta_{n-k-1,j}\tilde{F}_{1,i}$ is nonzero only for $i = n-k-1$ or $j = n-k-1$ and yields resp. $\tilde{F}_{1,j}$ or $-\tilde{F}_{1,i}$. Hence $\mathbb{C}\tilde{E}_{1,n-k} \oplus \cdots \oplus \mathbb{C}\tilde{E}_{1,n} \oplus \mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D^+_{n-1}$ is an ideal of $\mathbb{C}\tilde{E}_{1,n-k-1} \oplus (\mathbb{C}\tilde{E}_{1,n-k} \oplus \cdots \oplus \mathbb{C}\tilde{E}_{1,n} \oplus \mathbb{C}\tilde{F}_{1,n} \oplus \cdots \oplus \mathbb{C}\tilde{F}_{1,2} \oplus D^+_{n-1})$. The latter is $\mathcal{I}$-null since $\tilde{E}_{n-k-1,j}$ commutes with both $\tilde{F}_{n-k-1,j'}$, $\tilde{F}_{i,n-k-1}$ $(j' \geqslant n-k)$. The result follows by induction.

Case B_n. We may take B_n $(n \geqslant 2)$ as the Lie algebra of matrices

$$\left(\begin{array}{c|c|c} 0 & u & v \\ \hline -{}^t v & Z_1 & Z_2 \\ \hline -{}^t u & Z_3 & -{}^t Z_1 \end{array}\right) \tag{3.4}$$

with u, v complex $(1 \times n)$-matrices, $Z_i \in \mathfrak{gl}(n, \mathbb{C})$, Z_2, Z_3 skew symmetric, i.e.

$$\left(\begin{array}{c|ccc} 0 & u & & v \\ \hline -{}^t v & & & \\ & & \boxed{A} & \\ -{}^t u & & & \end{array}\right) \tag{3.5}$$

with $A \in D_n$. We identify $A \in D_n$ to the matrix

$$\left(\begin{array}{c|ccc} 0 & 0 & & 0 \\ \hline 0 & & & \\ & & \boxed{A} & \\ 0 & & & \end{array}\right) \in B_n.$$

The Cartan subalgebra of B_n is then simply that of D_n. B_n^+ consists of the matrices

$$\left(\begin{array}{c|c} 0 & 0 \quad\quad v \\ \hline -{}^t v & \\ & \boxed{A} \\ 0 & \end{array}\right) \tag{3.6}$$

with v complex $(1\times n)$-matrix and $A\in D_n^+$. For $1\leqslant q\leqslant n$, let v_q the $(1\times n)$-matrix $(0,\dots,1,\dots,0)$ (1 in q^{th} position), and

$$\tilde{v}_q=\left(\begin{array}{c|c} 0 & 0 \quad\quad v_q \\ \hline -{}^t v_q & \\ & \boxed{0} \\ 0 & \end{array}\right)$$

Hence $B_n^+=\left(\bigoplus_{q=1}^n\mathbb{C}\tilde{v}_q\right)\oplus D_n^+$. One has for $1\leqslant q\leqslant n,\ 1\leqslant i<j\leqslant n$

$$\begin{array}{rcl}[\tilde{v}_q,\tilde{E}_{i,j}] & = & -\delta_{q,j}\tilde{v}_i\\ [\tilde{v}_q,\tilde{F}_{i,j}] & = & 0\end{array}$$

and for $1\leqslant s<q\leqslant n$

$$[\tilde{v}_q,\tilde{v}_s]=\tilde{F}_{s,q}. \tag{3.7}$$

Consider $\mathbb{C}\tilde{v}_1\oplus D_n^+$. As $\tilde{v}_1$ commutes with $\tilde{E}_{i,j}$ and $\tilde{F}_{i,j}$, D_n^+ is an ideal of $\mathbb{C}\tilde{v}_1\oplus D_n^+$ and the latter is $\mathcal{I}$-null. Suppose that $\mathbb{C}\tilde{v}_{s-1}\oplus\cdots\oplus\mathbb{C}\tilde{v}_1\oplus D_n^+$ is an ideal of $\mathbb{C}\tilde{v}_s\oplus(\mathbb{C}\tilde{v}_{s-1}\oplus\cdots\oplus\mathbb{C}\tilde{v}_1\oplus D_n^+)$ and the latter is $\mathcal{I}$-null. Consider $\mathbb{C}\tilde{v}_{s+1}\oplus(\mathbb{C}\tilde{v}_s\oplus\mathbb{C}\tilde{v}_{s-1}\oplus\cdots\oplus\mathbb{C}\tilde{v}_1\oplus D_n^+)$. $[\tilde{v}_{s+1},\tilde{E}_{i,j}]=-\delta_{s+1,j}\tilde{v}_i$ hence $\tilde{v}_{s+1}$ commutes to all $\tilde{E}_{i,j}$'s except for $\tilde{E}_{i,s+1}$ $(i\leqslant s)$ and then yields $-\tilde{v}_i$. For $t\leqslant s$, $[\tilde{v}_{s+1},\tilde{v}_t]=\tilde{F}_{t,s+1}$. Hence $\mathbb{C}\tilde{v}_s\oplus\mathbb{C}\tilde{v}_{s-1}\oplus\cdots\oplus\mathbb{C}\tilde{v}_1\oplus D_n^+$ is an ideal of $\mathbb{C}\tilde{v}_{s+1}\oplus(\mathbb{C}\tilde{v}_s\oplus\mathbb{C}\tilde{v}_{s-1}\oplus\cdots\oplus\mathbb{C}\tilde{v}_1\oplus D_n^+)$. Now we cannot apply directly Corollary 2.7 to conclude that the latter is $\mathcal{I}$-null as the family $\mathcal{F}=\{\tilde{E}_{i,s+1},\tilde{v}_t;1\leqslant i\leqslant s,1\leqslant t\leqslant s\}$ is not commutative. The $\tilde{E}_{i,s+1}$'s $(i\leqslant s)$ commute to one another and to the $\tilde{v}_t$'s, but the $\tilde{v}_t$'s do not commute to one another. However, recall from the proof of Corollary 2.7 that one has to check that, for any invariant bilinear form B on $\mathbb{C}\tilde{v}_{s+1}\oplus(\mathbb{C}\tilde{v}_s\oplus\mathbb{C}\tilde{v}_{s-1}\oplus\cdots\oplus\mathbb{C}\tilde{v}_1\oplus D_n^+)$, $B(\tilde{v}_{s+1},[X,Y])=0$ for all $X,Y\in\mathcal{F}$. That reduces to $B(\tilde{v}_{s+1},[\tilde{v}_t,\tilde{v}_{t'}])=0\ \forall t,t',1\leqslant t<t'\leqslant s$. Now, $B(\tilde{v}_{s+1},[\tilde{v}_t,\tilde{v}_{t'}])=B([\tilde{v}_{s+1},\tilde{v}_t],\tilde{v}_{t'})=B(\tilde{F}_{t,s+1},\tilde{v}_{t'})=B([\tilde{E}_{t,s},\tilde{F}_{s,s+1}],\tilde{v}_{t'})=0$ since $\tilde{E}_{t,s},\tilde{F}_{s,s+1},\tilde{v}_{t'}\in\mathbb{C}\tilde{v}_s\oplus\mathbb{C}\tilde{v}_{s-1}\oplus\cdots\oplus\mathbb{C}\tilde{v}_1\oplus D_n^+$ which is $\mathcal{I}$-null. We conclude that $\mathbb{C}\tilde{v}_{s+1}\oplus(\mathbb{C}\tilde{v}_s\oplus\mathbb{C}\tilde{v}_{s-1}\oplus\cdots\oplus\mathbb{C}\tilde{v}_1\oplus D_n^+)$ is $\mathcal{I}$-null. By induction the property holds for $s=n$ and B_n^+ is $\mathcal{I}$-null.

Case C_n. This case is pretty similar to the case D_n. We may take C_n as the Lie algebra of matrices

$$\begin{pmatrix} Z_1 & Z_2\\ Z_3 & -{}^tZ_1\end{pmatrix} \tag{3.8}$$

with $Z_i \in \mathfrak{gl}(n,\mathbb{C}), Z_2, Z_3$ symmetric. $\tilde{E}_{i,j}$ and the Cartan subalgebra are identical to those of D_n. We denote for $1 \leqslant i,j \leqslant n : \hat{F}_{i,j} = \left(\begin{smallmatrix} 0 & E_{i,j}+E_{j,i} \\ 0 & 0 \end{smallmatrix}\right)$. Then

$$C_n^+ = \bigoplus_{1\leqslant i<j\leqslant n} \mathbb{C}\tilde{E}_{i,j} \oplus \bigoplus_{1\leqslant i\leqslant j\leqslant n} \mathbb{C}\hat{F}_{i,j}. \tag{3.9}$$

All $\hat{F}_{k,l}$'s commute to one another, and one has:

$$[\tilde{E}_{i,j}, \hat{F}_{k,l}] = \delta_{j,k}\hat{F}_{i,l} + \delta_{j,l}\hat{F}_{i,k}. \tag{3.10}$$

The case is step by step analogous to the case of D_n with (3.10) instead of (3.3) and (3.9) instead of (3.2).

Case G_2. The commutation relations for G_2 appear in [6], p. 346. G_2^+ is 6-dimensional with commutation relations $[x_1,x_2] = x_3$; $[x_1,x_3] = 2x_4$; $[x_1,x_4] = -3x_5$; $[x_2,x_5] = -x_6$; $[x_3,x_4] = -3x_6$. G_2^+ has the same adjoint cohomology $(1,4,7,8,7,5,2)$ as, and is isomorphic to, $\mathfrak{g}_{6,18}$, which is $\mathcal{I}$-null.

Case F_4. F_4^+ has 24 positive roots, and root vectors x_i $(1 \leqslant i \leqslant 24)$. From the root pattern, one gets with some calculations the commutation relations of F_4^+ : $[x_1,x_2] = x_5$; $[x_1,x_{13}] = x_{14}$; $[x_1,x_{15}] = -x_6$; $[x_1,x_{16}] = -x_7$; $[x_1,x_{17}] = -x_{23}$; $[x_1,x_{18}] = x_{19}$; $[x_1,x_{24}] = x_{22}$; $[x_2,x_3] = x_{15}$; $[x_2,x_7] = x_8$; $[x_2,x_{12}] = x_{13}$; $[x_2,x_{19}] = x_{20}$; $[x_2,x_{21}] = x_{24}$; $[x_2,x_{23}] = x_9$; $[x_3,x_4] = x_{21}$; $[x_3,x_5] = x_6$; $[x_3,x_6] = x_7$; $[x_3,x_9] = x_{10}$; $[x_3,x_{11}] = x_{12}$; $[x_3,x_{15}] = x_{16}$; $[x_3,x_{20}] = -2x_{11}$; $[x_3,x_{22}] = \frac{1}{2}x_{23}$; $[x_3,x_{24}] = -\frac{1}{2}x_{17}$; $[x_4,x_6] = x_{22}$; $[x_4,x_7] = x_{23}$; $[x_4,x_8] = x_9$; $[x_4,x_9] = -x_{20}$; $[x_4,x_{10}] = x_{11}$; $[x_4,x_{15}] = -x_{24}$; $[x_4,x_{16}] = x_{17}$; $[x_4,x_{17}] = x_{18}$; $[x_4,x_{23}] = -x_{19}$; $[x_5,x_{12}] = x_{14}$; $[x_5,x_{16}] = x_8$; $[x_5,x_{17}] = x_9$; $[x_5,x_{18}] = -x_{20}$; $[x_5,x_{21}] = x_{22}$; $[x_6,x_{11}] = -x_{14}$; $[x_6,x_{15}] = -x_8$; $[x_6,x_{17}] = x_{10}$; $[x_6,x_{18}] = 2x_{11}$; $[x_6,x_{21}] = \frac{1}{2}x_{23}$; $[x_6,x_{24}] = \frac{1}{2}x_9$; $[x_7,x_{18}] = 2x_{12}$; $[x_7,x_{20}] = -2x_{14}$; $[x_7,x_{24}] = x_{10}$; $[x_8,x_{18}] = 2x_{13}$; $[x_8,x_{19}] = 2x_{14}$; $[x_8,x_{21}] = -x_{10}$; $[x_9,x_{17}] = -2x_{13}$; $[x_9,x_{21}] = -x_{11}$; $[x_9,x_{23}] = 2x_{14}$; $[x_{10},x_{21}] = -x_{12}$; $[x_{10},x_{22}] = -x_{14}$; $[x_{10},x_{24}] = -x_{13}$; $[x_{11},x_{15}] = -x_{13}$; $[x_{15},x_{19}] = 2x_{11}$; $[x_{15},x_{21}] = \frac{1}{2}x_{17}$; $[x_{15},x_{22}] = \frac{1}{2}x_9$; $[x_{15},x_{23}] = -x_{10}$; $[x_{16},x_{19}] = 2x_{12}$; $[x_{16},x_{20}] = 2x_{13}$; $[x_{16},x_{22}] = x_{10}$; $[x_{17},x_{22}] = x_{11}$; $[x_{17},x_{23}] = 2x_{12}$; $[x_{21},x_{22}] = \frac{1}{2}x_{19}$; $[x_{21},x_{24}] = \frac{1}{2}x_{18}$; $[x_{22},x_{24}] = -\frac{1}{2}x_{20}$; $[x_{23},x_{24}] = x_{11}$.

Then the computation of all invariant bilinear forms on F_4^+ with the computer algebra system *Reduce* yields the conclusion that F_4^+ is $\mathcal{I}$-null.

Case E_6. In the case of E_6^+ the set Δ_+ of positive roots (associated to the set S of simple roots) has cardinality 36 ([6], p. 333):

$$\Delta_+ = \{\varepsilon_i + \varepsilon_j; 1 \leqslant i < j \leqslant 5\} \cup \{\varepsilon_i - \varepsilon_j; 1 \leqslant j < i \leqslant 5\}$$
$$\cup \{\frac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4 \pm \varepsilon_5 + \sqrt{3}\,\varepsilon_6); \#\text{ minus signs even}\}$$

(the (ε_j)'s an orthogonal basis of the Euclidean space). Instead of computing the commutation relations, we will use the following property $(\mathcal{P})$ of Δ_+.

$$(\mathcal{P}):\ \text{for } \alpha,\beta,\gamma \in \Delta_+ ,\ \text{if } \alpha+\beta \in \Delta_+ \text{ and } \alpha+\gamma \in \Delta_+ ,\ \text{then } \beta+\gamma \notin \Delta_+.$$

Introduce some Chevalley basis ([7], p. 19 ex. 7) of E_6^+ : $(X_\alpha)_{\alpha\in\Delta^+}$. One has

$$[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}\ \forall \alpha,\beta \in \Delta_+$$

$$N_{\alpha,\beta} = 0 \text{ if } \alpha+\beta \notin \Delta_+, N_{\alpha,\beta} \in \mathbb{Z}\setminus\{0\} \text{ if } \alpha+\beta\in\Delta_+.$$

Define inductively a sequence $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_{36} = E_6^+$ of $\mathcal{I}$-null subalgebras, each of which a codimension 1 ideal of the following, as follows. Start with $\mathfrak{g}_1 = \mathbb{C}X_{\delta_1}$, $\delta_1 \in \Delta_+$ of maximum height. Suppose $\mathfrak{g}_i$ defined. Then take $\mathfrak{g}_{i+1} = \mathbb{C}X_{\delta_{i+1}} \oplus \mathfrak{g}_i$ with $\delta_{i+1} \in \Delta_+ \setminus \{\delta_1,\dots,\delta_i\}$ of maximum height. Clearly, $\mathfrak{g}_i$ is a codimension 1 ideal of $\mathfrak{g}_{i+1}$. To prove that it is $\mathcal{I}$-null we only have to check that, for $1 \leqslant s,t \leqslant i$, if $\delta_{i+1}+\delta_s \in \Delta_+$ and $\delta_{i+1}+\delta_t\in\Delta_+$ then $\delta_s+\delta_t\notin\Delta_+$. That holds true because of property $(\mathcal{P})$.

Case E_7. In the case of E_7^+ the set Δ_+ of positive roots has cardinality 63 ([6], p. 333):

$$\Delta_+ = \{\varepsilon_i+\varepsilon_j; 1\leqslant i<j\leqslant 6\} \cup \{\varepsilon_i-\varepsilon_j; 1\leqslant j<i\leqslant 6\}\cup\{\sqrt{2}\varepsilon_7\}$$
$$\cup\{\frac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4\pm\varepsilon_5\pm\varepsilon_6+\sqrt{2}\,\varepsilon_7); \#\text{ minus signs odd}\}.$$

Property $(\mathcal{P})$ holds true for E_7^+ (see [14]). Hence the conclusion follows as in the case of E_6^+.

Case E_8. In the case of E_8^+ the set Δ_+ of positive roots has cardinality 120 ([6], p. 333):

$$\Delta_+ = \{\varepsilon_i+\varepsilon_j; 1\leqslant i<j\leqslant 8\} \cup \{\varepsilon_i-\varepsilon_j; 1\leqslant j<i\leqslant 8\}$$
$$\cup\{\frac{1}{2}(\pm\varepsilon_1\pm\varepsilon_2\pm\varepsilon_3\pm\varepsilon_4\pm\varepsilon_5\pm\varepsilon_6\pm\varepsilon_7+\varepsilon_8); \#\text{ minus signs even}\}.$$

Property $(\mathcal{P})$ holds true for E_8^+ (see [14]). Hence the conclusion follows as in the case of E_6^+. □

Remark 3.2. Property $(\mathcal{P})$ holds for A_n^+, hence we could have used it. However, it does not hold for F_4^+. One has for example in the above commutation relations of F_4^+ (with root vectors) $[x_3,x_4]\neq 0, [x_3,x_9]\neq 0$, yet $[x_4,x_9]\neq 0$.

Remark 3.3. In the *transversal to dimension* approach to the classification problem of nilpotent Lie algebras initiated in [18], one first associates a generalized Cartan matrix (abbr. GCM) A to any nilpotent finite dimensional complex Lie algebra $\mathfrak{g}$, and then looks at $\mathfrak{g}$ as the quotient $\hat{\mathfrak{g}}(A)_+/\mathfrak{I}$ of the nilradical of the Borel subalgebra of the Kac-Moody Lie algebra $\hat{\mathfrak{g}}(A)$ associated to A by some ideal $\mathfrak{I}$. Then one gets for any GCM A the subproblem of classifying (up to the action of a certain group) all ideals of $\hat{\mathfrak{g}}(A)_+$, thus getting all nilpotent Lie algebras of type A (see [2], [3], [4], [19], and the references therein). Any indecomposable GCM is of exactly one of the 3 types *finite*, *affine*, *indefinite* (among that last the hyperbolic GCMs, with the property that any connected proper subdiagram of the Dynkin diagram is of finite or affine type) ([1],[8],[20]). From Theorem 3.1, the nilpotent Lie algebras that are not $\mathcal{I}$-null all come from affine or indefinite types. Unfortunately, that is the case of many nilpotent Lie algebras, see Table 2. Finally, let us add some indications on how Table 2 was computed. The commutation relations for the nilpotent Lie algebras $\mathfrak{g}$ in Table 2 are given in [12], [13] in terms of a basis $(x_j)_{1\leqslant j\leqslant n}$ ($n = \dim \mathfrak{g}$) which diagonalizes a maximal torus T. We may suppose here that $(x_j)_{1\leqslant j\leqslant \ell}$, $\ell = \dim(\mathfrak{g}/\mathcal{C}^2\mathfrak{g})$, is a basis for $\mathfrak{g}$ modulo $\mathcal{C}^2\mathfrak{g}$. The associated weight pattern $R(T)$ and weight spaces decomposition $\mathfrak{g} = \bigoplus_{\beta\in R(T)} \mathfrak{g}^\beta$ appear in [13]. As in [18], one first introduces $R_1(T) = \{\beta \in R(T); \mathfrak{g}^\beta \not\subset \mathcal{C}^2\mathfrak{g}\} = \{\beta_1,\ldots,\beta_s\}$, $\ell_a = \dim\left(\mathfrak{g}^{\beta_a}/\left(\mathfrak{g}^{\beta_a}\cap\mathcal{C}^2\mathfrak{g}\right)\right)$, $d_a = \dim \mathfrak{g}^{\beta_a}$ $(1\leqslant a\leqslant s)$. By definition the GCM associated to $\mathfrak{g}$ is $A = (a^i_j)_{1\leqslant i,j\leqslant \ell}$ with $a^i_i = 2$ and, for $i\neq j$, $-a^i_j$ defined as follows. In the simplest case where $d_a = 1\,\forall a$ $(1\leqslant a\leqslant s)$, then, for $i\neq j$, $-a^i_j$ is the lowest $k\in\mathbb{N}$ such that $ad(x_i)^{k+1}(x_j) = 0$. If $d_a > 1$ for some $1\leqslant a\leqslant s$ (Lie algebras having that property are signalled by a ‡ in Table 2), one has (if $l_a > 1$ as well) to reorder $x_1,\ldots,x_\ell$ according to weights as $y_1,\ldots,y_\ell$ with y_j of weight $\beta_{f(j)}$, $f : \{1,\ldots,s\}\to\{1,\ldots,s\}$ some step function. Then, for $i\neq j$, $-a^i_j = \inf\{k\in\mathbb{N}; ad(v)^{k+1}(w) = 0\,\forall v\in\mathfrak{g}^{\beta_{f(i)}}\,\forall w\in\mathfrak{g}^{\beta_{f(j)}}\}$. The GCM A is an invariant of $\mathfrak{g}$, up to permutations of $\{\beta_1,\ldots,\beta_s\}$ that leave the d_β's invariant. The type of the GCM was identified either directly or through the associated Dynkin diagram. As an example to Table 2, there are (up to isomorphism) three 7-dimensional nilpotent Lie algebras that can be constructed from the GCM $D_4^{(3)}$: $\mathfrak{g}_{7,2.1(ii)}$, $\mathfrak{g}_{7,2.10}$, $\mathfrak{g}_{7,3.2}$. The 7-dimensional nilpotent Lie algebra $D_{4,42}^{(3),0}$ constructed from the GCM $D_4^{(3)}$ in [3] is isomorphic to $\mathfrak{g}_{7,3.2}$.

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Table 2. Kac-Moody types for indecomposable nilpotent Lie algebras of dimension ≤ 7. Notations for indefinite hyperbolic are those of [20], supplemented in parentheses for rank $3, 4$ by the notations of [1] (as there are misprints and omissions in [20]).

algebra	**GCM**	Finite	Affine	Indefinite Hyperbolic	Indefinite Not Hyperbolic
$\mathfrak{g}_3$	$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$	A_2			
$\mathfrak{g}_4$	$\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$	C_2			
$\mathfrak{g}_{5,1}$	$\begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$	$A_2 \times A_2$			
$\mathfrak{g}_{5,2}$	$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$	A_3			
$\mathfrak{g}_{5,3}$	$\begin{pmatrix} 2 & -2 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	B_3			
$\mathfrak{g}_{5,4}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$		$A_1^{(1)}$		
$\mathfrak{g}_{5,5}$	$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$	G_2			
$\mathfrak{g}_{5,6}$	$\begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$			$(3, 2)$	
$\mathfrak{g}_{6,1}$	$\begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$	A_4			
$\mathfrak{g}_{6,2}$	$\begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$	$B_2 \times A_2$			
$\mathfrak{g}_{6,3}$	$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$		$A_2^{(1)}$		
$\mathfrak{g}_{6,4}$	$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$	B_3			
$\mathfrak{g}_{6,5}$ ‡	$\begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_2^{(3)}$ (32)	
$\mathfrak{g}_{6,6}$	$\begin{pmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$	C_3			
$\mathfrak{g}_{6,7}$	$\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_1^{(3)}$ (1)	
$\mathfrak{g}_{6,8}$	$\begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$			$H_{96}^{(3)}$ (103)	
$\mathfrak{g}_{6,9}$	$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$	A_3			
$\mathfrak{g}_{6,10}$	$\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$		$A_4^{(2)}$		
$\mathfrak{g}_{6,11}$	$\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$		$G_2^{(1)}$		
$\mathfrak{g}_{6,12}$ ‡	$\begin{pmatrix} 2 & -3 & -2 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$				√
$\mathfrak{g}_{6,13}$	$\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$		$G_2^{(1)}$		
$\mathfrak{g}_{6,14}$	$\begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$			$(3, 2)$	
$\mathfrak{g}_{6,15}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$		$A_1^{(1)}$		
$\mathfrak{g}_{6,16}$	$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$		$A_2^{(2)}$		
$\mathfrak{g}_{6,17}$	$\begin{pmatrix} 2 & -4 \\ -2 & 2 \end{pmatrix}$			$(4, 2)$	

Table 2. continued

algebra	**GCM**	Finite	Affine	Indefinite Hyperbolic	Indefinite Not Hyperbolic
$\mathfrak{g}_{6,18}$	$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$	G_2			
$\mathfrak{g}_{6,19}$	$\begin{pmatrix} 2 & -4 \\ -2 & 2 \end{pmatrix}$			(4,2)	
$\mathfrak{g}_{6,20}$	$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$			$(3,3)$	
$\mathfrak{g}_{7,0.1}{}^{\ddagger}$	$\begin{pmatrix} 2 & -5 \\ -5 & 2 \end{pmatrix}$			$(5,5)$	
$\mathfrak{g}_{7,0.2}{}^{\ddagger}$	ditto			ditto	
$\mathfrak{g}_{7,0.3}{}^{\ddagger}$	ditto			ditto	
$\mathfrak{g}_{7,0.4(\lambda)}{}^{\ddagger}$	$\begin{pmatrix} 2 & -4 \\ -4 & 2 \end{pmatrix}$			$(4,4)$	
$\mathfrak{g}_{7,0.5}{}^{\ddagger}$	ditto			ditto	
$\mathfrak{g}_{7,0.6}{}^{\ddagger}$	$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$			$(3,3)$	
$\mathfrak{g}_{7,0.7}{}^{\ddagger}$	ditto			ditto	
$\mathfrak{g}_{7,0.8}{}^{\ddagger}$	$\begin{pmatrix} 2 & -3 & -3 \\ -3 & 2 & -3 \\ -3 & -3 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.01(i)}{}^{\ddagger}$	$\begin{pmatrix} 2 & 0 & -4 \\ 0 & 2 & -4 \\ -1 & -1 & 2 \end{pmatrix}$			$H_{123}^{(3)}$ (123)	
$\mathfrak{g}_{7,1.01(ii)}{}^{\ddagger}$	ditto			ditto	
$\mathfrak{g}_{7,1.02}{}^{\ddagger}$	$\begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}$			$(3,2)$	
$\mathfrak{g}_{7,1.03}{}^{\ddagger}$	$\begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$			$(3,2)$	
$\mathfrak{g}_{7,1.1(i_\lambda)}$ $\lambda \neq 0$	$\begin{pmatrix} 2 & -5 \\ -3 & 2 \end{pmatrix}$			$(5,3)$	
$\mathfrak{g}_{7,1.1(i_\lambda)}$ $\lambda = 0$	$\begin{pmatrix} 2 & -5 \\ -2 & 2 \end{pmatrix}$			$(5,2)$	
$\mathfrak{g}_{7,1.1(ii)}$	$\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}$			$(5,1)$	
$\mathfrak{g}_{7,1.1(iii)}$	$\begin{pmatrix} 2 & -4 \\ -3 & 2 \end{pmatrix}$			$(4,3)$	
$\mathfrak{g}_{7,1.1(iv)}$	$\begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}$			$(3,2)$	
$\mathfrak{g}_{7,1.1(v)}$	$\begin{pmatrix} 2 & 0 & -4 \\ 0 & 2 & -2 \\ -2 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.1(vi)}$	$\begin{pmatrix} 2 & -3 & -1 \\ -3 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.2(i_\lambda)}{}^{\ddagger}$	$\begin{pmatrix} 2 & -3 & -2 \\ -3 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.2(ii)}{}^{\ddagger}$	ditto				ditto
$\mathfrak{g}_{7,1.2(iii)}{}^{\ddagger}$	ditto				ditto
$\mathfrak{g}_{7,1.2(iv)}{}^{\ddagger}$	ditto				ditto

TABLE 2. continued

algebra	**GCM**	Finite	Affine	Indefinite Hyperbolic	Indefinite Not Hyperbolic
$\mathfrak{g}_{7,1.3(i_\lambda)}{}^{\ddagger}$	$\begin{pmatrix} 2 & -3 & -3 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.3(ii)}{}^{\ddagger}$	ditto				ditto
$\mathfrak{g}_{7,1.3(iii)}{}^{\ddagger}$	$\begin{pmatrix} 2 & -3 & -3 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.3(iv)}{}^{\ddagger}$	$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}$			$H_{18}^{(3)}$ (40)	
$\mathfrak{g}_{7,1.3(v)}{}^{\ddagger}$	$\begin{pmatrix} 2 & -3 & -3 & -2 \\ -2 & 2 & -1 & -1 \\ -2 & -1 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.4}$	$\begin{pmatrix} 2 & -5 \\ -2 & 2 \end{pmatrix}$			$(5,2)$	
$\mathfrak{g}_{7,1.5}$	$\begin{pmatrix} 2 & -4 \\ -2 & 2 \end{pmatrix}$			$(4,2)$	
$\mathfrak{g}_{7,1.6}$	$\begin{pmatrix} 2 & -5 \\ -2 & 2 \end{pmatrix}$			$(5,2)$	
$\mathfrak{g}_{7,1.7}$	$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_{8}^{(3)}$ (34)	
$\mathfrak{g}_{7,1.8}$	$\begin{pmatrix} 2 & -3 & 0 \\ -2 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.9}$	$\begin{pmatrix} 2 & -3 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.10}$	$\begin{pmatrix} 2 & -4 \\ -3 & 2 \end{pmatrix}$			$(4,3)$	
$\mathfrak{g}_{7,1.11}{}^{\ddagger}$	$\begin{pmatrix} 2 & -4 & -3 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.12}{}^{\ddagger}$	$\begin{pmatrix} 2 & -4 & -2 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.13}$	$\begin{pmatrix} 2 & -4 \\ -2 & 2 \end{pmatrix}$			$(4,2)$	
$\mathfrak{g}_{7,1.14}$	$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$			$(3,3)$	
$\mathfrak{g}_{7,1.15}{}^{\ddagger}$	$\begin{pmatrix} 2 & -4 & -3 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.16}{}^{\ddagger}$	$\begin{pmatrix} 2 & -3 & -2 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.17}{}^{\ddagger}$	$\begin{pmatrix} 2 & -4 \\ -4 & 2 \end{pmatrix}$			$(4,4)$	
$\mathfrak{g}_{7,1.18}{}^{\ddagger}$	$\begin{pmatrix} 2 & -3 & -2 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,1.19}{}^{\ddagger}$	$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}$			$H_{71}^{(3)}$ (80)	
$\mathfrak{g}_{7,1.20}$	$\begin{pmatrix} 2 & -1 & -2 \\ -3 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_{9}^{(3)}$ (6)	
$\mathfrak{g}_{7,1.21}{}^{\ddagger}$	$\begin{pmatrix} 2 & -3 & -2 \\ -3 & 2 & -2 \\ -1 & -1 & 2 \end{pmatrix}$				$\surd$

Table 2. continued

algebra	**GCM**	Finite	Affine	Indefinite Hyperbolic	Indefinite Not Hyperbolic
$\mathfrak{g}_{7,2.1(i_\lambda)}$	$\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_3^{(3)}$ (2)	
$\mathfrak{g}_{7,2.1(ii)}$	$\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$		$D_4^{(3)}$		
$\mathfrak{g}_{7,2.1(iii)}$	$\begin{pmatrix} 2 & -3 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$			$H_{27}^{(4)}$ (150)	
$\mathfrak{g}_{7,2.1(iv)}$	$\begin{pmatrix} 2 & 0 & -1 & -2 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$				$\checkmark$
$\mathfrak{g}_{7,2.1(v)}$	$\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_1^{(3)}$ (1)	
$\mathfrak{g}_{7,2.2}{}^{\ddagger}$	$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}$			$H_7^{(3)}$ (4)	
$\mathfrak{g}_{7,2.3}$	$\begin{pmatrix} 2 & -5 \\ -1 & 2 \end{pmatrix}$			$(5,1)$	
$\mathfrak{g}_{7,2.4}$	$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$		$A_2^{(2)}$		
$\mathfrak{g}_{7,2.5}$	$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$		$A_1^{(1)}$		
$\mathfrak{g}_{7,2.6}$	$\begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$			$(3,2)$	
$\mathfrak{g}_{7,2.7}$	$\begin{pmatrix} 2 & -4 \\ -2 & 2 \end{pmatrix}$			$(4,2)$	
$\mathfrak{g}_{7,2.8}$	$\begin{pmatrix} 2 & -3 \\ -2 & 2 \end{pmatrix}$			$(3,2)$	
$\mathfrak{g}_{7,2.9}$	$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$			$(3,3)$	
$\mathfrak{g}_{7,2.10}$	$\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$		$D_4^{(3)}$		
$\mathfrak{g}_{7,2.11}{}^{\ddagger}$	$\begin{pmatrix} 2 & -3 & -2 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$				$\checkmark$
$\mathfrak{g}_{7,2.12}{}^{\ddagger}$	$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$			$H_{109}^{(3)}$ (112)	
$\mathfrak{g}_{7,2.13}$	$\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$			$H_{100}^{(3)}$ (26)	
$\mathfrak{g}_{7,2.14}$	$\begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$			$H_{107}^{(3)}$ (111)	
$\mathfrak{g}_{7,2.15}$	$\begin{pmatrix} 2 & -4 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$			$H_{97}^{(3)}$ (104)	
$\mathfrak{g}_{7,2.16}$	ditto			ditto	
$\mathfrak{g}_{7,2.17}$	$\begin{pmatrix} 2 & -3 & 0 \\ -2 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$				$\checkmark$
$\mathfrak{g}_{7,2.18}$	ditto				ditto
$\mathfrak{g}_{7,2.19}$	$\begin{pmatrix} 2 & -3 & -1 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$				$\checkmark$
$\mathfrak{g}_{7,2.20}$	$\begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$				$\checkmark$
$\mathfrak{g}_{7,2.21}$	$\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_3^{(3)}$ (2)	
$\mathfrak{g}_{7,2.22}$	$\begin{pmatrix} 2 & 0 & -3 \\ 0 & 2 & -1 \\ -2 & -1 & 2 \end{pmatrix}$				$\checkmark$
$\mathfrak{g}_{7,2.23}$	$\begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$		$D_4^{(2)}$		

Table 2. continued

algebra	**GCM**	Finite	Affine	Indefinite Hyperbolic	Indefinite Not Hyperbolic
$\mathfrak{g}_{7,2.24}$	$\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$		$G_2^{(1)}$		
$\mathfrak{g}_{7,2.25}$ ‡	$\begin{pmatrix} 2 & -3 & 0 & -2 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,2.26}$ ‡	$\begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_2^{(3)}$ (32)	
$\mathfrak{g}_{7,2.27}$ ‡	$\begin{pmatrix} 2 & -2 & -1 & -1 \\ -2 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,2.28}$	$\begin{pmatrix} 2 & -1 & -2 & 0 \\ -2 & 2 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$			$H_{40}^{(4)}$ (164)	
$\mathfrak{g}_{7,2.29}$	$\begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,2.30}$	$\begin{pmatrix} 2 & -3 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$			$(3,2) \times A_2$	
$\mathfrak{g}_{7,2.31}$	$\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix}$			$H_{100}^{(3)}$ (26)	
$\mathfrak{g}_{7,2.32}$	$\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$			$H_{106}^{(3)}$ (25)	
$\mathfrak{g}_{7,2.33}$	$\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}$			$H_{105}^{(3)}$ (28)	
$\mathfrak{g}_{7,2.34}$	$\begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$			$H_{104}^{(3)}$ (107)	
$\mathfrak{g}_{7,2.35}$	$\begin{pmatrix} 2 & -1 & -2 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$		$A_4^{(2)}$		
$\mathfrak{g}_{7,2.36}$	$\begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -2 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$			$H_8^{(4)}$ (131)	
$\mathfrak{g}_{7,2.37}$ ‡	$\begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_2^{(3)}$ (32)	
$\mathfrak{g}_{7,2.38}$	$\begin{pmatrix} 2 & -2 & 0 & -1 \\ -2 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$				$\surd$
$\mathfrak{g}_{7,2.39}$	$\begin{pmatrix} 2 & -2 & -2 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_5^{(3)}$ (3)	
$\mathfrak{g}_{7,2.40}$	$\begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_2^{(3)}$ (32)	
$\mathfrak{g}_{7,2.41}$	$\begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$			$H_{99}^{(3)}$ (106)	
$\mathfrak{g}_{7,2.42}$	$\begin{pmatrix} 2 & -1 & -2 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_6^{(3)}$ (5)	
$\mathfrak{g}_{7,2.43}$	$\begin{pmatrix} 2 & -2 & -2 \\ -1 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$			$H_{99}^{(3)}$ (106)	
$\mathfrak{g}_{7,2.44}$	$\begin{pmatrix} 2 & -1 & -2 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_6^{(3)}$ (5)	
$\mathfrak{g}_{7,2.45}$	$\begin{pmatrix} 2 & -2 & 0 & -1 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix}$				$\surd$

Table 2. continued

algebra	**GCM**	Finite	Affine	Indefinite Hyperbolic	Indefinite Not Hyperbolic
$\mathfrak{g}_{7,3.1(i_\lambda)}$	$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$		$A_2^{(1)}$		
$\mathfrak{g}_{7,3.1(iii)}$	$\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$	D_4			
$\mathfrak{g}_{7,3.2}$	$\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$		$D_4^{(3)}$		
$\mathfrak{g}_{7,3.3}$	$\begin{pmatrix} 2 & -3 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$		$G_2^{(1)}$		
$\mathfrak{g}_{7,3.4}$	$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$		$D_3^{(2)}$		
$\mathfrak{g}_{7,3.5}$	$\begin{pmatrix} 2 & -1 & -1 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$	B_3			
$\mathfrak{g}_{7,3.6}$	$\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$			$H_1^{(3)}$ (1)	
$\mathfrak{g}_{7,3.7}$	$\begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$	B_4			
$\mathfrak{g}_{7,3.8}$	$\begin{pmatrix} 2 & -2 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$	F_4			
$\mathfrak{g}_{7,3.9}$	$\begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$		$B_3^{(1)}$		
$\mathfrak{g}_{7,3.10}$	$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$	C_4			
$\mathfrak{g}_{7,3.11}$	$\begin{pmatrix} 2 & -2 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$	F_4			
$\mathfrak{g}_{7,3.12}$	$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$		$A_3^{(1)}$		
$\mathfrak{g}_{7,3.13}$	$\begin{pmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$		$A_1^{(1)} \times A_2$		
$\mathfrak{g}_{7,3.14}$	$\begin{pmatrix} 2 & -1 & -2 & 0 \\ -1 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$	C_4			
$\mathfrak{g}_{7,3.15}$	$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -2 & 2 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$	B_4			
$\mathfrak{g}_{7,3.16}$	$\begin{pmatrix} 2 & -2 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}$	$B_2 \times B_2$			
$\mathfrak{g}_{7,3.17}$	$\begin{pmatrix} 2 & -3 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$	$G_2 \times A_2$			
$\mathfrak{g}_{7,3.18}$	$\begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 & 2 \end{pmatrix}$	$B_3 \times A_2$			
$\mathfrak{g}_{7,3.19}$	$\begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$	A_5			

Table 2. continued

algebra	**GCM**	Finite	Affine	Indefinite Hyperbolic	Indefinite Not Hyperbolic
$\mathfrak{g}_{7,3.20}$	$\begin{pmatrix} 2 & -2 & -2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$		$C_2^{(1)}$		
$\mathfrak{g}_{7,3.21}$	$\begin{pmatrix} 2 & -2 & -1 \\ -1 & 2 & 0 \\ -2 & 0 & 2 \end{pmatrix}$		$A_4^{(2)}$		
$\mathfrak{g}_{7,3.22}$	$\begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$	C_3			
$\mathfrak{g}_{7,3.23}$	$\begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$			$H_{96}^{(3)}$ (103)	
$\mathfrak{g}_{7,3.24}$	$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$			$H_3^{(4)}$ (126)	
$\mathfrak{g}_{7,4.1}$	$\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$	A_4			
$\mathfrak{g}_{7,4.2}$	$\begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$	D_4			
$\mathfrak{g}_{7,4.3}$	$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$	$A_2 \times A_3$			
$\mathfrak{g}_{7,4.4}$	$\begin{pmatrix} 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$	$A_2 \times A_2 \times A_2$			