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AN ELEMENTARY PROOF OF THE CONTINUITY FROM $L^2_0(\Omega)$ TO $H^1_0(\Omega)^n$ OF BOGOVSKII'S RIGHT INVERSE OF THE DIVERGENCE

RICARDO G. DURÁN

ABSTRACT. The existence of right inverses of the divergence as an operator from $H^1_0(\Omega)^n$ to $L^2_0(\Omega)$ is a problem that has been widely studied because of its importance in the analysis of the classic equations of fluid dynamics. When Ω is a bounded domain which is star-shaped with respect to a ball B, a right inverse given by an integral operator was introduced by Bogovskii, who also proved its continuity using the Calderón-Zygmund theory of singular integrals.

In this paper we give an alternative elementary proof of the continuity using the Fourier transform. As a consequence, we obtain estimates for the constant in the continuity in terms of the ratio between the diameter of Ω and that of B. Moreover, using the relation between the existence of right inverses of the divergence with the Korn and improved Poincaré inequalities, we obtain estimates for the constants in these two inequalities. We also show that one can proceed in the opposite way, that is, the existence of a continuous right inverse of the divergence, as well as estimates for the constant in that continuity, can be obtained from the improved Poincaré inequality. We give an interesting example of this situation in the case of convex domains.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Given a smooth vector field $\mathbf{u}$ defined in Ω we will denote with $D\mathbf{u}$ its differential matrix, namely,

$$D\mathbf{u} = \left(\frac{\partial u_i}{\partial x_j}\right)$$

and for a tensor field (a_{ij}) we define its norm by

$$\|a\|^2_{L^2(\Omega)} = \sum_{i,j=1}^{n} \|a_{ij}\|^2_{L^2(\Omega)}.$$

The existence of solutions $\mathbf{u} \in H^1_0(\Omega)^n$ of

$$\operatorname{div} \mathbf{u} = f \tag{1.1}$$

satisfying

$$\|D\mathbf{u}\|_{L^2(\Omega)} \leq C_{div,\Omega} \|f\|_{L^2(\Omega)}, \tag{1.2}$$

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where $f \in L^2_0(\Omega) := \{f \in L^2(\Omega) : \int_\Omega f = 0\}$ and the constant $C_{div,\Omega}$ depends only on Ω, is a problem that has been widely analyzed because of its several applications and connections with other important results.

Assume that $\Omega \subset \mathbb{R}^n$ is a domain with diameter R which is star-shaped with respect to a ball $B \subset \Omega$, which we assume centered at the origin and of radius ρ. For a function $\omega \in C_0^\infty(B)$ such that $\int_\Omega \omega = 1$, a solution of (1.1) is given by

$$\mathbf{u}(x) = \int_\Omega G(x,y) f(y)\, dy \tag{1.3}$$

where $G = (G_1, \cdots, G_n)$ is defined by

$$G(x,y) = \int_0^1 \frac{(x-y)}{t} \omega\left(y + \frac{x-y}{t}\right) \frac{dt}{t^n} \quad .$$

Moreover, $\mathbf{u} \in H^1_0(\Omega)^n$ and (1.2) is satisfied.

This formula was introduced in [6] by Bogovskii who proved the estimate (1.2), as well as its generalization for L^p, $1 < p < \infty$, using the general Calderón-Zygmund theory of singular integrals developed in [7].

More recently, several papers have considered extensions and applications of this formula. In [10], a weighted version of (1.2), which is of interest in finite element analysis, was proved. In [1], an extension of Bogovskii's formula was introduced for the rather general class of John domains and the estimate (1.2) was proved using again the Calderón-Zygmund theory. Also, extensions of (1.2) for fractional order positive and negative Sobolev norms have been obtained in [8, 13].

The goal of this paper is twofold:

First, we want to give a simple proof of the estimate (1.2) for the solution given by (1.3) using elementary properties of the Fourier transform. In this way we avoid the use of the complicated general theory of singular integral operators. We believe that this can be interesting for teaching purposes.

Second, we are interested in obtaining some information on the constant in terms of the ratio R/ρ. As a byproduct, this result can be used to give estimates for the constants in some Korn and improved Poincaré inequalities.

The paper is organized in such a way that the reader interested only in the first part needs to read only up to the end of Section 2, which deals with the continuity of the singular integral operator. In Section 3 we modify the proof of the continuity in order to obtain a sharper estimate of the constant in (1.2). In Section 4 we obtain estimates for the constant in the so called second case of Korn inequality. Finally, Section 5 deals with the improved Poincaré inequality. First, we recall that the existence of solutions of (1.1) satisfying (1.2) can be proved by assuming the improved Poincaré inequality. Moreover, tracing constants in this proof, it is possible to obtain information for the constant in (1.2) from that in the improved Poincaré inequality. As an interesting example of this situation we show how sharp estimates for the constant in (1.2) can be obtained for the case of convex domains. On the other hand, we show that in some cases one can proceed the other way around, namely, it is possible to obtain estimates for the constant in the improved Poincaré inequality knowing estimates for the constant in (1.2). This is, for example, the case of planar star-shaped domains. Consequently, we obtain new estimates for the constant in the improved Poincaré inequality in this case.

2. Boundedness of the singular integral operator

In order to work with functions defined in $\mathbb{R}^n$ we extend f by zero outside of Ω in (1.3).

Let us recall the basic properties of the Fourier transform that we will need (see for example [24]). The Fourier transform is defined for $f \in L^1(\mathbb{R}^n)$ by

$$\widehat{f}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x)dx.$$

Here and in the rest of the paper, when we do not indicate the domain of integration it is understood that it is $\mathbb{R}^n$. The Fourier transform can be extended to f in the class of tempered distributions $\mathcal{S}'$, in particular, it is defined in $L^2(\mathbb{R}^n)$ and it is an isometry, i.e.,

$$\|\widehat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}.$$

We will use the well known equality

$$\widehat{\frac{\partial f}{\partial x_j}}(\xi) = 2\pi i \xi_j \widehat{f}(\xi).$$

The k-component of $\mathbf{u}$ is given by

$$u_k = u_{k,1} - u_{k,2},$$

where

$$u_{k,1}(x) = \int_0^1 \int \left(y_k + \frac{(x_k - y_k)}{t} \right) \omega \left(y + \frac{x-y}{t} \right) f(y)\, dy\, \frac{dt}{t^n}.$$

and

$$u_{k,2}(x) = \int_0^1 \int y_k \omega \left(y + \frac{x-y}{t} \right) f(y)\, dy\, \frac{dt}{t^n}.$$

These double integrals exist, if for example we assume that $f \in L^1(\mathbb{R}^n)$ and has compact support. Indeed, if $\operatorname{supp} f \subset B(0,M)$ then both integrands vanish unless $\left| y + \frac{x-y}{t} \right| < \rho$ and $|y| < M$, and so, assuming that $\rho < M$, we can restrict the domain of integration to $|x-y| < 2Mt$. Therefore, integrating first in the t variable, it follows that, for $i = 1, 2$,

$$|u_{k,i}(x)| \le C \int \frac{|f(y)|}{|x-y|^{n-1}}\, dy,$$

where the constant C depends only on ω, n, and M. Since $f \in L^1(\mathbb{R}^n)$ the last integral is finite for almost every x.

In order to take the derivatives of $u_{k,i}$ it is convenient to write

$$u_{k,1}(x) = \lim_{\varepsilon \to 0} \int_\varepsilon^1 \int \left(y_k + \frac{(x_k - y_k)}{t} \right) \omega \left(y + \frac{x-y}{t} \right) f(y)\, dy\, \frac{dt}{t^n},$$

and

$$u_{k,2}(x)=\lim_{\varepsilon\to 0}\int_\varepsilon^1\int y_k\omega\left(y+\frac{x-y}{t}\right)f(y)\,dy\,\frac{dt}{t^n},$$

where, as we will see, the limits exist in $\mathcal{S}'$. Consider the first integral and, to simplify notation, define $\varphi(x)=x_k\omega(x)$. Then, given $g\in\mathcal{S}$ we have to show that

$$\int\left(\int_\varepsilon^1\int\varphi\left(y+\frac{x-y}{t}\right)f(y)\,dy\,\frac{dt}{t^n}\right)g(x)\,dx$$
$$\to\int\left(\int_0^1\int\varphi\left(y+\frac{x-y}{t}\right)f(y)\,dy\,\frac{dt}{t^n}\right)g(x)\,dx$$

when $\varepsilon\to 0$. It is enough to see that

$$I_\varepsilon:=\int_0^\varepsilon\int\int\left|\varphi\left(y+\frac{x-y}{t}\right)\right|\,|f(y)|\,|g(x)|\,dx\,dy\,\frac{dt}{t^n}\to 0. \tag{2.1}$$

But, making the change of variable $z=\frac{x-y}{t}$ in the interior integral we have

$$I_\varepsilon=\int_0^\varepsilon\int\int|\varphi(y+z)|\,|f(y)|\,|g(y+tz)|\,dz\,dy\,dt\le\|g\|_{L^\infty(\mathbb{R}^n)}\|\varphi\|_{L^1(\mathbb{R}^n)}\|f\|_{L^1(\mathbb{R}^n)}\varepsilon$$

which proves (2.1). The integral defining $u_{k,2}$ can be treated in the same way, indeed, defining now $\varphi(x)=\omega(x)$, the only difference with the case of $u_{k,1}$ is the factor y_k appearing in the integrand, but it can be bounded assuming again that f has compact support.

Now, for $\varepsilon>0$ fixed, we can take the derivative inside the integral, and therefore,

$$\frac{\partial u_k}{\partial x_j}=T_{kj,1}f+T_{kj,2}(y_kf) \tag{2.2}$$

where $T_{kj,1}$ and $T_{kj,2}$ are of the form

$$Tf(x)=\lim_{\varepsilon\to 0}\int_\varepsilon^1\int\frac{\partial}{\partial x_j}\left[\varphi\left(y+\frac{x-y}{t}\right)\right]f(y)\,dy\,\frac{dt}{t^n} \tag{2.3}$$

with $\varphi(x)=x_k\omega(x)$ for $T_{kj,1}$ and $\varphi(x)=\omega(x)$ for $T_{kj,2}$.

We are going to prove continuity of operators of the form given in (2.3) where $\varphi\in C_0^\infty(B)$ with $B=B(0,\rho)$. With this goal we decompose the operator as

$$Tf=T_1f+T_2f$$

where

$$T_1f(x)=\lim_{\varepsilon\to 0}\int_\varepsilon^{\frac12}\int\frac{\partial}{\partial x_j}\left[\varphi\left(y+\frac{x-y}{t}\right)\right]f(y)\,dy\,\frac{dt}{t^n} \tag{2.4}$$

and

$$T_2f(x)=\int_{\frac12}^1\int\frac{\partial}{\partial x_j}\left[\varphi\left(y+\frac{x-y}{t}\right)\right]f(y)\,dy\,\frac{dt}{t^n}$$

An estimate of $\|T_2f\|_{L^2(\mathbb{R}^n)}$ for L^2-functions f vanishing outside Ω can be obtained easily as we show in the following lemma.

Lemma 2.1. *If $f \in L^2(\mathbb{R}^n)$ vanishes outside Ω then*

$$\|T_2 f\|_{L^2(\Omega)} \leq 2^n |\Omega| \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\Omega)}$$

Proof. We have

$$T_2 f(x) = \int \left\{ \int_{\frac{1}{2}}^{1} \left(\frac{\partial \varphi}{\partial x_j} \right) \left(y + \frac{x-y}{t} \right) \frac{dt}{t^{n+1}} \right\} f(y)\, dy. \tag{2.5}$$

Then,

$$|T_2 f(x)| \leq 2^n \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^\infty(\mathbb{R}^n)} \int_\Omega |f(y)|\, dy$$

and the result follows immediately using the Schwarz inequality. □

We now proceed to bound the operator T_1 in L^2. This will be done using the Fourier transform. By standard density arguments it is enough to bound the operator acting on f smooth enough. In the following lemma we give a simple form for T_1 in terms of Fourier transforms.

Lemma 2.2. *For $f \in C_0^\infty(\mathbb{R}^n)$ we have*

$$\widehat{T_1 f}(\xi) = 2\pi i \xi_j \int_0^{\frac{1}{2}} \widehat{\varphi}(t\xi) \widehat{f}((1-t)\xi)\, dt \tag{2.6}$$

Proof. From (2.4) we have

$$T_1 f = \lim_{\varepsilon \to 0} T_{1,\varepsilon} f,$$

where

$$T_{1,\varepsilon} f(x) = \int_\varepsilon^{\frac{1}{2}} \int \frac{\partial}{\partial x_j} \left[\varphi \left(y + \frac{x-y}{t} \right) \right] f(y)\, dy\, \frac{dt}{t^n}$$

and the limit is taken in $\mathcal{S}'$.

Now, we have

$$\widehat{T_{1,\varepsilon} f}(\xi) = \int \int_\varepsilon^{\frac{1}{2}} \int \frac{\partial}{\partial x_j} \left[\varphi \left(y + \frac{x-y}{t} \right) \right] f(y)\, e^{-2\pi i x \cdot \xi}\, dy\, \frac{dt}{t^n}\, dx,$$

and, since this triple integral exists, we can interchange the order of integration. Therefore, integrating by parts we obtain

$$\widehat{T_{1,\varepsilon} f}(\xi) = 2\pi i \xi_j \int_\varepsilon^{\frac{1}{2}} \int \int \varphi \left(y + \frac{x-y}{t} \right) f(y)\, e^{-2\pi i x \cdot \xi}\, dx\, dy\, \frac{dt}{t^n},$$

and making the change of variable

$$z = y + \frac{(x-y)}{t}$$

in the interior integral,

$$\widehat{T_{1,\varepsilon} f}(\xi) = 2\pi i \xi_j \int_\varepsilon^{\frac{1}{2}} \int \int \varphi(z)\, e^{-2\pi i (tz+(1-t)y)\cdot\xi}\, f(y) dz\, dy\, dt$$
$$= 2\pi i \xi_j \int_\varepsilon^{\frac{1}{2}} \int \int \widehat{\varphi}(t\xi)\, e^{-2\pi i (1-t) y\cdot\xi}\, f(y)\, dy\, dt,$$

and therefore,

$$\widehat{T_{1,\varepsilon} f}(\xi) = 2\pi i \xi_j \int_\varepsilon^{\frac{1}{2}} \widehat{\varphi}(t\xi) \widehat{f}((1-t)\xi)\, dt,$$

and taking $\varepsilon \to 0$ we conclude the proof. □

Using the expression given in (2.6) we will give an estimate for the operator T_1 in L^2. First we prove an auxiliary result.

Lemma 2.3. *Define* $C_{\varphi,\rho} = \rho^{-1} \|\varphi\|_{L^1(\mathbb{R}^n)} + \rho \left\| \frac{\partial^2 \varphi}{\partial x_j^2} \right\|_{L^1(\mathbb{R}^n)}$. *Then,*

$$2\pi |\xi_j| \int_0^\infty |\widehat{\varphi}(t\xi)|\, dt \le C_{\varphi,\rho}$$

Proof. We have

$$2\pi|\xi_j| \int_0^\infty |\widehat{\varphi}(t\xi)|\, dt = 2\pi|\xi_j| \int_0^{\frac{1}{2\pi\rho|\xi_j|}} |\widehat{\varphi}(t\xi)|\, dt + 2\pi|\xi_j| \int_{\frac{1}{2\pi\rho|\xi_j|}}^\infty |\widehat{\varphi}(t\xi)|\, dt := I + II$$

Now,

$$I \le \rho^{-1} \|\widehat{\varphi}\|_{L^\infty(\mathbb{R}^n)} \le \rho^{-1} \|\varphi\|_{L^1(\mathbb{R}^n)}$$

and

$$II = 2\pi \int_{\frac{1}{2\pi\rho|\xi_j|}}^\infty \frac{t^2|\xi_j|^2 |\widehat{\varphi}(t\xi)|}{t^2|\xi_j|}\, dt \le 2\pi \|\xi_j^2 \widehat{\varphi}\|_{L^\infty(\mathbb{R}^n)} \int_{\frac{1}{2\pi\rho|\xi_j|}}^\infty \frac{1}{t^2|\xi_j|}\, dt$$

but

$$-4\pi^2 \xi_j^2 \widehat{\varphi} = \widehat{\frac{\partial^2 \varphi}{\partial x_j^2}}$$

and therefore,

$$II \le \frac{1}{2\pi} \left\| \widehat{\frac{\partial^2 \varphi}{\partial x_j^2}} \right\|_{L^\infty(\mathbb{R}^n)} \int_{\frac{1}{2\pi\rho|\xi_j|}}^\infty \frac{1}{t^2|\xi_j|}\, dt \le \rho \left\| \frac{\partial^2 \varphi}{\partial x_j^2} \right\|_{L^1(\mathbb{R}^n)}$$

and the lemma is proved. □

As a consequence of this lemma we obtain the following estimate for the operator T_1.

Lemma 2.4. *If $C_{\varphi,\rho}$ is the constant defined in the previous lemma, then*

$$\|T_1 f\|_{L^2(\mathbb{R}^n)} \le 2^{\frac{n-1}{2}} C_{\varphi,\rho} \|f\|_{L^2(\mathbb{R}^n)}.$$

Proof. Applying the Schwarz inequality in (2.6) we have

$$|\widehat{T_1 f}(\xi)|^2 \leq \left(\int_0^{\frac{1}{2}} 2\pi|\xi_j||\widehat{\varphi}(t\xi)|\,dt\right)\left(\int_0^{\frac{1}{2}} 2\pi|\xi_j||\widehat{\varphi}(t\xi)||\widehat{f}((1-t)\xi)|^2\,dt\right)$$

and so, from Lemma 2.3,

$$|\widehat{T_1 f}(\xi)|^2 \leq C_{\varphi,\rho}\int_0^{\frac{1}{2}} 2\pi|\xi_j||\widehat{\varphi}(t\xi)||\widehat{f}((1-t)\xi)|^2\,dt$$

Then, integrating in ξ and making the change of variable $\eta = (1-t)\xi$, we obtain

$$\int |\widehat{T_1 f}(\xi)|^2\,d\xi \leq C_{\varphi,\rho}\int_0^{\frac{1}{2}}\int \frac{2\pi}{(1-t)^{n+1}}|\eta_j|\left|\widehat{\varphi}\left(\frac{t\eta}{1-t}\right)\right||\widehat{f}(\eta)|^2\,d\eta\,dt$$

and, integrating first in the variable t and making now the change $s = t/(1-t)$, we get

$$\int |\widehat{T_1 f}(\xi)|^2\,d\xi \leq 2^{n-1}C_{\varphi,\rho}\int\left(\int_0^1 2\pi|\eta_j||\widehat{\varphi}(s\eta)|\,ds\right)|\widehat{f}(\eta)|^2\,d\eta.$$

Therefore, applying again Lemma 2.3,

$$\int |\widehat{T_1 f}(\xi)|^2\,d\xi \leq 2^{n-1}C_{\varphi,\rho}^2\int |\widehat{f}(\eta)|^2\,d\eta,$$

and we conclude the proof recalling that the Fourier transform is an isometry in $L^2(\mathbb{R}^n)$. □

Summing up the lemmas we obtain the main result of this section.

Theorem 2.1. *If T is the operator given in (2.3) and f vanishes outside Ω, then*

$$\|Tf\|_{L^2(\Omega)} \leq C_{\varphi,\rho,\Omega}\|f\|_{L^2(\Omega)}$$

with

$$C_{\varphi,\rho,\Omega} = 2^{\frac{n-1}{2}}\rho^{-1}\|\varphi\|_{L^1(\mathbb{R}^n)} + 2^{\frac{n-1}{2}}\rho\left\|\frac{\partial^2\varphi}{\partial x_j^2}\right\|_{L^1(\mathbb{R}^n)} + 2^n|\Omega|\left\|\frac{\partial\varphi}{\partial x_j}\right\|_{L^\infty(\mathbb{R}^n)}.$$

3. Dependence of the constant on Ω

An interesting question is what can be said, in terms of the geometry of the domain Ω, about the behavior of the constant $C_{div,\Omega}$ in the estimate (1.2). Recall that we are assuming that the domain Ω has diameter R and that it is star-shaped with respect to a ball of radius ρ which, to simplify notation, we assume centered at the origin.

It is known that the constant cannot be bounded independently of the ratio R/ρ. Indeed, this can be seen by the following elementary example which also shows that, in some cases,

$$C_{div,\Omega} \geq c_1(R/\rho) \tag{3.1}$$

where c_1 is a constant independent of Ω.

Given positive numbers a and ε, consider the rectangular domain $\Omega_{a,\varepsilon} := (-a, a)\times(-\varepsilon, \varepsilon)$ and suppose that, for any $f \in L^2_0(\Omega_{a,\varepsilon})$, there exists $\mathbf{u} \in H^1_0(\Omega_{a,\varepsilon})$ solving (1.1) and satisfying the estimate (1.2) with a constant $C_{div,\Omega} = C_{a,\varepsilon}$. Take $f(x_1, x_2) = x_1$ and the corresponding solution $\mathbf{u}$, then

$$\|x_1\|^2_{L^2(\Omega_{a,\varepsilon})} = \int_{\Omega_{a,\varepsilon}} x_1 \operatorname{div}\mathbf{u} = -\int_{\Omega_{a,\varepsilon}} u_1 = \int_{\Omega_{a,\varepsilon}} x_2 \frac{\partial u_1}{\partial x_2}$$
$$\leq \|x_2\|_{L^2(\Omega_{a,\varepsilon})} \left\|\frac{\partial u_1}{\partial x_2}\right\|_{L^2(\Omega_{a,\varepsilon})} \leq C_{a,\varepsilon}\|x_2\|_{L^2(\Omega_{a,\varepsilon})}\|x_1\|_{L^2(\Omega_{a,\varepsilon})}$$

and so,

$$\|x_1\|_{L^2(\Omega_{a,\varepsilon})} \leq C_{a,\varepsilon}\|x_2\|_{L^2(\Omega_{a,\varepsilon})}$$

but,

$$\|x_1\|_{L^2(\Omega_{a,\varepsilon})} = \frac{2}{\sqrt{3}}\varepsilon^{\frac12}a^{\frac32} \qquad \text{and} \qquad \|x_2\|_{L^2(\Omega_{a,\varepsilon})} = \frac{2}{\sqrt{3}}\varepsilon^{\frac32}a^{\frac12}$$

and therefore,

$$C_{a,\varepsilon} \geq (a/\varepsilon).$$

Consequently, if $a > \varepsilon$, it follows that in this example (3.1) holds.

For the kind of domains that we are considering the following estimate for the constant $C_{div,\Omega}$ is given in [12]

$$C_{div,\Omega} \leq C_0(R/\rho)^{n+1}$$

with a constant C_0 independent of Ω. The reader can check that the result given in Theorem 2.1 recovers this estimate. However, as we will show, this result can be improved.

Indeed, Theorem 2.1 does not give a good estimate of the constant in terms of the function φ (or equivalently on ρ). Curiously, this is due to the estimate obtained in Lemma 2.1 for the operator T_2 which in some sense is easier to handle than T_1. Then, in order to obtain a sharper bound, we will give in the following lemmas a different argument to bound T_2.

Lemma 3.1. *If* $1 \leq p < \frac{n}{n-1}$ *then*

$$\|T_2 f\|_{L^p(\mathbb{R}^n)} \leq \frac{2^{\frac{n}{p'}}}{(1-\frac{n}{p'})}\left\|\frac{\partial\varphi}{\partial x_j}\right\|_{L^1(\mathbb{R}^n)}\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. From (2.5) we have

$$|T_2 f(x)| \leq \int_{\frac12}^1 \int \left|\left(\frac{\partial\varphi}{\partial x_j}\right)\left(y + \frac{x-y}{t}\right)\right| |f(y)|\, dy\, \frac{dt}{t^{n+1}}.$$

Making the change of variable

$$z = y + \frac{x-y}{t}$$

in the interior integral, we obtain

$$|T_2 f(x)| \leq 2 \int_{\frac{1}{2}}^{1} \int \left| \frac{\partial \varphi}{\partial x_j}(z) \right| \left| f\left(\frac{tz - x}{t-1} \right) \right| \frac{1}{(1-t)^n} \, dz \, dt.$$

Applying now the Minkowski inequality for integrals we have

$$\|T_2 f\|_{L^p(\mathbb{R}^n)} \leq 2 \int_{\frac{1}{2}}^{1} \int \left| \frac{\partial \varphi}{\partial x_j}(z) \right| \left(\int \left| f\left(\frac{tz - x}{t-1} \right) \right|^p dx \right)^{\frac{1}{p}} \frac{1}{(1-t)^n} \, dz \, dt$$

and, by the change of variable

$$\overline{x} = \frac{tz - x}{t - 1}$$

in the interior integral, it follows that

$$\|T_2 f\|_{L^p(\mathbb{R}^n)} \leq 2 \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^1(\mathbb{R}^n)} \|f\|_{L^p(\mathbb{R}^n)} \int_{\frac{1}{2}}^{1} \frac{1}{(1-t)^{\frac{n}{p'}}} \, dt$$

therefore, since $p' > n$, the integral on the right hand side of this inequality is finite and so we obtain the lemma. □

Unfortunately the restriction for the value of p in the previous lemma excludes the case $p = 2$. However, using well known interpolation theorems we can obtain an estimate for the L^2 case.

Lemma 3.2. *If $f \in L^2(\mathbb{R}^n)$ vanishes outside Ω then, for $1 \leq p < \frac{n}{n-1}$,*

$$\|T_2 f\|_{L^2(\Omega)} \leq \frac{2^{\frac{n}{2}}}{(1 - \frac{n}{p'})^{\frac{p}{2}}} |\Omega|^{1-\frac{p}{2}} \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^1(\mathbb{R}^n)}^{\frac{p}{2}} \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{p}{2}} \|f\|_{L^2(\Omega)}$$

Proof. From the definition of T_2 (2.5) it is easy to see that

$$\|T_2 f\|_{L^\infty(\Omega)} \leq 2^n |\Omega| \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^\infty(\Omega)}.$$

Then, the result follows immediately from this estimate together with Lemma 3.1 and the well known Riesz-Thorin interpolation theorem (see for example [14, p. 34]) which states that

$$\|T_2\|_{\mathcal{L}(L^2, L^2)} \leq \|T_2\|_{\mathcal{L}(L^p, L^p)}^{\frac{p}{2}} \|T_2\|_{\mathcal{L}(L^\infty, L^\infty)}^{1-\frac{p}{2}}. \qquad \square$$

Summing up we obtain the following estimate in terms of the function φ.

Theorem 3.1. *If T is the operator given in (2.3), f vanishes outside Ω, and $1 \leq p < \frac{n}{n-1}$, then*

$$\|Tf\|_{L^2(\Omega)} \leq \left(2^{\frac{n-1}{2}} C_{\varphi,\rho} + \frac{2^{\frac{n}{2}}}{(1 - \frac{n}{p'})^{\frac{p}{2}}} \widetilde{C}_{\varphi,p} |\Omega|^{1-\frac{p}{2}} \right) \|f\|_{L^2(\Omega)}$$

where

$$C_{\varphi,\rho} = \rho^{-1} \|\varphi\|_{L^1(\mathbb{R}^n)} + \rho \left\| \frac{\partial^2 \varphi}{\partial x_j^2} \right\|_{L^1(\mathbb{R}^n)}$$

and

$$\widetilde{C}_{\varphi,p} = \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^1(\mathbb{R}^n)}^{\frac{p}{2}} \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^\infty(\Omega)}^{1-\frac{p}{2}}$$

Proof. The result follows immediately from Lemmas 2.4 and 3.2. □

We want to bound $\|\frac{\partial u_k}{\partial x_j}\|_{L^2(\mathbb{R}^n)}$ using the expression (2.2). This is the goal of the following theorem.

In what follows C_n denotes a constant depending only on n, not necessarily the same at each occurrence, and $A \sim B$ means that A/B is bounded by above and below by positive constants which may depend on n and p only.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of diameter R which is star-shaped with respect to a ball $B \subset \Omega$ of radius ρ and $\mathbf{u} \in H_0^1(\Omega)$ be the solution of (1.1) given by (1.3). Then, there exists a constant C_n depending only on n such that*

$$\|D\mathbf{u}\|_{L^2(\Omega)} \le C_n \frac{R}{\rho} \left(\frac{|\Omega|}{|B|} \right)^{\frac{n-2}{2(n-1)}} \left(\log \frac{|\Omega|}{|B|} \right)^{\frac{n}{2(n-1)}} \|f\|_{L^2(\Omega)}$$

Proof. As we have mentioned, both operators on the right hand side of (2.2) are of the form given in (2.3). We will estimate the term $T_{kj,2}(y_k f)$ which is the worst part due to the presence of y_k. The reader can check that the term $T_{kj,1} f$ can be bounded analogously.

For $T_{kj,2}$ the function φ is exactly ω, which is supported in $B(0,\rho)$ and has integral equal to one. Therefore, φ can be taken as

$$\varphi(x) = \rho^{-n} \psi(\rho^{-1} x),$$

where ψ is a smooth function supported in the unit ball and with integral equal to one. Then,

$$\frac{\partial \varphi}{\partial x_j}(x) = \rho^{-n-1} \frac{\partial \psi}{\partial x_j}(\rho^{-1} x), \qquad \frac{\partial^2 \varphi}{\partial x_j^2}(x) = \rho^{-n-2} \frac{\partial^2 \psi}{\partial x_j^2}(\rho^{-1} x)$$

and so,

$$C_{\varphi,\rho} = \rho^{-1} \|\varphi\|_{L^1(\mathbb{R}^n)} + \rho \left\| \frac{\partial^2 \varphi}{\partial x_j^2} \right\|_{L^1(\mathbb{R}^n)} \sim \rho^{-1} \tag{3.2}$$

and

$$\widetilde{C}_{\varphi,p} = \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^1(\mathbb{R}^n)}^{\frac{p}{2}} \left\| \frac{\partial \varphi}{\partial x_j} \right\|_{L^\infty(\mathbb{R}^n)}^{1-\frac{p}{2}} \sim \rho^{-1-n(1-\frac{p}{2})}. \tag{3.3}$$

Therefore, applying Theorem 3.1 for $T = T_{kj,2}$, using $|y_k| \le R$ and the relations (3.2) and (3.3), we obtain, for $1 \le p < \frac{n}{n-1}$,

$$\begin{aligned} \|D\mathbf{u}\|_{L^2(\Omega)} &\le C_n \frac{R}{\rho} \frac{1}{(1-\frac{n}{p'})^{\frac{p}{2}}} \left(\frac{|\Omega|}{|B|} \right)^{1-\frac{p}{2}} \|f\|_{L^2(\Omega)} \\ &= C_n \frac{R}{\rho} \frac{1}{(1-\frac{n}{p'})^{\frac{p}{2}}} \left(\frac{|\Omega|}{|B|} \right)^{\frac{n-2}{2(n-1)}} \left(\frac{|\Omega|}{|B|} \right)^{\frac{1}{2}\left(\frac{n}{n-1}-p\right)} \|f\|_{L^2(\Omega)} \end{aligned}$$

Now, assuming that $\frac{|\Omega|}{|B|}$ is large enough, we can choose p such that

$$\frac{1}{2}\left(\frac{n}{n-1}-p\right)=\frac{1}{\log\frac{|\Omega|}{|B|}}$$

obtaining

$$\|D\mathbf{u}\|_{L^2(\Omega)}\leq C_n\,\frac{R}{\rho}\,\frac{1}{(1-\frac{n}{p'})^{\frac{p}{2}}}\left(\frac{|\Omega|}{|B|}\right)^{\frac{n-2}{2(n-1)}}e\|f\|_{L^2(\Omega)},$$

and so, we conclude the proof using that

$$1-\frac{n}{p'}=\frac{(n-1)}{p}\left(\frac{n}{n-1}-p\right)=\frac{2(n-1)}{p\log\left(\frac{|\Omega|}{|B|}\right)}$$

and $p<\frac{n}{n-1}$. □

Remark 3.1. *In the particular case $n=2$ the theorem gives*

$$\|D\mathbf{u}\|_{L^2(\Omega)}\leq C\left(\frac{R}{\rho}\right)\log\left(\frac{R}{\rho}\right)\|f\|_{L^2(\Omega)}$$

In view of the example given above this estimate is almost optimal (i.e., optimal up to the logarithmic factor).

4. The Korn inequality

As it is well known, Korn type inequalities are strongly connected with the existence of solutions of (1.1) satisfying (1.2). For example, in the particular case of two dimensional simple connected domains with a C^1 boundary, the explicit relation between the best constant in (1.2) and that in the so-called second case of Korn inequality was given in [17]. More generally, for arbitrary domains in n dimensions, $n\geq 2$, the Korn inequality can be derived from the existence of solutions of the divergence satisfying (1.2), and therefore, information on the constant in the Korn inequality can be obtained from estimates for the constant in (1.2).

A lot of work has been done in order to obtain the behavior of the constant in the different versions of Korn inequality in terms of the domain (see [16] and its references).

We are going to show how our results in the previous section can be used to obtain estimates for the constant in the second case of Korn inequality. Let us mention that domains which are star-shaped with respect to a ball were considered by Kondratiev and Oleinik in [20, 21] where the authors obtain sharp estimates for the constant in a Korn inequality in terms of R/ρ. However, their results are for a different type of Korn inequality than the one that we are considering and it is not clear what is the relation between the constants in the two different Korn type inequalities.

For a vector field $\mathbf{v}\in H^1(\Omega)^n$, $\varepsilon(\mathbf{v})$ and $\mu(\mathbf{v})$ denote the symmetric and skew symmetric part of $D\mathbf{v}$ respectively, i.e.,

$$\varepsilon_{ij}(\mathbf{v})=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)$$

and

$$\mu_{ij}(\mathbf{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

Then, the so-called second case of Korn inequality states that there exists a constant $C_{K,\Omega}$ such that

$$\|D\mathbf{v}\|_{L^2(\Omega)} \le C_{K,\Omega}\|\varepsilon(\mathbf{v})\|_{L^2(\Omega)}$$

for vector fields $\mathbf{v} \in H^1(\Omega)^n$ satisfying

$$\int_\Omega \mu_{ij}(\mathbf{v}) = 0\,, \qquad \text{for} \quad i,j = 1,\dots,n. \tag{4.1}$$

The argument used in the proof of the following theorem is known but we include it for the sake of completeness. For an arbitrary domain Ω we will say that it admits a right inverse of the divergence with constant $C_{div,\Omega}$ if, for any $f \in L^2_0(\Omega)$, there exists $\mathbf{u} \in H^1_0(\Omega)^n$ satisfying

$$\operatorname{div}\mathbf{u} = f$$

and

$$\|D\mathbf{u}\|_{L^2(\Omega)} \le C_{div,\Omega}\|f\|_{L^2(\Omega)}.$$

Theorem 4.1. *If Ω admits a right inverse of the divergence with constant $C_{div,\Omega}$, then the second case of Korn inequality holds in Ω with a constant $C_{K,\Omega}$ which satisfies*

$$C_{K,\Omega} \le (1+4n^2)^{1/2}C_{div,\Omega}$$

Proof. Let $\mathbf{v} \in H^1(\Omega)^n$ such that (4.1) holds. By density we can assume that $\mathbf{v}$ is smooth. By orthogonality we have

$$\|D\mathbf{v}\|^2_{L^2(\Omega)} = \|\varepsilon(\mathbf{v})\|^2_{L^2(\Omega)} + \|\mu(\mathbf{v})\|^2_{L^2(\Omega)}$$

and so, observing that $C_{div,\Omega} \ge 1$, it is enough to prove that

$$\|\mu(\mathbf{v})\|^2_{L^2(\Omega)} \le 4n^2C^2_{div,\Omega}\|\varepsilon(\mathbf{v})\|^2_{L^2(\Omega)}. \tag{4.2}$$

Given i and j, since

$$\int_\Omega \mu_{ij}(\mathbf{v}) = 0,$$

there exists $\mathbf{u}^{ij} \in H^1_0(\Omega)^n$ such that

$$\operatorname{div}\mathbf{u}^{ij} = \mu_{ij}(\mathbf{v})$$

and

$$\left\|D\mathbf{u}^{ij}\right\|_{L^2(\Omega)} \le C_{div,\Omega}\|\mu_{ij}(\mathbf{v})\|_{L^2(\Omega)}. \tag{4.3}$$

Then,

$$\|\mu_{ij}(\mathbf{v})\|^2_{L^2(\Omega)} = \int_\Omega \mu_{ij}(\mathbf{v})\operatorname{div}\mathbf{u}^{ij} = -\int_\Omega \nabla\mu_{ij}(\mathbf{v})\cdot\mathbf{u}^{ij}$$

but,

$$\frac{\partial\mu_{ij}(\mathbf{v})}{\partial x_k} = \left(\frac{\partial\varepsilon_{ik}(\mathbf{v})}{\partial x_j} - \frac{\partial\varepsilon_{jk}(\mathbf{v})}{\partial x_i}\right)$$

and so,

$$\begin{aligned}\|\mu_{ij}(\mathbf{v})\|^2_{L^2(\Omega)} &= -\sum_{k=1}^n \int_\Omega \left(\frac{\partial \varepsilon_{ik}(\mathbf{v})}{\partial x_j} - \frac{\partial \varepsilon_{jk}(\mathbf{v})}{\partial x_i}\right) u_k^{ij} \\ &= \sum_{k=1}^n \int_\Omega \left(\varepsilon_{ik}(\mathbf{v})\frac{\partial u_k^{ij}}{\partial x_j} - \varepsilon_{jk}(\mathbf{v})\frac{\partial u_k^{ij}}{\partial x_i}\right),\end{aligned}$$

and using now (4.3) we obtain,

$$\|\mu_{ij}(\mathbf{v})\|^2_{L^2(\Omega)} \le C_{div,\Omega}\|\mu_{ij}(\mathbf{v})\|_{L^2(\Omega)} \sum_{k=1}^n \left(\|\varepsilon_{ik}(\mathbf{v})\|_{L^2(\Omega)} + \|\varepsilon_{jk}(\mathbf{v})\|_{L^2(\Omega)}\right).$$

Therefore,

$$\|\mu_{ij}(\mathbf{v})\|_{L^2(\Omega)} \le C_{div,\Omega}\, n^{1/2} \left\{\sum_{k=1}^n \left(\|\varepsilon_{ik}(\mathbf{v})\|_{L^2(\Omega)} + \|\varepsilon_{jk}(\mathbf{v})\|_{L^2(\Omega)}\right)^2\right\}^{1/2}$$

and then

$$\|\mu_{ij}(\mathbf{v})\|^2_{L^2(\Omega)} \le 2C^2_{div,\Omega}\, n \sum_{k=1}^n \left(\|\varepsilon_{ik}(\mathbf{v})\|^2_{L^2(\Omega)} + \|\varepsilon_{jk}(\mathbf{v})\|^2_{L^2(\Omega)}\right).$$

Finally, summing now in i and j we obtain (4.2). □

Consequently, using the results of the previous section we obtain an estimate for the Korn inequality in star-shaped domains.

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of diameter R which is star-shaped with respect to a ball $B \subset \Omega$ of radius ρ. Then, there exists a constant C_n depending only on n such that, for all $\mathbf{v} \in H^1(\Omega)^n$ satisfying $\int_\Omega \mu_{ij}(\mathbf{v}) = 0$, for $i, j = 1, \dots, n$,*

$$\|D\mathbf{v}\|_{L^2(\Omega)} \le C_n \frac{R}{\rho}\left(\frac{|\Omega|}{|B|}\right)^{\frac{n-2}{2(n-1)}}\left(\log\frac{|\Omega|}{|B|}\right)^{\frac{n}{2(n-1)}} \|\varepsilon(\mathbf{v})\|_{L^2(\Omega)}$$

Proof. The result follows immediately from Theorems 3.2 and 4.1. □

5. The improved Poincaré inequality

In this section we consider another well known result usually called improved Poincaré inequality. To recall this inequality we need to introduce some notation. For a bounded domain $\Omega \subset \mathbb{R}^n$ and any $x \in \Omega$ we denote with $d(x)$ the distance from x to the boundary of Ω. Then, the improved Poincaré inequality states that there exists a constant $C_{iP,\Omega}$ such that, for any $f \in H^1(\Omega) \cap L^2_0(\Omega)$,

$$\|f\|_{L^2(\Omega)} \le C_{iP,\Omega}\|d\nabla f\|_{L^2(\Omega)}. \tag{5.1}$$

It is known that this inequality is valid for Lipschitz domains and, more generally, for John domains (see for example [5, 9, 18]).

For the star-shaped domains that we are considering in this paper, the argument given in [9], applied in this particular case, can be used to show that

$$C_{iP,\Omega} \leq C_n (R/\rho)^{n+1}; \tag{5.2}$$

indeed, this was done in [4, Prop. 5.2] for the analogous inequality in L^1, but it is easy to see that the arguments extend straightforward to the L^2 case. We are going to show that the dependence on R/ρ can be improved using our estimates of Section 3, at least in the two dimensional case.

In [11] the relation between Poincaré type inequalities and solutions of the divergence was analyzed in a very general context. A particular case of the results in that paper says that the improved Poincaré inequality (5.1) implies the existence of a right inverse of the divergence as an operator from $H_0^1(\Omega)^n$ to $L_0^2(\Omega)$. The interest of this result is that it allows us to obtain information on the constant in (1.2) from that for the constant in the improved Poincaré inequality. We will give an example of this situation for the case of convex domains.

Let us reproduce the argument given in [11] for the sake of completeness. With this purpose we need to use a Whitney decomposition of Ω, i.e., a sequence of cubes $\{Q_j\}$ with pairwise disjoint interiors and such that, if d_j and ℓ_j are the distance of Q_j to the boundary of Ω and the length of its edges respectively, then d_j/ℓ_j is bounded by above and below by positive constants depending only on n. Associated with this decomposition there is a partition of unity $\{\phi_j\}$, namely, $\sum_j \phi_j = 1$ in Ω, with $\phi_j \in C_0^\infty(\widetilde{Q}_j)$ where $\widetilde{Q}_j$ is an expansion of Q_j still with diameter proportional to its distance to the boundary of Ω. A Whitney decomposition exists for any domain (see for example [24] for a proof).

Lemma 5.1. *If the improved Poincaré inequality (5.1) is satisfied in Ω then, given $f \in L_0^2(\Omega)$ and a Whitney decomposition of Ω, there exists a sequence $\{f_j\}$ such that $f_j \in L_0^2(\widetilde{Q}_j)$, $f = \sum_j f_j$, and*

$$\|f\|^2_{L^2(\Omega)} \leq C_n \sum_j \|f_j\|^2_{L^2(\widetilde{Q}_j)} \tag{5.3}$$

and

$$\sum_j \|f_j\|^2_{L^2(\widetilde{Q}_j)} \leq C_n (1 + C_{iP,\Omega}) \|f\|^2_{L^2(\Omega)}. \tag{5.4}$$

Proof. First we observe that, by duality, (5.1) implies that, for all $f \in L_0^2(\Omega)$, there exists $\mathbf{v} \in L^2(\Omega)^n$ such that

$$\operatorname{div} \mathbf{v} = f \quad \text{in } \Omega\ , \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \tag{5.5}$$

and

$$\left\| \frac{\mathbf{v}}{d} \right\|_{L^2(\Omega)} \leq C_{iP,\Omega} \|f\|_{L^2(\Omega)}, \tag{5.6}$$

where both equations in (5.5) have to be understood in a distributional sense.

Indeed,

$$L(\nabla g) = \int_\Omega f g$$

defines a linear form on the subspace of $L^2(\Omega)^n$ formed by the gradient vector fields. L is well defined because $\int_\Omega f = 0$. Moreover, it follows from (5.1) that

$$|L(\nabla g)| = \left| \int_\Omega f(g - \overline{g}) \right| \leq C_{iP,\Omega} \|f\|_{L^2(\Omega)} \|d\nabla g\|_{L^2(\Omega)}$$

where $\overline{g}$ is the average of g in Ω.

By the Hahn-Banach theorem L can be extended as a linear continuous functional to the space $L^2_d(\Omega)^n$, where $L^2_d(\Omega)$ denotes the Hilbert space with norm $\|f\|_{L^2_d} := \|df\|_{L^2}$, and therefore, there exists $\mathbf{v} \in L^2_{d^{-1}}(\Omega)^n$ satisfying (5.6) and such that

$$L(\mathbf{w}) = \int_\Omega \mathbf{v} \cdot \mathbf{w} \qquad \forall\, \mathbf{w} \in L^2_d(\Omega)^n;$$

in particular,

$$\int_\Omega \mathbf{v} \cdot \nabla g = \int fg \quad \forall g \in H^1(\Omega)$$

which is equivalent to (5.5).

Given now $f \in L^2_0(\Omega)$, let $\mathbf{v} \in L^2(\Omega)^n$ satisfying (5.5) and (5.6), and define

$$f_j = \operatorname{div}(\phi_j \mathbf{v}).$$

Then, we have

$$f = \operatorname{div} \mathbf{v} = \operatorname{div}\Big(\mathbf{v} \sum_j \phi_j\Big) = \sum_j \operatorname{div}(\phi_j \mathbf{v}) = \sum_j f_j.$$

Since $\operatorname{supp} \phi_j \subset \widetilde{Q}_j$ we have $\operatorname{supp} f_j \subset \widetilde{Q}_j$ and $\int f_j = 0$.

Moreover, using the finite superposition (with constant depending only on n) of the expanded cubes $\widetilde{Q}_j$, we obtain immediately (5.3). On the other hand, using again the finite superposition and that $\|\phi_j\|_{L^\infty} \leq 1$ and $\|\nabla \phi_j\|_{L^\infty} \leq C/d_j$, we have

$$\|f_j\|^2_{L^2(\widetilde{Q}_j)} \leq C_n \left\{ \|f\|^2_{L^2(\widetilde{Q}_j)} + \left\| \frac{\mathbf{v}}{d} \right\|^2_{L^2(\widetilde{Q}_j)} \right\},$$

and therefore, (5.4) follows from (5.6). $\square$

Theorem 5.1. *If the improved Poincaré inequality (5.1) is satisfied in Ω, then Ω admits a right inverse of the divergence with constant $C_{div,\Omega}$ which satisfies*

$$C_{div,\Omega} \leq C_n(1 + C_{iP,\Omega}). \tag{5.7}$$

Proof. Given $f \in L^2_0(\Omega)$ let f_j be the functions given in the previous lemma. Since $f_j \in L^2_0(\widetilde{Q}_j)$, there exists $\mathbf{u}_j \in H^1_0(\widetilde{Q}_j)^n$ such that

$$\operatorname{div} \mathbf{u}_j = f_j \quad \text{and} \quad \|D\mathbf{u}_j\|_{L^2(\widetilde{Q}_j)} \leq C_n \|f_j\|_{L^2(\widetilde{Q}_j)},$$

indeed, a scaling argument shows that the constant in this inequality is independent of the size of the cube. Then, $\mathbf{u} = \sum_j \mathbf{u}_j \in H^1_0(\Omega)^n$ is a solution of $\operatorname{div} \mathbf{u} = f$. Moreover, it follows from (5.4) that

$$\|D\mathbf{u}\|_{L^2(\Omega)} \leq C_n(1 + C_{iP,\Omega}) \|f\|_{L^2(\Omega)}$$

and the theorem is proved. $\square$

Let us now consider the case of convex domains. For these domains it was proved in [4, Cor. 5.3] that, for $f \in W^{1,1}(\Omega)$,

$$\|f - f_{\widetilde{S}}\|_{L^1(\Omega)} \leq C_n \frac{R}{\rho} \|d\nabla f\|_{L^1(\Omega)}, \tag{5.8}$$

where C_n is a constant depending only on n and $f_{\widetilde{S}}$ is the average of f over a particular subset of Ω. From this inequality it is easy to see that, for $f \in W^{1,1}(\Omega)$ with vanishing mean value,

$$\|f\|_{L^1(\Omega)} \leq 2C_n \frac{R}{\rho} \|d\nabla f\|_{L^1(\Omega)}. \tag{5.9}$$

Indeed, if $\int_\Omega f = 0$, we have

$$\|f_{\widetilde{S}}\|_{L^1(\Omega)} = \Big| \int_\Omega (f_{\widetilde{S}} - f) \Big| \leq \|f - f_{\widetilde{S}}\|_{L^1(\Omega)},$$

and therefore,

$$\|f\|_{L^1(\Omega)} \leq \|f - f_{\widetilde{S}}\|_{L^1(\Omega)} + \|f_{\widetilde{S}}\|_{L^1(\Omega)} \leq 2\|f - f_{\widetilde{S}}\|_{L^1(\Omega)}$$

which together with (5.8) implies (5.9).

It is known that (5.9) implies the improved Poincaré in L^2. In fact, this was proved in a more general context in [11, Prop. 3.3]. In the following lemma we reproduce, for a particular case, the argument given in that paper but tracing constants in the proof in order to obtain an explicit dependence of the constant in the L^2 estimate in terms of the constant in (5.9).

Lemma 5.2. *Let Ω be an arbitrary bounded domain. If there exists a constant C_1 such that, for any $f \in W^{1,1}(\Omega)$ with $\int_\Omega f = 0$,*

$$\|f\|_{L^1(\Omega)} \leq C_1 \|d\nabla f\|_{L^1(\Omega)}, \tag{5.10}$$

then, for any $f \in H^1(\Omega) \cap L^2_0(\Omega)$,

$$\|f\|_{L^2(\Omega)} \leq 6\sqrt{2}\, C_1 \|d\nabla f\|_{L^2(\Omega)}. \tag{5.11}$$

Proof. The first step in the proof given in [11] is to show that for any measurable $E \subset \Omega$ such that $|E| \geq |\Omega|/2$ and any $f \in W^{1,1}(\Omega)$ which vanishes on E, it follows from (5.10) that

$$\|f\|_{L^1(\Omega)} \leq 3\, C_1 \|d\nabla f\|_{L^1(\Omega)}. \tag{5.12}$$

Indeed, calling f_Ω the average of f over Ω and using that f vanishes on E, we have

$$|f_\Omega| = \frac{1}{|E|} \Big| \int_E (f_\Omega - f) \Big| \leq \frac{1}{|E|} \int_E |f_\Omega - f|,$$

and therefore,

$$\|f_\Omega\|_{L^1(\Omega)} \leq \frac{|\Omega|}{|E|} \|f_\Omega - f\|_{L^1(\Omega)}.$$

Then,

$$\|f\|_{L^1(\Omega)} \leq \|f - f_\Omega\|_{L^1(\Omega)} + \|f_\Omega\|_{L^1(\Omega)} \leq \|f - f_\Omega\|_{L^1(\Omega)} + \frac{|\Omega|}{|E|} \|f_\Omega - f\|_{L^1(\Omega)}$$

and using that $|E| \geq |\Omega|/2$ we obtain

$$\|f\|_{L^1(\Omega)} \leq 3\|f - f_\Omega\|_{L^1(\Omega)},$$

which together with (5.10) applied to $f - f_\Omega$ gives (5.12).

Now, for $E \subset \Omega$ as above and $f \in H^1(\Omega)$ vanishing on E we can apply (5.12) for f^2 to obtain,

$$\int_\Omega f^2 \leq 3\,C_1 \int_\Omega d|\nabla f^2| \leq 6\,C_1 \int_\Omega d|f||\nabla f| \leq 6\,C_1\|f\|_{L^2(\Omega)}\|d\nabla f\|_{L^2(\Omega)},$$

and therefore,

$$\|f\|_{L^2(\Omega)} \leq 6\,C_1\|d\nabla f\|_{L^2(\Omega)}. \tag{5.13}$$

For a function f we define f_+ as the function which agrees with f where $f \geq 0$ and is equal to zero otherwise, and $f_- := f_+ - f$. Now, given $f \in H^1(\Omega)$, it is easy to see by continuity arguments, that there exists $\lambda \in \mathbb{R}$ such that

$$\int_\Omega (f-\lambda)_+^2 = \int_\Omega (f-\lambda)_-^2. \tag{5.14}$$

On the other hand one of the two functions $(f-\lambda)_+$ or $(f-\lambda)_-$ vanishes in a set E such that $|E| \geq |\Omega|/2$, suppose that it is $(f-\lambda)_+$ (if not we apply the same argument to the other function). Then, using (5.13) applied to $(f-\lambda)_+$ we obtain

$$\|(f-\lambda)_+\|_{L^2(\Omega)} \leq 6\,C_1\|d\nabla f\|_{L^2(\Omega)},$$

but, in view of (5.14) we have

$$\int_\Omega (f-\lambda)^2 = \int_\Omega (f-\lambda)_+^2 + \int_\Omega (f-\lambda)_-^2 = 2\int_\Omega (f-\lambda)_+^2 \leq 72\,C_1^2\|d\nabla f\|_{L^2(\Omega)}^2,$$

and the proof concludes by recalling that, for $f \in L_0^2(\Omega)$,

$$\|f\|_{L^2(\Omega)} \leq \|f - \lambda\|_{L^2(\Omega)}. \qquad \square$$

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded convex domain of diameter R which contains a ball B of radius ρ. Then, given $f \in L_0^2(\Omega)$, there exists a solution $\mathbf{u} \in H_0^1(\Omega)^n$ of (1.1) such that*

$$\|D\mathbf{u}\|_{L^2(\Omega)} \leq C_n \frac{R}{\rho}\|f\|_{L^2(\Omega)},$$

where C_n is a constant which depends only on n.

Proof. The result follows immediately from Theorem 5.1, inequality (5.9), and Lemma 5.2. $\square$

Now, a natural question is whether the converse of Theorem 5.1 is true, namely, if the improved Poincaré inequality can be proved assuming the existence of continuous right inverses of the divergence. To the author's knowledge this is not known. However, a weaker result will allow us to obtain an estimate for the constant in the improved Poincaré inequality for planar star-shaped domains using the results of Section 3. In fact, we will see that the converse can be proved if we assume that the following inequality is satisfied in Ω,

$$\left\|\frac{g}{d}\right\|_{L^2(\Omega)} \leq C_{H,\Omega}\|\nabla g\|_{L^2(\Omega)} \qquad \forall\, g \in H_0^1(\Omega). \tag{5.15}$$

This is one of the many results called “Hardy inequality” although, at least to the author’s knowledge, Hardy proved only the one dimensional case. It is known that this inequality is valid for a very large class of domains (see for example [15, 19, 22, 23]).

Theorem 5.3. *If Ω admits a right inverse of the divergence with constant $C_{div,\Omega}$ and the Hardy inequality (5.15) is satisfied in Ω, then the improved Poincaré inequality (5.1) is valid in Ω with a constant $C_{iP,\Omega}$ such that*

$$C_{iP,\Omega} \le C_{H,\Omega} C_{div,\Omega}. \tag{5.16}$$

Proof. Given $f \in H^1(\Omega) \cap L^2_0(\Omega)$ let $\mathbf{u} \in H^1_0(\Omega)^n$ be such that

$$\operatorname{div} \mathbf{u} = f \quad \text{and} \quad \|D\mathbf{u}\|_{L^2(\Omega)} \le C_{div,\Omega} \|f\|_{L^2(\Omega)}. \tag{5.17}$$

Then,

$$\begin{aligned} \|f\|^2_{L^2(\Omega)} = \int_\Omega f \operatorname{div} \mathbf{u} = -\int_\Omega \nabla f \cdot \mathbf{u} &\le \|d\nabla f\|_{L^2(\Omega)} \left\| \frac{\mathbf{u}}{d} \right\|_{L^2(\Omega)} \\ &\le C_{H,\Omega} \|d\nabla f\|_{L^2(\Omega)} \|D\mathbf{u}\|_{L^2(\Omega)} \end{aligned}$$

and using (5.17) we conclude the proof. □

In order to apply this theorem together with our results of Section 3 we need to know estimates for $C_{H,\Omega}$. For example, for simply connected (in particular for star-shaped) planar domains it has been proved that

$$C_{H,\Omega} \le 4; \tag{5.18}$$

see [2, 3].

Therefore, using this estimate and the results of Section 3, we obtain an estimate for the constant C_{iP} which improves (5.2).

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of diameter R which is star-shaped with respect to a ball $B \subset \Omega$ of radius ρ. Then, there exists a positive constant C such that, for all $f \in H^1(\Omega) \cap L^2_0(\Omega)$, we have*

$$\|f\|_{L^2(\Omega)} \le C \left(\frac{R}{\rho}\right) \log\left(\frac{R}{\rho}\right) \|d\nabla f\|_{L^2(\Omega)}.$$

Proof. The result follows immediately from Theorems 3.2 and 5.3 and inequality (5.18). □

To finish the paper let us mention that the bound given in the previous theorem is almost optimal. Indeed, in view of Theorem 5.1, the same example given in Section 3 shows that in some cases $C_{iP} \ge c_1(R/\rho)$, where c_1 is a constant.

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Ricardo G. Durán
Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
and IMAS, CONICET,
Buenos Aires, Argentina.
rduran@dm.uba.ar

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