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THE N_0^1-MATRIX COMPLETION PROBLEM

GU-FANG MOU AND TING-ZHU HUANG

ABSTRACT. An $n \times n$ matrix is called an N_0^1-matrix if all its principal minors are non-positive and each entry is non-positive. In this paper, we study general combinatorially symmetric partial N_0^1-matrix completion problems and prove that a combinatorially symmetric partial N_0^1-matrix with all specified off-diagonal entries negative has an N_0^1-matrix completion if the graph of its specified entries is an undirected cycle or a 1-chordal graph.

1. INTRODUCTION

An $n \times n$ real matrix is called an N_0^1-*matrix* if all its principal minors are non-positive and each entry is non-positive (see [1]). Obviously, the diagonal entries of an N_0^1-matrix are non-positive. An $n \times n$ real matrix is called an N_0-matrix if all its principal minors are non-positive. N_0-matrices arise in multivariate analysis [18], in linear complementary problems [16, 17], in the theory of global univalence of functions [15] and in completion problems [20]. The following simple facts are very useful in the study of N_0^1-matrices:

Proposition 1.1. *Let A be an N_0^1-matrix. Then*
(1) If P is a permutation matrix, then PAP^T is an N_0^1-matrix;
(2) If D is a positive diagonal matrix, then DA, DA is an N_0^1-matrix;
(3) Any principal submatrix of A is an N_0^1-matrix.

By Proposition 1.1, the set of N_0^1-matrices is closed under permutation similarity and left and right positive diagonal multiplication. In this paper, we may take all diagonal entries to be -1 or 0.

The submatrix of an $n \times n$ matrix A, lying in rows α and columns β, α, $\beta \subseteq \{1, 2, \ldots, n\}$, is denoted by $A[\alpha|\beta]$, and the principal submatrix $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$. Therefore, a real matrix A, of size $n \times n$, is an N_0^1-matrix only if $\det A[\alpha] \leq 0$, for all $\alpha \subseteq \{1, 2, \ldots, n\}$ and each entry is non-positive.

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A *partial matrix* is an array in which some entries are specified, while others are free to be chosen from a certain set. A partial matrix is said to be a *partial N_0^1-matrix* if every completely specified principal submatrix with each entry non-positive is an N_0^1-matrix. A *matrix completion problem* asks whether a partial matrix has a completion to a conventional matrix that has a desired property. Matrix completion problems have been studied for many classes of matrices, such as P-matrices [3, 4, 5], P_0-matrices [6, 7], M-matrices [8], inverse M-matrices [8, 9, 10], N-matrices [11, 12, 13] and N_0-matrices [20]. Applications of matrix completion problems arise in situations where some data are known but other data are not available, and it is known that the full data matrix must have a certain property.

An $n \times n$ partial matrix $A = (a_{ij})$ it said to be *combinatorially symmetric* when a_{ij} is specified if and only if a_{ji} is, and *non-combinatorially symmetric* in other cases. For a combinatorially symmetric partial matrix, all main diagonal entries are specified. A natural way to describe an $n \times n$ combinatorially symmetric partial matrix is via a graph $G_A = (V, E)$, where the set of vertices V is $\{1, 2, \ldots, n\}$ and $\{i, j\}$, $i \neq j$, is an edge or arc when the (i, j) entry is specified. In general, an undirected graph is associated with a combinatorially symmetric partial matrix, and when the partial matrix is non-combinatorially symmetric, a directed graph is used. The graph theoretic techniques used to study matrix completion problems have been discussed by Hogben [14]. In this paper, we will work on combinatorially symmetric partial matrices with undirected graphs.

In this paper, we study general combinatorially symmetric partial N_0^1-matrix completion problems and prove that a combinatorially symmetric partial N_0^1-matrix with all specified off-diagonal entries negative has an N_0^1-matrix completion if the graph of its specified entries is an undirected cycle or a 1-chordal graph. In [20] the authors study the N_0-matrix completion problem, and they show that a combinatorially symmetric partial N_0-matrix, with no null main diagonal entries and all (i, j) specified entry $\mathrm{sign}(a_{ij}) = (-1)^{i+j+1}$, has an N_0-matrix completion. Our interest here is in the N_0^1-matrix completion problem, that is does a partial N_0^1-matrix have an N_0^1- matrix completion? The study of this problem is different from the previous one since some main diagonal entries can be zero and each specified off-diagonal entry is negative.

Throughout the paper we denote the entries of a partial matrix A as follows: the entry d_i denotes a specified diagonal entry, a_{ij} denotes a specified off-diagonal entry, and the entry x_{ij} an unspecified entry, $1 \leq i, j \leq n$. The entry c_{ij} denotes a value assigned to the unspecified entry x_{ij} during the process of completing a partial matrix. A_c is the completion of the partial matrix A.

2. Preliminary results

Proposition 2.1. *Every 2×2 partial N_0^1-matrix with all specified off-diagonal entries negative has an N_0^1-matrix completion.*

Proof. Let A be a combinatorially symmetric partial N_0^1-matrix with all diagonal entries specified.

We denote by K the number of unspecified entries of A. Consider the following cases:

(a) $K = 1$

Using (1) of Proposition 1.1, we can assume that A has either the form

$$A = \begin{pmatrix} -a_{11} & -x_{12} \\ -a_{21} & -a_{22} \end{pmatrix}, \quad \text{with } a_{11}, a_{22} \geq 0, a_{21} > 0,$$

or the form

$$A = \begin{pmatrix} -x_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}, \quad \text{with } a_{12}, a_{21} > 0, a_{22} \geq 0.$$

In the first case, it suffices to consider the completion of A is

$$A_c = \begin{pmatrix} -a_{11} & -c_{12} \\ -a_{21} & -a_{22} \end{pmatrix}.$$

If we choose $c_{12} \geq a_{11}a_{22}/a_{21}$, then $\det A_c \leq 0$.

In the second case, consider a completion

$$A_c = \begin{pmatrix} -c_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{pmatrix}$$

of A with $a_{22}c_{11} \leq a_{12}a_{21}$. There are two cases to consider.

Case 1: If a_{22} >0, then $0 \leq c_{11} \leq a_{12}a_{21}/a_{22}$.

Case 2: If a_{22} =0, then $c_{11} \geq 0$.

(b) $1 < K < 4$

In this case, we can complete some entries of matrix A in order to obtain a partial N_0^1-matrix with exactly one unspecified entry and, then, use case (a). □

Proposition 2.2. *Every 3×3 combinatorially symmetric partial N_0^1-matrix with all specified off-diagonal entries negative has N_0^1-matrix completion.*

Proof. Let A be a combinatorially symmetric partial N_0^1-matrix with all diagonal entries specified. We denote by K the number of unspecified entries of A. Consider the following cases: Firstly, let us consider the case in which A has exactly $K = 2$. By Proposition 1.1, we can assume, without loss of generality, that A has the form:

$$A = \begin{pmatrix} -d_1 & -x_{12} & -a_{13} \\ -x_{21} & -d_2 & -a_{23} \\ -a_{31} & -a_{32} & -d_3 \end{pmatrix}$$

with each a_{ij} positive and with each d_i nonnegative. Our goal is to prove the existence of c_{12}, $c_{21} > 0$, such that the completion

$$A_c = \begin{pmatrix} -d_1 & -c_{12} & -a_{13} \\ -c_{21} & -d_2 & -a_{23} \\ -a_{31} & -a_{32} & -d_3 \end{pmatrix}$$

of A.

We will show that $\det A_c[\alpha] \leq 0$ for any $\alpha \subseteq \{1, 2, 3\}$.

Choose $c_{12} > 0$, $c_{21} > 0$ and large enough, then

$$\det A_c[\{1, 2\}] = d_1 d_2 - c_{12}c_{21} \leq 0,$$

$$\begin{aligned}\det A_c &= -d_1d_2d_3 + d_1a_{23}a_{32} + d_3c_{21}c_{12} - a_{13}a_{32}c_{21} - a_{23}a_{31}c_{12} + d_2a_{13}a_{31} \\ &\quad + c_{12}c_{21} \\ &= -d_3(d_1d_2 - c_{12}c_{21}) - c_{12}a_{23}a_{32} - c_{21}a_{13}a_{32} + d_2a_{13}a_{31} + d_1a_{23}a_{32} \\ &\le 0.\end{aligned}$$

So, it is always possible to choose c_{12}, $c_{21} \ge 0$ such that $\det A_c[\{1,2\}] \le 0$ and $\det A_c \le 0$.

In cases $K = 4$ and $K = 6$, we can complete some entries of matrix A in a way to obtain a partial N_0^1-matrix with exactly two unspecified entries, then the formulation of these problems reduces to that of $K = 2$. $\square$

Remark 2.3. *For a 3×3 combinatorially symmetric partial N_0^1-matrix, the condition of all specified off-diagonal entries negative is sufficient but not necessary to obtain N_0^1-matrix completion, as the following examples show.*

Example 2.4. *The matrix*

$$A = \begin{pmatrix} -1 & -1 & -x_{13} \\ -1 & 0 & 0 \\ -x_{31} & 0 & -1 \end{pmatrix}$$

is a partial N_0^1-matrix, but A does not have N_0^1-completion since $\det A = 1$ *for all* $x_{13}, x_{31} > 0$.

Example 2.5. *The matrix*

$$A = \begin{pmatrix} 0 & -1 & -x_{13} \\ 0 & 0 & 0 \\ -x_{31} & 0 & -1 \end{pmatrix}$$

is a partial N_0^1-matrix that A has an N_0^1-matrix completion, since $\det A[\{1,3\}] = 0$, $\det A = 0$ *for all* $x_{13}, x_{31} \ge 0$.

3. Cycles

A path is a sequence of edges $\{i_1, i_2\}$, $\{i_2, i_3\}, \ldots, \{i_{k-1}, i_k\}$ in which all vertices are distinct. A cycle is a closed path, that is, a path in which the first and the last vertices coincide.

In the section we show the existence of N_0^1-matrix completions of an $n \times n$ partial N_0^1-matrix A, whose associated graph (of specified entries) is a n-cycle.

Lemma 3.1. *Let A be a 4×4 combinatorially symmetric partial N_0^1-matrix with all specified off-diagonal entries negative, whose associated graph is a 4-cycle. Then, there exists an N_0^1-matrix completion.*

Proof. Without loss of generality, we may assume

$$A = \begin{pmatrix} -d_1 & -a_{12} & -x_{13} & -a_{14} \\ -a_{21} & -d_2 & -a_{23} & -x_{24} \\ -x_{31} & -a_{32} & -d_3 & -a_{34} \\ -a_{41} & -x_{42} & -a_{43} & -d_4 \end{pmatrix},$$

where each a_{ij} is positive and each d_i is nonnegative.

Our aim is to prove the existence of nonnegative c_{13}, c_{24}, c_{31} and c_{42} such that the completion

$$A_c = \begin{pmatrix} -d_1 & -a_{12} & -c_{13} & -a_{14} \\ -a_{21} & -d_2 & -a_{23} & -c_{24} \\ -c_{31} & -a_{32} & -d_3 & -a_{34} \\ -a_{41} & -c_{42} & -a_{43} & -d_4 \end{pmatrix}$$

is N_0^1. By Proposition 1.1, we can assume, without loss of generality, that $c_{31} = a_{32} > 0$, $c_{24} = a_{41} > 0$, $a_{21} = 1$.

We are going to prove that there exist nonnegative values for c_{13}, c_{24}, such that

$$A_c = \begin{pmatrix} -d_1 & -a_{12} & -c_{13} & -a_{14} \\ -1 & -d_2 & -a_{23} & -c_{24} \\ -a_{32} & -a_{32} & -d_3 & -a_{34} \\ -a_{41} & -a_{41} & -a_{43} & -d_4 \end{pmatrix}$$

is an N_0^1-matrix. By Proposition 1.1, we have five different cases to consider:

Case 1: $d_1 = d_2 = d_3 = d_4 = 1$, with $a_{12} \geq 1$, $a_{23}a_{32} \geq 1$, $a_{34}a_{43} \geq 1$ and $a_{14}a_{41} \geq 1$.

If $c_{13} \geq a_{32}^{-1}$ and $c_{24} \geq a_{41}^{-1}$, then

$$\begin{aligned} \det A_c[\{1,3\}] &= 1 - c_{13}a_{32} \leq 0, \\ \det A_c[\{2,4\}] &= 1 - c_{24}a_{41} \leq 0, \\ \det A_c[\{1,2,4\}] &= (a_{12} - 1) \det A_c[\{2,4\}] \leq 0, \\ \det A_c[\{1,2,3\}] &= (a_{12} - 1) \det A_c[\{2,3\}] \leq 0. \end{aligned}$$

Moreover, our aim is to prove that there exists $c_{13} \geq a_{32}^{-1}$ such that $\det A_c[\{1,3, 4\}] \leq 0$, and that there exists $c_{24} \geq a_{41}^{-1}$ such that $\det A_c[\{2,3,4\}] \leq 0$.

$$\begin{aligned} \det A_c[\{2,3,4\}] &= (a_{41} - a_{32}a_{43})c_{24} + a_{34}a_{43} - 1 - a_{23}a_{34}a_{41} + a_{23}a_{32} \\ &= (a_{41} - a_{32}a_{43})c_{24} + \det A_c[\{3,4\}] + a_{23}a_{32}a_{43} + a_{23}a_{32} \\ &\quad - a_{23}a_{34}a_{41} \\ &\leq (a_{41} - a_{32}a_{43})c_{24} + a_{23}a_{32} \\ &\leq 0 \end{aligned}$$

if and only if $(a_{32}a_{43} - a_{41})c_{24} \geq a_{23}a_{32}$.

We can assume that $a_{32}a_{43}(1 + a_{23}a_{32})^{-1} \leq a_{41} < a_{32}a_{43}$, such that $c_{24} \geq a_{23}a_{32}(a_{32}a_{43} - a_{41})^{-1} \geq a_{41}^{-1}$.

So, we can choose $c_{24} \geq a_{23}a_{32}(a_{32}a_{43} - a_{41})^{-1}$ such that $\det A_c[\{2,3,4\}] \leq 0$ and $\det A_c = (a_{12} - 1) \det A_c[\{2,3,4\}] \leq 0$. Since the calculation of c_{24} does not involve c_{13} and vice versa, we can obtain the desired c_{13} to make $\det A_c[\{1,3,4\}] \leq 0$ by a dual argument.

Case 2: $d_1 = d_2 = d_3 = 1$, $d_4 = 0$, with $a_{12} \geq 1$, $a_{23}a_{32} \geq 1$.

If $c_{13} \geq a_{32}^{-1}$ and $c_{24} \geq 0$, then

$$\begin{aligned}
\det A_c[\{1,3\}] &= 1 - c_{13}a_{32} \leq 0,\\
\det A_c[\{2,4\}] &= -c_{24}a_{41} \leq 0,\\
\det A_c[\{1,2,4\}] &= (a_{12}-1)\det A_c[\{2,4\}] \leq 0,\\
\det A_c[\{1,2,3\}] &= (a_{12}-1)\ \det A_c[\{2,3\}] \leq 0.
\end{aligned}$$

Moreover, our aim is to prove that there exists $c_{13} \geq a_{32}^{-1}$ such that $\det A_c[\{1,3,4\}] \leq 0$, and that there exists $c_{24} \geq 0$ such that $\det A_c[\{2,3,4\}] \leq 0$.

$$\begin{aligned}
\det A_c[\{2,3,4\}] &= c_{24}(-a_{32}a_{43} + a_{41}) + a_{43}a_{34} - a_{23}a_{41}a_{34}\\
&\leq c_{24}(-a_{32}a_{43} + a_{41}) + a_{43}a_{34},
\end{aligned}$$

if and only if $c_{24}(a_{32}a_{43} - a_{41}) \geq a_{43}a_{34}$.

We can assume $0 < a_{41} < a_{32}a_{43}$ such that $c_{24} \geq a_{23}a_{32}(a_{32}a_{43} - a_{41})^{-1}$.

So, it is always possible to choose $c_{24} \geq a_{23}a_{32}(a_{32}a_{43} - a_{41})^{-1}$ such that $\det A_c[\{2,3,4\}] \leq 0$ and $\det A_c = (a_{12}-1)\det A_c[\{2,3,4\}]$. Since the calculation of c_{24} does not involve c_{13} and vice versa, we can obtain the desired c_{13} to make $\det A_c[\{1,3,4\}] \leq 0$ by a dual argument.

Case 3: $d_1 = d_2 = 1$, $d_3 = d_4 = 0$, with $a_{12} \geq 1$.
If $c_{13} \geq a_{43}/a_{41}$ and $c_{24} \geq a_{34}/a_{32}$, then

$$\begin{aligned}
\det A_c[\{1,3\}] &= -c_{13}a_{32} \leq 0,\\
\det A_c[\{2,4\}] &= -c_{24}a_{41} \leq 0,\\
\det A_c[\{1,2,3\}] &= (a_{12}-1)\det A_c[\{2,3\}] \leq 0,\\
\det A_c[\{1,2,4\}] &= (a_{12}-1)\det A_c[\{2,4\}] \leq 0,\\
\det A_c[\{2,3,4\}] &= -c_{24}a_{32}a_{43} - a_{23}a_{32}a_{41} + a_{34}a_{43} \leq 0,\\
\det A_c[\{1,3,4\}] &= a_{34}a_{43} - a_{14}a_{43}a_{32} - c_{13}a_{41}a_{34} \leq 0.\\
\det A_c &= (a_{12}-1)\det A_c[\{2,3,4\}] \leq 0.
\end{aligned}$$

Case 4: $d_2 = 1$, $d_1 = d_3 = d_4 = 0$.
If $c_{13} \geq 0$ and $c_{24} \geq a_{34}/a_{32}$, then

$$\begin{aligned}
\det A_c[\{1,3\}] &= -c_{13}a_{32} \leq 0,\\
\det A_c[\{2,4\}] &= -c_{24}a_{41} \leq 0,\\
\det A_c[\{1,2,3\}] &= -a_{12}a_{23}a_{32} \leq 0,\\
\det A_c[\{1,2,4\}] &= \det A_c[\{2,4\}] \leq 0,\\
\det A_c[\{2,3,4\}] &= -c_{24}a_{32}a_{43} - a_{23}a_{34}a_{14} + a_{34}a_{43} \leq a_{34}a_{43} - c_{24}a_{32}a_{43} \leq 0,\\
\det A_c[\{1,3,4\}] &= -a_{14}a_{43}a_{32} - c_{13}a_{41}a_{34} \leq 0.\\
\det A_c &= \det A_c[\{2,3,4\}] \leq 0.
\end{aligned}$$

Case 5: $d_1 = d_2 = d_3 = d_4 = 0$.
It is easy to prove that all the 2×2 principal minors are non-positive.

If $c_{24} \geq 0$ and $c_{13} \geq (a_{12}a_{31})(a_{41})^{-1}$, then

$$\begin{aligned}
\det A_c[\{1,2,3\}] &= -a_{12}a_{31}a_{23} - c_{13}a_{32} \leq 0,\\
\det A_c[\{1,3,4\}] &= -a_{14}a_{43}c_{13} - a_{41}a_{32}a_{34} \leq 0,\\
\det A_c[\{1,2,4\}] &= -a_{12}a_{41}c_{24} - a_{14}a_{41} \leq 0,\\
\det A_c[\{2,3,4\}] &= -a_{32}a_{43}c_{24} - a_{34}a_{23}a_{41} \leq 0,\\
\det A_c &= a_{21}(a_{12}a_{34}a_{43} - a_{14}a_{32}a_{43} - c_{13}a_{41}a_{34}) \leq 0.
\end{aligned}$$

□

Lemma 3.2. [2]. *Let A be an $n \times n$ matrix and D be a diagonal matrix with diagonal entries $d_1, d_2, \ldots, d_n$. Then*

$$\begin{aligned}
|A+D| = |A| &+ \sum_i d_i A_i + \sum\sum_{i<j} d_i d_j A_{ij}\\
&+ \sum\sum\sum_{i<j<k} d_i d_j d_k A_{ijk} + \cdots + d_1 d_2 \ldots d_n,
\end{aligned}$$

where A_i is the determinant of the submatrix obtained by deleting the i-th row and i-th column, A_{ij} is the determinant obtained by i-th and j-th rows and the i-th and j-th columns, and so on.

Theorem 3.3. *Let A be an $n \times n$ combinatorially symmetric partial N_0^1-matrix with all specified off-diagonal entries negative, whose associated graph is a n-cycle. Then, there exists an N_0^1-matrix completion.*

Proof. Let

$$A = \begin{pmatrix}
-d_1 & -a_{12} & -x_{13} & \cdots & -x_{2,n-1} & -a_{1n}\\
-a_{21} & -d_2 & -a_{23} & \cdots & -x_{2,n-1} & -x_{12}\\
-x_{31} & -a_{32} & -d_3 & \cdots & -x_{3,n-1} & -c_{3n}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
-x_{n-1,2} & -x_{n-1,2} & -x_{n-1,3} & \cdots & -d_{n-1} & -a_{n-1,n}\\
-a_{n1} & -x_{n2} & -x_{n3} & \cdots & -a_{n,n-1} & -d_n
\end{pmatrix}$$

be an $n \times n$ partial N_0^1-matrix whose graph of specified entries is a n-cycle. We can assume A with each a_{ij} positive and d_i nonnegative.

The proof is by induction on n. The case in which $n = 4$ is shown in the proof of Lemma 3.1. Assume true for $n-1$.

We will complete A to an N_0^1 matrix A_c in the following four steps:

Step 1: Choose $x_{2n} = c_{2n}$ and $x_{n2} = c_{n2}$ in an appropriate way so that $A_c[\{2,n\}]$ is an N_0^1-matrix. Then, the principal submatrix

$$C = \begin{pmatrix} -d_2 & -a_{23} & -x_{24} & \cdots & -x_{2,n-1} & -c_{2n} \\ -a_{32} & -d_3 & -a_{34} & \cdots & -x_{3,n-1} & -x_{3n} \\ -x_{24} & -a_{43} & -d_4 & \cdots & -x_{4,n-1} & -x_{4n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{n-1,2} & -x_{n-1,3} & -x_{n-1,4} & \cdots & -d_{n-1} & -a_{n-1,n} \\ -c_{n2} & -x_{n3} & -x_{n4} & \cdots & -a_{n,n-1} & -d_n \end{pmatrix},$$

obtained by deleting row one and column one is a partial N_0^1-matrix whose graph of specified entries is an $(n-1)$-cycle.

By Proposition 1.1, without loss of generality, we assume that $a_{21} = 1$ and $d_i = 0$ or 1 for all i. Using permutation similarities, the proof is divided into two cases:

Case 1: $d_2 = 1$, with $1 \le a_{12} \le a_{1n}a_{n1}$.

Case 2: $d_2 = 0$.

For Case 1, choose $c_{n2} = a_{n1}$, $c_{2n} = a_{1n}/a_{12}$; for Case 2, choose $c_{n2} = a_{n1}$, $c_{2n} = a_{1n}$.

We show $C[\{2,n\}]$ is an N_0^1-matrix.

For Case 1, $\det C[\{2,n\}] = d_n - (a_{1n}a_{n1})/a_{12} \le 0$.

For Case 2, $\det C[\{2,n\}] = -a_{1n}a_{n1} < 0$.

Step 2: Using the induction hypothesis, C can be completed to an N_0^1-matrix, denoted by $A_c[\{2,\dots,n\}]$.

Step 3: For $2 < i,j < n$, choose $x_{i1} = c_{i2}$ and $x_{1j} = c_{2j}a_{12}$ for Case 1 and $x_{1j} = c_{2j}$ for Case 2 to obtain the completion A_c of A.

Step 4: Show A_c is an N_0^1-matrix. We must show that $\det A_c[\alpha] \le 0$ for any $\alpha \subseteq \{1,2,\dots,n\}$. For Case 1 and Case 2, for $1 \notin \alpha$, $A_c[\alpha]$ is principal submatrix of the N_0^1-matrix $A_c[\{2,\dots,n\}]$, so $\det A_c[\alpha] \le 0$. Thus, assume $1 \in \alpha$.

Case 1: $d_2 = 1$, with $1 \le a_{12} \le a_{1n}$.

Case 1.1: $d_1 = 1$

$$A_c = \begin{pmatrix} -1 & -a_{12} & -a_{12}a_{23} & \cdots & -a_{12}c_{2,n-1} & -a_{1n} \\ -1 & -1 & -a_{23} & \cdots & -c_{2,n-1} & -a_{1n}/a_{12} \\ -a_{32} & -a_{32} & -d_3 & \cdots & -c_{3,n-1} & -c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{n-1,2} & -c_{n-1,2} & -c_{n-1,3} & \cdots & -d_{n-1} & -a_{n-1,n} \\ -a_{n1} & -a_{n1} & -c_{n3} & \cdots & -a_{n,n-1} & -d_n \end{pmatrix}.$$

For $2 \in \alpha$, add $-a_{12}$ times row 2 to row 1 (which does not change the determinant) to obtain

$$\begin{pmatrix} a_{12-1} & 0 & 0 & \cdots & 0 & 0 \\ -1 & -1 & -a_{23} & \cdots & -c_{2,n-1} & -a_{1n}/a_{12} \\ -a_{32} & -a_{32} & -d_3 & \cdots & -c_{3,n-1} & -c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{n-1,2} & -c_{n-1,2} & -c_{n-1,3} & \cdots & -d_{n-1} & -a_{n-1,n} \\ -a_{n1} & -a_{n1} & -c_{n3} & \cdots & -a_{n,n-1} & -d_n \end{pmatrix}.$$

$$\det A_c[\alpha] = (a_{12}-1)\det A_c[\alpha - \{1\}] \leq 0.$$

For $2 \notin \alpha$: $A_c[\alpha]$ can be obtained from $A_c[(\alpha - \{1\}) \cup \{2\}]$ by multiplying the first row by $a_{12} \geq 1$, and adding $\mathrm{diag}(a_{12}-1, 0, \ldots, 0)$. According to Lemma 3.2,

$$\begin{aligned} \det A_c[\alpha] &= a_{12}\det A_c[(\alpha - \{1\}) \cup \{2\}] + (a_{12}-1)\det A_c[\alpha - \{1,2\}] \\ &\leq a_{12} A_c[(\alpha - \{1\}) \cup \{2\}] \\ &\leq 0. \end{aligned}$$

Case 1.2: $d_1 = 0$

$$A_c = \begin{pmatrix} 0 & -a_{12} & -a_{12}a_{23} & \cdots & -a_{12}c_{2,n-1} & -a_{1n} \\ -1 & -1 & -a_{23} & \cdots & -c_{2,n-1} & -a_{1n}/a_{12} \\ -a_{32} & -a_{32} & -d_3 & \cdots & -c_{3,n-1} & -c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{n-1,2} & -c_{n-1,2} & -c_{n-1,3} & \cdots & -d_{n-1} & -a_{n-1,n} \\ -a_{n1} & -a_{n1} & -c_{n3} & \cdots & -a_{n,n-1} & -d_n \end{pmatrix}.$$

For $2 \in \alpha$, add $-a_{12}$ times row 2 to row 1 (which does not change the determinant) to obtain

$$\det A_c[\alpha] = a_{12}\det A_c[\alpha - \{1\}] \leq 0.$$

For $2 \notin \alpha$: $A_c[\alpha]$ can be obtained from $A_c[(\alpha - \{1\}) \cup \{2\}]$ by multiplying the first row by $a_{12} > 0$, and adding $\mathrm{diag}(a_{12}, 0, \ldots, 0)$. According to Lemma 3.2,

$$\begin{aligned} \det A_c[\alpha] &= a_{12}\det A_c[(\alpha - \{1\}) \cup \{2\}] + a_{12}\det A_c[\alpha - \{1,2\}] \\ &\leq a_{12}\det A_c[(\alpha - \{1\}) \cup \{2\}] \\ &\leq 0. \end{aligned}$$

Case 2: $d_2 = 0$.

$$A_c = \begin{pmatrix} 0 & -a_{12} & -a_{23} & \cdots & -c_{2,n-1} & -a_{1n} \\ -1 & 0 & -a_{23} & \cdots & -c_{2,n-1} & -a_{1n} \\ -a_{32} & -a_{32} & -d_3 & \cdots & -c_{3,n-1} & -c_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -c_{n-1,2} & -c_{n-1,2} & -c_{n-1,3} & \cdots & -d_{n-1} & -a_{n-1,n} \\ -a_{n1} & -a_{n1} & -c_{n3} & \cdots & -a_{n,n-1} & -d_n \end{pmatrix}.$$

For $2 \in \alpha$:

$$\det A_c[\alpha] = a_{21}\det A_c[\alpha - \{1\}] + a_{12}\det A_c[\alpha - \{1\}|\alpha - \{2\}],$$

in which $A_c[\alpha-\{1\}|\alpha-\{2\}]$ can be obtained from $A_c[\alpha-\{1\}]$ by adding diag$(-a_{21}, 0,\dots,0)$. According to Lemma 3.2,

$$\det A_c[\alpha] = a_{21}\det A_c[\alpha-\{1\}] + a_{12}(-a_{21}\det A_c[\alpha-\{1,2\}] + \det A_c[\alpha-\{1\}]).$$

If $\det A_c[\alpha-\{1,2\}]=0$, then

$$\det A_c[\alpha] = (1+a_{12})\det A_c[\alpha-\{1\}] \le 0.$$

If $\det A_c[\alpha-\{1,2\}]\neq 0$, it is possible to choose $\det A_c[\alpha-\{1\}] \le a_{21}\det A_c[\alpha-\{1,2\}]$, then $\det A_c[\alpha]\le 0$.

For $2\notin\alpha$:

$$\det A_c[\alpha] = \det A_c[(\alpha-\{1\})\cup\{2\}] \le 0.$$

□

Remark 3.4. *For an $n\times n$ $(n\ge 4)$ combinatorially symmetric partial N_0^1-matrix, the condition of all specified off-diagonal entries negative is sufficient but not necessary to obtain N_0^1-matrix completion, as the following examples show.*

Example 3.5. *The matrix*

$$A=\begin{pmatrix} 0 & -1 & -x_{13} & 0\\ -1 & -1 & -1 & -x_{24}\\ -x_{31} & -1 & -1 & -1\\ 0 & -x_{42} & -1 & -1\end{pmatrix}$$

is a partial N_0^1-matrix whose graph of specified entries is a 4-cycle, but A does not have the N_0^1-matrix completion since $\det A[\{1,2,4\}]=1$ *for all $x_{24},x_{42}\ge 0$.*

Example 3.6. *The matrix*

$$A=\begin{pmatrix} 0 & 0 & -x_{13} & 0\\ -1 & 0 & 0 & -x_{24}\\ -x_{31} & -1 & 0 & 0\\ -1 & -x_{42} & -1 & 0\end{pmatrix}$$

is a partial N_0^1-matrix whose graph of specified entries is a 4-cycle. The reader can easily verify that the matrix A_c obtained by setting $x_{13}=x_{24}=1$ is an N_0^1-matrix completion.

4. 1-chordal graphs

A graph is *chordal* if every cycle of length 4 or more has a chord or, equivalently, if it has no minimal induced cycles of length 4 or more.

A *clique* in an undirected graph G is simply a complete (all possible edges) induced subgraph. We also use clique to refer to a complete graph and use K_p to indicate a clique on p vertices. If G_1 is the clique, denoted by K_q and G_2 is any chordal graph containing the clique, denoted by K_p, $p<q$, then the clique sum of G_1 and G_2 along K_p is also chordal. The clique M is called a *maximal clique* if M is not a proper subset of any clique. The cliques that are used to build chordal graphs are the maximal cliques of the resulting chordal graph and the cliques along which the summing takes place are the so-called *minimal vertex separators* of the

resulting chordal graph. If the maximum number of vertices in a minimal vertex separator is p, then the chordal graph is said to be *p-chordal*. For more on chordal graphs see [19].

In the section we show the existence of N_0^1-matrix completions of an $n\times n$ partial N_0^1-matrix A, whose associated graph is a 1-chordal.

Lemma 4.1. *Let A be a combinatorially symmetric partial N_0^1-matrix with all specified off-diagonal entries negative, whose associated graph is 1-chordal with two maximal cliques, one of them with two vertices. Then, there exists an N_0^1-matrix completion of A.*

Proof. We may assume the partial N_0^1-matrix of A has the following three forms:

Form 1:

$$A=\begin{pmatrix} -1 & -a_{12} & -x_{13} & \cdots & -x_{1,n-1} & -x_{1n} \\ -a_{21} & -1 & -a_{23} & \cdots & -a_{2,n-1} & -a_{2n} \\ -x_{31} & -a_{32} & -d_3 & \cdots & -a_{3,n-1} & -a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{n-1,1} & -a_{n-1,2} & -a_{n-3,3} & \cdots & -d_{n-1} & -a_{n-1,n} \\ -x_{n1} & -a_{n2} & -a_{n3} & \cdots & -a_{n,n-1} & -d_n \end{pmatrix}.$$

The proof is the same as the Case 1.1 of Theorem 3.3.

Form 2:

$$A=\begin{pmatrix} 0 & -a_{12} & -x_{13} & \cdots & -x_{1,n-1} & -x_{1n} \\ -a_{21} & -1 & -a_{23} & \cdots & -a_{2,n-1} & -a_{2n} \\ -x_{31} & -a_{32} & -d_3 & \cdots & -a_{3,n-1} & -a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{n-1,1} & -a_{n-1,2} & -a_{n-3,3} & \cdots & -d_{n-1} & -a_{n-1,n} \\ -x_{n1} & -a_{n2} & -a_{n3} & \cdots & -a_{n,n-1} & -d_n \end{pmatrix}.$$

The proof is the same as the Case 1.2 of Theorem 3.3.

Form 3:

$$A=\begin{pmatrix} 0 & -a_{12} & -x_{13} & \cdots & -x_{1,n-1} & -x_{1n} \\ -a_{21} & 0 & -a_{23} & \cdots & -a_{2,n-1} & -a_{2n} \\ -x_{31} & -a_{32} & -d_3 & \cdots & -a_{3,n-1} & -a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -x_{n-1,1} & -a_{n-1,2} & -a_{n-3,3} & \cdots & -d_{n-1} & -a_{n-1,n} \\ -x_{n1} & -a_{n2} & -a_{n3} & \cdots & -a_{n,n-1} & -d_n \end{pmatrix}.$$

The proof is the same as the Case 2 of Theorem 3.3. □

Theorem 4.2. *Let A be a combinatorially symmetric partial N_0^1-matrix with all specified off-diagonal entries negative, whose associated graph is 1-chordal with two maximal cliques. Then, there exists an N_0^1-matrix completion of A.*

Proof. Let A be a combinatorially symmetric partial N_0^1-matrix, whose graph of specified entries is G. Without loss of generality, we may assume that A has the following forms:

Form 1:

$$A = \begin{pmatrix} A_{11} & A_{12} & X \\ A_{21}^T & -1 & A_{23}^T \\ Y & A_{32} & A_{33} \end{pmatrix}.$$

Form 2:

$$A = \begin{pmatrix} A_{11} & A_{12} & X \\ A_{21}^T & 0 & A_{23}^T \\ Y & A_{32} & A_{33} \end{pmatrix}.$$

where $A_{12}, A_{21} \in \mathbb{R}^p$, $A_{23}, A_{32} \in \mathbb{R}^q$ $(p+q=n-1)$ and X, Y are completely unspecified and the remaining entries of A are prescribed and negative.

By Proposition 1.1, without loss of generality, we may assume all the entries of A_{21}^T are -1.

For the Form 1, consider the completion

$$A_c = \begin{pmatrix} A_{11} & A_{12} & -A_{12}A_{23}^T \\ A_{21}^T & -1 & A_{23}^T \\ -A_{32} & A_{32} & A_{33} \end{pmatrix}.$$

Therefore, in analogous way to *Case 1* of Theorem 3.3 we can obtain that A is N_0^1.

For the Form 2, consider the completion

$$A_c = \begin{pmatrix} A_{11} & A_{12} & A_{23}^T \\ A_{21}^T & 0 & A_{23}^T \\ A_{32} & A_{32} & A_{33} \end{pmatrix}.$$

Therefore, in analogous way to *Case 2* of Theorem 3.3 we can obtain that A is N_0^1. □

Theorem 4.3. *Let A be a combinatorially symmetric partial N_0^1-matrix with all specified off-diagonal entries negative, whose associated graph is 1-chordal with l maximal cliques ($l \geq 2$). Then, there exists an N_0^1-matrix completion of A.*

Proof. Let A be a combinatorially symmetric partial N_0^1-matrix, whose graph of specified entries is G. The proof is by induction on the number l of maximal cliques in G.

For $l = 2$, we obtain the desired completion by applying Theorem 4.2.

For $l > 2$, suppose that the result is true for 1-chordal graph with $l-1$ maximal cliques and we are going to prove the result for the case of p maximal cliques. Let G_1 be the subgraph induced by two maximal cliques with a common vertex. Applying Theorem 4.2 to submatrix A_1 of A, we obtain an N_0^1 completion A_{1c} of A_1. Replacing A_1 with the completion A_{1c} in A, we obtain a partial N_0^1-matrix whose graph of the specified entries is a 1-chordal graph with $l-1$ maximal cliques. The induction hypothesis allows us to obtain the result. □

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Gu-Fang Mou
School of Mathematical Sciences, University of Electronic Science and Technology of China,
Qingshuihe Campus: No.2006, Xiyuan Ave, West Hi-Tech Zone,
611731, Chengdu, Sichuan, People's Republic of China
`mougufang1010@163.com`

Ting-Zhu Huang
School of Mathematical Sciences, University of Electronic Science and Technology of China,
Qingshuihe Campus: No.2006, Xiyuan Ave, West Hi-Tech Zone,
611731, Chengdu, Sichuan, People's Republic of China
`tzhuang@uestc.edu.cn`

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