REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 54, No. 1, 2013, Pages 15–25
Published online: May 10, 2013

DIFFERENTIABILITY AND BEST LOCAL APPROXIMATION

HÉCTOR H. CUENYA AND CLAUDIA N. RODRIGUEZ

Abstract. In this paper we give sufficient conditions over the differentiability of a function to assure existence of the best local approximant in L^p-spaces, $0 < p \leq \infty$. These conditions are weaker than those given in previous papers. For $p = 2$ we show that, in a certain way, they are also necessary. In addition, we characterize the best local approximant.

1. Introduction

Let $x_1, x_2, \dots, x_k$ be k points in $\mathbb{R}$ and let $r > 0$ be such that the intervals $[x_i - r, x_i + r]$, $1 \leq i \leq k$, are pairwise disjoint. Let $\mathcal{L}$ be the space of real Lebesgue measurable functions defined on $A_r := \bigcup_{i=1}^{k} [x_i - r, x_i + r]$. For each Lebesgue measurable set $A \subset A_r$, with $|A| > 0$, we consider the semi-norms on $\mathcal{L}$,

$$\|h\|_{p,A} := \left(\frac{1}{|A|} \int_A |h(x)|^p dx \right)^{1/p}, \ 0 < p < \infty, \qquad \text{and} \qquad \|h\|_{\infty,A} := \max_{x \in A} |h(x)|,$$

where $|A|$ denotes the measure of the set A. Note that for $0 < p < 1$, $\|\cdot\|_{p,A}$ is not a semi-norm, but we will also call it in this way.

If $0 < \epsilon \leq r$, $A_{-\epsilon,i} := [x_i - \epsilon, x_i]$, $A_{+\epsilon,i} := [x_i, x_i + \epsilon]$, and $A_{\epsilon,i} := [x_i - \epsilon, x_i + \epsilon]$, we write $\|h\|_{p,-\epsilon,i} = \|h\|_{p,A_{-\epsilon,i}}$, $\|h\|_{p,+\epsilon,i} = \|h\|_{p,A_{+\epsilon,i}}$, and $\|h\|_{p,\epsilon,i} = \|h\|_{p,A_{\epsilon,i}}$. So, we have

$$\|h\|_{p,\epsilon,i}^p = \frac{1}{2} \left(\|h\|_{p,-\epsilon,i}^p + \|h\|_{p,+\epsilon,i}^p \right),$$

and

$$\|h\|_{p,\epsilon}^p := \|h\|_{p,A_\epsilon}^p = \sum_{i=1}^{k} \|h\|_{p,\epsilon,i}^p.$$

With the obvious modifications we have the notation for $p = \infty$. Suppose n a non negative fixed integer and $n + 1 = kq$. For a non negative integer s, we denote by Π^s the linear space of polynomials of degree at most s.

If h has left derivatives up to order $q-1$ at x_i, $1 \leq i \leq k$, say $h_-^{(j)}(x_i)$, we denote by $H_-(h)$ the *left Hermite interpolate polynomial of degree* n, i.e., $H_-(h)$ is the

---

2010 *Mathematics Subject Classification.* Primary 41A50. Secondary 41A10.

Key words and phrases. Best approximation, L^p-norm, local approximant.

This work was supported by Universidad Nacional de Río Cuarto, and CONICET ANPCyT.

unique polynomial in Π^n such that $H_-^{(j)}(h)(x_i) = h_-^{(j)}(x_i)$, $1 \le i \le k$, $0 \le j \le q-1$. The right Hermite interpolate polynomial of degree n, $H_+(h)$, is defined in a similar way.

If $h \in \mathcal{L}$, it is well known that there exists a best $\|\cdot\|_{p,\epsilon}$-approximant of h from Π^n. Let $P_\epsilon(h)$ denote such a best approximant, i.e., $P_\epsilon(h) \in \Pi^n$ satisfies

$$\|h - P_\epsilon(h)\|_{p,\epsilon} \le \|h - P\|_{p,\epsilon}, \quad \text{for all} \quad P \in \Pi^n.$$

If there exists the $\lim_{\epsilon \to 0} P_\epsilon(h) =: P_0(h) \in \Pi^n$, then $P_0(h)$ is called the *best local approximant of h at x_i, $1 \le i \le k$, from Π^n.*

The existence of the best local approximant to a function f from Π^n, in one variable, has been extensively studied. This problem was investigated in a single point and several points. In L^p-spaces, where $1 \le p \le \infty$, we can mention [3] and [2], while if $0 < p \le \infty$ we can mention [8]; in Orlicz spaces we cite [6], [7] and [4]. Recently the existence was considered for more general norms in [11] and [10]. We emphasize that in all these works, the usual differentiability or the Peano differentiability of a function f up to order $q-1$ at x_i, $1 \le i \le k$, was required.

In Section 2 of this paper we give a sufficient condition weaker than the differentiability up to order $q-1$, in order to assure the existence of the best local approximant of a function f at x_i, $1 \le i \le k$, from Π^n in L^p-spaces, $0 < p \le \infty$. More precisely, we show that the differentiability of f at x_i, up to the order $q-2$, and the existence of the lateral derivatives of order $q-1$ at x_i, $1 \le i \le k$, are sufficient conditions for the existence of the best local approximant, whenever certain minimization problem has a unique solution. In particular, if $p > 1$, or $p = 1$ and $q \ne 1$, that minimization problem has a unique solution. We also give a characterization of the best local approximant, i.e., it is just the average of the left and right Hermite interpolate polynomials at the points x_i, $1 \le i \le k$. To prove these results we will adapt the technique used in [8], which was employed to show the existence of the best local approximant in several points.

In Section 3, we will consider $k = 1$, and we will prove that the established conditions in Section 2 are also necessary in the setting of the L^2-space, when we approximate to a function which has lateral derivatives up to order n at x_1.

2. Sufficient conditions in L^p

Henceforward, f shall be a fixed function in $\mathcal{L}$, and K a positive constant, not necessarily the same at each occurrence. Assume that f has left and right derivatives up to order $q-1$ at $x_i, 1 \le i \le k$.

If $f_-^{(j)}(x_i) = f_+^{(j)}(x_i) =: f^{(j)}(x_i)$, $0 \le j \le q-2$, $1 \le i \le k$, and $\epsilon \to 0$, under certain conditions we shall show that the best approximants of f on A_ϵ from Π^n converge to the polynomial $H \in \Pi^n$ defined by

$$H = \frac{H_-(f) + H_+(f)}{2}.$$

If $q = 1$ we do not require any condition over the derivatives of f.

We write $f_-^{(j)}(x_i) = p_{j,i}$ and $f_+^{(j)}(x_i) = t_{j,i}$, $0 \le j \le q-1$, $1 \le i \le k$. We also denote by $T^s_{-,i}(f)$ and $T^s_{+,i}(f)$ the left and right Taylor polynomials of f at x_i, of degree s, respectively. If $T^s_{-,i}(f) = T^s_{+,i}(f)$ we denote it by $T^s_i(f)$.

Suppose that $p_{j,i} = t_{j,i}$, $0 \le j \le q-2$, $1 \le i \le k$, and consider the following set:

$$\mathcal{H}(f) = \{Q \in \Pi^n : Q^{(j)}(x_i) = f^{(j)}(x_i),\ 0 \le j \le q-2,\ 1 \le i \le k\}.$$

If $q = 1$ we put $\mathcal{H}(f) = \{0\}$. Let $H_0 \in \mathcal{H}(f)$. If $g = f - H_0$, then $g^{(j)}(x_i) = 0$, $1 \le j \le q-2$, $1 \le i \le k$. The next lemma immediately follows.

Lemma 2.1. *Let $P \in \Pi^n$, $0 < p \le \infty$, and $0 < \epsilon \le r$. Then P is a best $\|\cdot\|_{p,\epsilon}$-approximant to f from Π^n if and only if $P = H_0 + Q$, with Q a best $\|\cdot\|_{p,\epsilon}$-approximant to g from Π^n. In addition, the best local approximant to f exists if and only if the best local approximant to g exists, and $P_0(f) = P_0(g) + H_0$.*

Therefore, it shall be sufficient for our purposes to prove the existence and characterization of $P_0(g)$.

Lemma 2.2. *Let $0 < p \le \infty$. The set of polynomials $\{P_\epsilon(g)\}$, for ϵ small, is uniformly bounded on compact sets.*

Proof. Let $0 < p < \infty$. It is easy to show that

$$\|g - T^{q-1}_{\pm,i}(g)\|_{p,\pm\epsilon,i} + \|T^{q-1}_{\pm,i}(g)\|_{p,\pm\epsilon,i} = o(\epsilon^{q-1}) + O(\epsilon^{q-1}) = O(\epsilon^{q-1}),$$

$1 \le i \le k$. So, we get

$$\begin{aligned}\|g\|^p_{p,\epsilon,i} \le \frac{1}{2}\left(\|g - T^{q-1}_{-,i}(g)\|_{p,-\epsilon,i} + \|T^{q-1}_{-,i}(g)\|_{p,-\epsilon,i}\right)^p \\ + \frac{1}{2}\left(\|g - T^{q-1}_{+,i}(g)\|_{p,+\epsilon,i} + \|T^{q-1}_{+,i}(g)\|_{p,+\epsilon,i}\right)^p = O(\epsilon^{p(q-1)}),\quad 0 < p < \infty.\end{aligned} \tag{2.1}$$

Analogously, we obtain $\|g\|_{\infty,\epsilon,i} = O(\epsilon^{q-1})$. Now, we recall the Pólya inequality (see [5], Lemma 2.1): There exists a constant $K = K(p) > 0$ $(0 < p \le \infty)$ such that

$$|P_\epsilon(g)^{(j)}(x_i)| \le \frac{K}{\epsilon^j}\|P_\epsilon(g)\|_{p,\epsilon,i},\quad 0 \le j \le q-1,\quad 0 < \epsilon \le r,\quad 1 \le i \le k. \tag{2.2}$$

As a consequence of (2.2) and (2.1), we have

$$\begin{aligned}|P_\epsilon(g)^{(j)}(x_i)| &\le \frac{K}{\epsilon^j}\left(\|P_\epsilon(g) - g\|_{p,\epsilon,i} + \|g\|_{p,\epsilon,i}\right) \\ &\le \frac{2K}{\epsilon^j}\|g\|_{p,\epsilon,i} = O(\epsilon^{q-1-j}),\quad 0 \le j \le q-1,\quad 1 \le i \le k.\end{aligned} \tag{2.3}$$

So, the lemma is proved. □

Let $p'_i = p_{q-1,i} - H_0^{(q-1)}(x_i)$ and $t'_i = t_{q-1,i} - H_0^{(q-1)}(x_i)$. We write

$$u_i = \frac{p'_i}{(q-1)!} \quad\text{and}\quad v_i = \frac{t'_i}{(q-1)!}.$$

Henceforward, without loss of generality we assume $r = 1$. If $P \in \Pi^n$ we consider the following function defined on A_1,

$$P^*(x) = \begin{cases} u_i(x - x_i)^{q-1} - T_i^{q-1}(P)(x), & \text{if } x \in [x_i - 1, x_i], \\ v_i(x - x_i)^{q-1} - T_i^{q-1}(P)(x), & \text{if } x \in [x_i, x_i + 1]. \end{cases}$$

Lemma 2.3. *Let* $0 < p \leq \infty$.
a) Every uniformly bounded net $\{Q_\epsilon\} \subset \Pi^n$ *verifies*

$$\begin{aligned} \frac{\|g - Q_\epsilon\|_{p,\epsilon}}{\epsilon^{q-1}} &= \frac{\|Q_\epsilon^*\|_{p,\epsilon}}{\epsilon^{q-1}} + o(1), \text{ if } 1 \leq p \leq \infty; \\ \frac{\|g - Q_\epsilon\|_{p,\epsilon}^p}{\epsilon^{p(q-1)}} &= \frac{\|Q_\epsilon^*\|_{p,\epsilon}^p}{\epsilon^{p(q-1)}} + o(1), \text{ if } 0 < p < 1. \end{aligned} \tag{2.4}$$

b) $\dfrac{\|g - P_\epsilon(g)\|_{p,\epsilon}}{\epsilon^{q-1}} = O(1)$.

Proof. Let $\{Q_\epsilon\} \subset \Pi^n$ be uniformly bounded. We have

$$(g - Q_\epsilon - Q_\epsilon^*)(x) = \begin{cases} g(x) - u_i(x - x_i)^{q-1} + O(x - x_i)^q, & \text{if } x \in [x_i - 1, x_i], \\ g(x) - v_i(x - x_i)^{q-1} + O(x - x_i)^q, & \text{if } x \in [x_i, x_i + 1]. \end{cases}$$

On the other hand,

$$\begin{aligned} &\|g(x) - u_i(x - x_i)^{q-1} + O(x - x_i)^q\|_{p,-\epsilon,i} \\ &\qquad = \|g(x) - T_{-,i}^{q-1}(g)(x) + O(x - x_i)^q\|_{p,-\epsilon,i} = o(\epsilon^{q-1}) \end{aligned}$$

and

$$\begin{aligned} &\|g(x) - v_i(x - x_i)^{q-1} + O(x - x_i)^q\|_{p,+\epsilon,i} \\ &\qquad = \|g(x) - T_{+,i}^{q-1}(g)(x) + O(x - x_i)^q\|_{p,+\epsilon,i} = o(\epsilon^{q-1}). \end{aligned}$$

Therefore,

$$\frac{\|g - Q_\epsilon - Q_\epsilon^*\|_{p,\epsilon}}{\epsilon^{q-1}} = o(1).$$

For $1 \leq p \leq \infty$, the triangular inequality implies (2.4). For $0 < p < 1$, (2.4) follows from the following inequality:

$$\left| \|g - Q_\epsilon\|_{p,\epsilon}^p - \|Q_\epsilon^*\|_{p,\epsilon}^p \right| \leq \|g - Q_\epsilon - Q_\epsilon^*\|_{p,\epsilon}^p.$$

To prove b) we observe that

$$\begin{aligned} \frac{\|P_\epsilon(g)^*\|_{p,\epsilon}^p}{\epsilon^{(q-1)p}} &= \sum_{i=1}^k \int_{A_{\epsilon,i}} \frac{|P_\epsilon(g)^*(t)|^p}{\epsilon^{(q-1)p}} \frac{dt}{2k\epsilon} \\ &= \sum_{i=1}^k \int_{[-1,1]} \frac{|P_\epsilon(g)^*(x_i + \epsilon t)|^p dt}{2k\epsilon^{(q-1)p}}, \end{aligned} \tag{2.5}$$

and

$$\frac{P_\epsilon(g)^*}{\epsilon^{q-1}}(x_i+\epsilon t)=\begin{cases} u_i t^{q-1}-\sum\limits_{j=0}^{q-1}\frac{1}{j!}\epsilon^{j-q+1}t^j P_\epsilon(g)^{(j)}(x_i), & \text{if } t\in[-1,0];\\ v_i t^{q-1}-\sum\limits_{j=0}^{q-1}\frac{1}{j!}\epsilon^{j-q+1}t^j P_\epsilon(g)^{(j)}(x_i), & \text{if } t\in[0,1].\end{cases} \tag{2.6}$$

Now, (2.3) implies that $\frac{P_\epsilon(g)^*}{\epsilon^{q-1}}(x_i+\epsilon t)$ is uniformly bounded on $[-1,1]$, as $\epsilon\to 0$. As a consequence, from (2.5) and the part *a*) for $Q_\epsilon=P_\epsilon(g)$, we get *b*). The case $p=\infty$ follows with the obvious modifications. □

Next, we introduce k real functions on $\mathbb{R}^q$. If $0<p<\infty$ and $1\le i\le k$ we define

$$\begin{aligned} F_{p,i}(c_{0,i},c_{1,i},\dots,c_{q-1,i}) = &\int\limits_{[-1,0]}\left|u_i t^{q-1}-\sum_{j=0}^{q-1}c_{j,i}t^j\right|^p\frac{dt}{2k}\\ &+\int\limits_{[0,1]}\left|v_i t^{q-1}-\sum_{j=0}^{q-1}c_{j,i}t^j\right|^p\frac{dt}{2k}, \end{aligned} \tag{2.7}$$

and for $p=\infty$ we define

$$\begin{aligned} &F_{p,i}(c_{0,i},c_{1,i},\dots,c_{q-1,i})\\ &\quad=\max\left\{\max_{t\in[-1,0]}\left|u_i t^{q-1}-\sum_{j=0}^{q-1}c_{j,i}t^j\right|,\ \max_{t\in[0,1]}\left|v_i t^{q-1}-\sum_{j=0}^{q-1}c_{j,i}t^j\right|\right\}. \end{aligned}$$

If $\max\limits_{0\le j\le q-1}|c_{j,i}|\to\infty$, it is easy to see that $F_{p,i}(c_{0,i},c_{1,i},\dots,c_{q-1,i})\to\infty$, $0<p\le\infty$. Since $F_{p,i}$ is a continuous function, then $F_{p,i}$ has a minimum value. We write

$$B_{p,i}=\min_{\mathbb{R}^q}F_{p,i}(c_{0,i},c_{1,i},\dots,c_{q-1,i}),\quad 1\le i\le k. \tag{2.8}$$

If B_p is the minimum of $\sum\limits_{i=1}^{k}F_{p,i}(c_{0,i},c_{1,i},\dots,c_{q-1,i})$, where the minimum is taken over all the k-tuples of vectors in $\mathbb{R}^q$, clearly $B_p=\sum\limits_{i=1}^{k}B_{p,i}$.

If there is a unique q-tuple minimizing (2.8), we denote it by $(c_{0,i}(p),c_{1,i}(p),\dots,c_{q-1,i}(p))$.

Lemma 2.4. *Let $1\le p\le\infty$. If $p\ne 1$ or $q\ne 1$, then there exists a unique q-tuple that minimizes the problem (2.8).*

Proof. Let h_i be the function defined by $h_i(t)=u_i\,t^{q-1}$ if $t\in[-1,0)$, and $h_i(t)=v_i\,t^{q-1}$ if $t\in(0,1]$. We observe that the minimizing problem (2.8) is equivalent to finding the best $\|\cdot\|_{p,[-1,1]}$-approximant of h_i from Π^{q-1}.

If $1<p<\infty$ and $q\ge 1$, the Lemma is a consequence of the strict convexity of the norm.

If $q \geq 2$, h_i is an essentially continuous function. Therefore, for $p = 1$ or $p = \infty$, the Lemma is a consequence of the uniqueness of the best approximant to a continuous function from Π^{q-1} (see [9]).

Finally, if $q = 1$ and $p = \infty$ it is easy to see that $c_{0,i}(p) = \frac{u_i+v_i}{2}$. □

Remark 2.5. If $p = q = 1$, in general we have not uniqueness for the minimizing problem (2.8).

The following theorem gives us an expression for the asymptotic behavior of the normalized error.

Lemma 2.6. *Let $P_\epsilon(g)$ be a best $\|\cdot\|_{p,\epsilon}$-approximant to g from Π^n. Then*

$$\lim_{\epsilon\to 0} \frac{\|g - P_\epsilon(g)\|_{p,\epsilon}}{\epsilon^{q-1}} = B_p^{1/p}, \quad \text{if } 0 < p < \infty, \quad \text{and}$$

$$\lim_{\epsilon\to 0} \frac{\|g - P_\epsilon(g)\|_{p,\epsilon}}{\epsilon^{q-1}} = B_p, \quad \text{if } p = \infty.$$

Proof. Suppose $0 < p < \infty$ and let $(a_{0,i}, \dots, a_{q-1,i})$, $1 \leq i \leq k$, be such that $F_{p,i}(a_{0,i}, \dots, a_{q-1,i}) = B_{p,i}$. Let $Q_\epsilon \in \Pi^n$ be the polynomial satisfying $Q_\epsilon^{(j)}(x_i) = a_{j,i} j! \epsilon^{q-1-j}$, $0 \leq j \leq q-1$, $1 \leq i \leq k$. As a consequence of Lemma 2.3, a), we get

$$\begin{aligned} \frac{\|g - P_\epsilon(g)\|_{p,\epsilon}}{\epsilon^{q-1}} &\leq \frac{\|g - Q_\epsilon\|_{p,\epsilon}}{\epsilon^{q-1}} = \frac{\|Q_\epsilon^*\|_{p,\epsilon}}{\epsilon^{q-1}} + o(1) = B_p^{1/p} + o(1), \ 1 \leq p < \infty \\ \frac{\|g - P_\epsilon(g)\|_{p,\epsilon}^p}{\epsilon^{p(q-1)}} &\leq \frac{\|g - Q_\epsilon\|_{p,\epsilon}^p}{\epsilon^{p(q-1)}} = \frac{\|Q_\epsilon^*\|_{p,\epsilon}^p}{\epsilon^{p(q-1)}} + o(1) = B_p + o(1), \ 0 < p < 1, \end{aligned} \tag{2.9}$$

where the last equality follows from the definitions of $(a_{0,i}, a_{1,i}, \dots, a_{q-1,i})$ and Q_ϵ. On the other hand, if $b_{j,i} = \frac{1}{j!}\epsilon^{j-q+1} P_\epsilon(g)^{(j)}(x_i)$, using again Lemma 2.3, a), for the net $\{P_\epsilon(g)\}$, we have

$$B_p^{1/p} \leq \left(\sum_{i=1}^{k} F_{p,i}(b_{0,i}, b_{1,i}, \dots, b_{q-1,i}) \right)^{1/p} = \frac{\|P_\epsilon^*(g)\|_{p,\epsilon}}{\epsilon^{q-1}} = \frac{\|g - P_\epsilon(g)\|_{p,\epsilon}}{\epsilon^{q-1}} + o(1), \tag{2.10}$$

for $1 \leq p < \infty$, and

$$B_p \leq \sum_{i=1}^{k} F_{p,i}(b_{0,i}, b_{1,i}, \dots, b_{q-1,i}) = \frac{\|P_\epsilon^*(g)\|_{p,\epsilon}^p}{\epsilon^{p(q-1)}} = \frac{\|g - P_\epsilon(g)\|_{p,\epsilon}^p}{\epsilon^{p(q-1)}} + o(1), \tag{2.11}$$

for $0 < p < 1$. Now the lemma is a consequence of (2.9)–(2.11). The case $p = \infty$ follows with the obvious modifications. □

Lemma 2.7. *Let $0 < p \leq \infty$. Assume that the minimization problem (2.8) has a unique solution. Let $Q_i \in \Pi^{q-1}$, $1 \leq i \leq k$, be defined by $Q_i(t) = \sum_{j=0}^{q-1} c_{j,i}(p) t^j =: \widetilde{Q_i}(t) + c_{q-1,i}(p) t^{q-1}$. Then we have $c_{q-1,i}(p) = \frac{u_i+v_i}{2}$.*

Proof. We give the proof for $q-1$ even. The case $q-1$ odd is similar. Let $d_i = u_i + v_i - c_{q-1,i}(p)$. From definition (2.7) and the change of variable $x = -t$ we get

$$\begin{aligned} &F_{p,i}(c_{0,i}(p), c_{1,i}(p), \ldots, c_{q-1,i}(p)) \\ &\quad = \int_{[-1,0]} \left|(d_i - v_i)t^{q-1} - \widetilde{Q_i}(t)\right|^p \frac{dt}{2k} + \int_{[0,1]} \left|(d_i - u_i)t^{q-1} - \widetilde{Q_i}(t)\right|^p \frac{dt}{2k} \\ &\quad = \int_{[0,1]} \left|(v_i - d_i)x^{q-1} + \widetilde{Q_i}(-x)\right|^p \frac{dx}{2k} + \int_{[-1,0]} \left|(u_i - d_i)x^{q-1} + \widetilde{Q_i}(-x)\right|^p \frac{dx}{2k} \\ &\quad = F_{p,i}(-c_{0,i}(p), c_{1,i}(p), \ldots, (-1)^{j+1}c_{j,i}(p), \ldots, c_{q-2,i}(p), d_i). \end{aligned} \tag{2.12}$$

Therefore, the uniqueness of the q-tuple $(c_{0,i}(p), c_{1,i}(p), \ldots, c_{q-1,i}(p))$ which minimizes (2.8) implies that $\widetilde{Q_i}(t) = -\widetilde{Q_i}(-t)$, for all $t \in \mathbb{R}$, and $d_i = c_{q-1,i}(p)$. Therefore $c_{q-1,i}(p) = \frac{u_i+v_i}{2}$.

If $p = \infty$, the proof follows with the obvious modifications. □

Lemma 2.8. *Let $0 < p \leq \infty$. If the minimization problem (2.8) has a unique solution then the best local approximant of g exists, and it satisfies $P^{(j)}(x_i) = 0$, $0 \leq j \leq q-2$, $P^{(q-1)}(x_i) = \frac{(u_i+v_i)(q-1)!}{2}$, $1 \leq i \leq k$.*

Proof. Suppose $0 < p < \infty$ and let $\{\epsilon_m\}$ be a sequence tending to zero. Let $c_{j,i}(\epsilon_m) := \frac{1}{j!}\epsilon_m^{j-q+1} P^{(j)}_{\epsilon_m}(g)(x_i)$, $0 \leq j \leq q-1$. From (2.6) and (2.7), we obtain

$$\sum_{i=1}^{k} F_{p,i}(c_{0,i}(\epsilon_m), c_{1,i}(\epsilon_m), \ldots, c_{q-1,i}(\epsilon_m)) = \left\| \frac{P^*_{\epsilon_m}(g)}{\epsilon_m^{q-1}} \right\|^p_{p,\epsilon_m}. \tag{2.13}$$

By Lemma 2.3, a), and Lemma 2.6 we have $\lim_{m\to\infty} \frac{\|P^*_{\epsilon_m}(g)\|_{p,\epsilon_m}}{\epsilon_m^{q-1}} = B_p^{1/p}$. So, from (2.13) we obtain

$$\lim_{m\to\infty} \sum_{i=1}^{k} F_{p,i}(c_{0,i}(\epsilon_m), c_{1,i}(\epsilon_m), \ldots, c_{q-1,i}(\epsilon_m)) = B_p. \tag{2.14}$$

Lemma 2.2 implies that there is a subsequence $\epsilon_{m_l} \to 0$, as $l \to \infty$, such that $(P_{\epsilon_{m_l}}(g)(x_i), P_{\epsilon_{m_l}}(g)^{(1)}(x_i), \ldots, \frac{1}{(q-1)!} P_{\epsilon_{m_l}}(g)^{(q-1)}(x_i))$ converges to a q-tuple $(a_{0,i}, a_{1,i}, \ldots, a_{q-1,i})$. On the other hand, as a consequence of (2.3), the sequence $(c_{0,i}(\epsilon_{m_l}), c_{1,i}(\epsilon_{m_l}), \ldots, c_{q-1,i}(\epsilon_{m_l}))$ is bounded, therefore it has a subsequence which we denote in the same way, such that it converges to a q-tuple $(b_{0,i}, b_{1,i}, \ldots, b_{q-1,i})$. Clearly, by definition of $c_{j,i}(\epsilon_{m_l})$ we get $a_{j,i} = 0$, $0 \leq j \leq q-2$, and $a_{q-1,i} = b_{q-1,i}$.

The continuity of $F_{p,i}$ (2.14) and the definition of $B_{p,i}$ imply that $c_{j,i}(p) = b_{j,i}$, $0 \leq j \leq q-1$, $1 \leq i \leq k$. By Lemma 2.7 we obtain $a_{q-1,i} = \frac{u_i+v_i}{2}$. So, there exists $P(x) = \lim_{l\to\infty} P_{\epsilon_{m_l}}(x)$ verifying $P^{(j)}(x_i) = 0$, $0 \leq j \leq q-2$, $P^{(q-1)}(x_i) =$

$\frac{(u_i+v_i)(q-1)!}{2}$, $1 \leq i \leq k$. Note that the polynomial $P \in \Pi^n$ is univocally determined by these conditions. Since $\{\epsilon_m\}$ is arbitrary, we have proved the lemma for $0 < p < \infty$.

For $p = \infty$ the proof follows the same patterns of above. □

Now, we establish one of our main results.

Theorem 2.9. *Let $0 < p \leq \infty$. Let f be a function with derivatives up to order $q-2$ at x_i, and with lateral derivatives of order $q-1$ at x_i, $1 \leq i \leq k$. If the minimization problem (2.8) has a unique solution, then the best local approximant of f exists, and it is equal to*

$$\frac{H_-(f) + H_+(f)}{2}.$$

Proof. From Lemma 2.8 we have $P_\epsilon(g)(x) \to \frac{H_-(g)+H_+(g)}{2}$, as $\epsilon \to 0$. Therefore, by Lemma 2.1 the best local approximant of f exists and it is equal to $\frac{H_-(g)+H_+(g)}{2} + H_0 = \frac{H_-(f)+H_+(f)}{2}$. □

The next corollary immediately follows from Lemma 2.4 and Theorem 2.9.

Corollary 2.10. *Let f be a function with derivatives up to order $q-2$ at x_i, and with lateral derivatives of order $q-1$ at x_i, $1 \leq i \leq k$, and let $1 \leq p \leq \infty$. If $p \neq 1$ or $q \neq 1$, then the best local approximant of f exists and it is*

$$\frac{H_-(f) + H_+(f)}{2}.$$

Remark 2.11. Suppose that f is Peano differentiable up to order $q-2$ and left and right Peano differentiable of order $q-1$ (see [1]), i.e., there exist polynomials $R_i \in \Pi^{q-2}$, and real numbers p_i and q_i, $1 \leq i \leq k$, such that

$$\|f(x) - R_i(x) - p_i(x - x_i)^{q-1}\|_{p,-\epsilon,i} = o(\epsilon^{q-1}),$$

and

$$\|f(x) - R_i(x) - q_i(x - x_i)^{q-1}\|_{p,+\epsilon,i} = o(\epsilon^{q-1}).$$

If $1 \leq p \leq \infty$, and $p \neq 1$ or $q \neq 1$, we can prove in a similar way to the proof of Corollary 2.10, that the best local approximant of f from Π^n is the polynomial H satisfying $H^{(j)}(x_i) = R_i^{(j)}(x_i), 0 \leq j \leq q-2$, and $H^{(q-1)}(x_i) = \frac{(q-1)!(p_i+q_i)}{2}$, $1 \leq i \leq k$.

3. A necessary condition in L^2

In this section we assume $k = 1$, $x_1 = 0$ and $n \geq 1$. Let $f \in \mathcal{L}$ be a function which has left and right derivatives up to order n at 0. We shall prove in this section that a necessary condition for the existence of the best local approximant of f from Π^n, is that f be differentiable up to order $n-1$ at 0.

We write $f_-^{(j)}(0)/j! = p_j$ and $f_+^{(j)}(0)/j! = q_j$, $0 \leq j \leq n$. Let $P_\epsilon(f)$ be the best $\|\cdot\|_{2,\epsilon}$-best approximant of f on $[-\epsilon, \epsilon]$ from Π^n.

We assume that the best local approximant exists.

It is well known that $P_\epsilon(f) := \sum_{j=0}^{n} a_j(\epsilon)x^j$ verifies

$$\int_{[-\epsilon,\epsilon]} (f - P_\epsilon(f))(x)x^i dx = 0, \quad 0 \le i \le n. \tag{3.1}$$

Now we can write $f(x) = \sum_{j=0}^{n} p_j x^j + o(x^n)$, if $x \in [-1,0]$, and $f(x) = \sum_{j=0}^{n} q_j x^j + o(x^n)$, if $x \in [0,1]$. Therefore, from (3.1) we get for $0 \le i \le n$,

$$\sum_{j=0}^{n} \int_{[-\epsilon,0]} p_j x^{i+j} dx + \int_{[0,\epsilon]} q_j x^{i+j} dx = \sum_{j=0}^{n} a_j(\epsilon) \int_{[-\epsilon,\epsilon]} x^{i+j} dx + o(\epsilon^{n+1+i}). \tag{3.2}$$

A straightforward computation shows that (3.2) is equivalent to

$$\sum_{j=0}^{n} \frac{\epsilon^{i+j+1}}{i+j+1}[q_j - (-1)^{i+j+1}p_j] = \sum_{j=0}^{n} \frac{\epsilon^{i+j+1}}{i+j+1} a_j(\epsilon)[1 - (-1)^{i+j+1}] + o(\epsilon^{n+1+i}), \tag{3.3}$$

$0 \le i \le n$.

We shall need the following auxiliary lemma.

Lemma 3.1. *Let $\alpha \in \mathbb{R}$, $m \in \mathbb{N}$, and let $A(\alpha)$ and $A'(\alpha)$ be the $m \times m$ matrices defined by*

$$a_{ij} = \begin{cases} \frac{\epsilon^{2(i+j-1)-1}}{2(i+j-1)-1} & 1 \le i \le m,\ 1 \le j \le m-1, \\ \epsilon^{2(i+m-1)-\alpha} O_i(1) & 1 \le i \le m,\ j = m, \end{cases} \tag{3.4}$$

$$a'_{ij} = \begin{cases} \frac{\epsilon^{2(i+j)-1}}{2(i+j)-1} & 1 \le i \le m,\ 1 \le j \le m-1, \\ \epsilon^{2(i+m)-2+\alpha} O_i(1) & 1 \le i \le m,\ j = m, \end{cases} \tag{3.5}$$

respectively, where $O_i(1)$ means a bounded function as $\epsilon \to 0$. Then

$$\det(A(\alpha)) = \epsilon^{2m^2-m-\alpha+1}O(1), \quad \textit{and} \quad \det(A'(\alpha)) = \epsilon^{2m^2+m-1+\alpha}O(1).$$

If $a_{i,m} = \frac{\epsilon^{2(i+m-1)-\alpha}}{2(i+m-1)-\alpha}$, $1 \le i \le m$, then $O(1) = K(\alpha)$, and if $a'_{i,m} = \frac{\epsilon^{2(i+m)-2+\alpha}}{2(i+m)-2+\alpha}$, $1 \le i \le m$, then $O(1) = K'(\alpha)$, where $K(\alpha)$ and $K'(\alpha)$ are real numbers not dependent on ϵ. In addition, $K(2) > 0$ and $K'(0) > 0$.

Proof. We denote B_{ij} the matrix $A(\alpha)$ where we have omitted the i-th file and the j-th column. If we develop the determinant of $A(\alpha)$ by the last column, we obtain

$$\det(A(\alpha)) = \sum_{i=1}^{m} (-1)^{i+m} a_{im} \det(B_{im}) = \sum_{i=1}^{m} \epsilon^{2(i+m-1)-\alpha} O_i(1) \det(B_{im}). \tag{3.6}$$

If p denotes an arbitrary permutation of $\{1, 2, \ldots, m-1\}$, we have

$$\begin{aligned}\det(B_{im}) &= \sum_p \operatorname{sgn}(p)\, a_{1p_1} \dots a_{(i-1)p_{i-1}} a_{(i+1)p_i} \dots a_{mp_{m-1}} \\ &= \epsilon^{2m^2-3m-2i+3} C_i,\end{aligned} \tag{3.7}$$

where C_i, $1 \le i \le m$, are real numbers not dependent on ϵ and α.

From (3.6) and (3.7) it follows that $\det(A(\alpha)) = \epsilon^{2m^2-m-\alpha+1}O(1)$. Following the same patterns as before we can show that $\det(A'(\alpha)) = \epsilon^{2m^2+m-1+\alpha}O(1)$.

Next we suppose $a_{i,m} = \frac{\epsilon^{2(i+m-1)-\alpha}}{2(i+m-1)-\alpha}$, $1 \le i \le m$. Clearly, we have that the functions $O_i(1)$, $1 \le i \le m$, in (3.4) are not dependent on ϵ. In consequence, we can write $\det(A(\alpha)) = \epsilon^{2m^2-m-\alpha+1}K(\alpha)$. Analogously, $\det(A'(\alpha)) = \epsilon^{2m^2+m-1+\alpha}K'(\alpha)$.

On the other hand, $K(2)$ is the determinant of a sub-matrix of a Hilbert matrix. It is well known that a Hilbert matrix is totally positive, i.e., every sub-matrix has a positive determinant, so $K(2) > 0$. Analogously, $K'(0) > 0$. □

Theorem 3.2. *Let f be a function with lateral derivatives up to order n at 0 and suppose that there exists the best local approximant of f from Π^n. Then f is differentiable up to order $n-1$ at 0.*

Proof. By hypothesis (3.3) holds. In particular, if $i = 1$ we obtain

$$\sum_{j=0}^{n} \frac{\epsilon^{2+j}}{2+j}[q_j - (-1)^{2+j}p_j] = \sum_{j=1, j \text{ odd}}^{n} \frac{\epsilon^{2+j}}{2+j} 2a_j(\epsilon) + o(\epsilon^{n+2}). \tag{3.8}$$

Since we have assumed the existence of the best local approximant, the coefficients $a_j(\epsilon)$, $1 \le j \le n$, are uniformly bounded as $\epsilon \to 0$. Therefore, from 3.8 we get $q_0 - p_0 = O(\epsilon)$, i.e., $p_0 = q_0$.

Next, we proceed with an inductive argument. Suppose that $q_j = p_j$ for all $0 \le j < s$ with $s \le n-1$, and we shall show that $q_s = p_s$. First we assume s odd, say $s = 2l-1$, $l \in \mathbb{N}$, and we consider the sub-system of (3.3) for the values $i = 0, 2, \dots, s+1$, where we omit the columns corresponding to $q_j - p_j$, $j = 1, 3, \dots, s-2$, i.e., we consider the linear equations system in the variables $q_j + p_j - 2a_j(\epsilon)$, for j even, $0 \le j \le s$, and $q_s - p_s$. Since the coefficients $a_j(\epsilon)$, $s+1 \le j \le n$, are uniformly bounded as $\epsilon \to 0$, the principal matrix and the matrix of non-homogeneous terms now are of the form

$$\begin{pmatrix} \epsilon & \frac{\epsilon^3}{3} & \cdots & \frac{\epsilon^s}{s} & \frac{\epsilon^{s+1}}{s+1} \\ \frac{\epsilon^3}{3} & \frac{\epsilon^5}{5} & \cdots & \frac{\epsilon^{s+2}}{s+2} & \frac{\epsilon^{s+3}}{s+3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\epsilon^{s+2}}{s+2} & \frac{\epsilon^{s+4}}{s+4} & \cdots & \frac{\epsilon^{2s+1}}{2s+1} & \frac{\epsilon^{2s+2}}{2s+2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \epsilon^{s+2}O(1) \\ \epsilon^{s+4}O(1) \\ \vdots \\ \epsilon^{2s+3}O(1) \end{pmatrix}.$$

By the Cramer rule and Lemma 3.1, (3.4), for $m = l + 1$, $\alpha_1 = 1$, and $\alpha_2 = 2$ we get

$$q_s - p_s = \frac{\det(A(\alpha_1))}{\det(A(\alpha_2))} = \frac{O(1)\epsilon}{K(\alpha_2)},$$

i.e., $q_s = p_s$.

Finally, if s is even, i.e., $s = 2l$, $l \in \mathbb{N}$, we consider the sub-system of (3.3) for the values $i = 1, 3, \ldots, s + 1$, where we omit the columns corresponding to $q_j - p_j$, $j = 0, 2, \ldots, s - 2$. Then we proceed inductively in a similar way to the odd case, using Lemma 3.1, (3.5), for $m = l + 1$, $\alpha_1 = 1$, and $\alpha_2 = 0$. □

References

[1] A. P. Calderon and A. Zygmund. *Local properties of solutions of elliptic partial differential equations.* Studia Math. **20** (1961), 171–225.

[2] C. Chui, H. Diamond and R. Raphael. *On best approximation.* Approx. Theory Appl., **1**(1) (1984), 37–56.

[3] C. Chui, O. Shisha and P. Smith. *Best local approximation.* J. Approx. Theory, **15** (1975), 371–381.

[4] H. Cuenya, S. Favier, F. Levis and C. Ridolfi. *Weighted best local $\|\cdot\|$-approximation in Orlicz spaces.* Jaén J. Approx., **2**(1) (2010), 113–127.

[5] H. Cuenya and F. Levis. *Pólya-type polynomial inequalities in L^p spaces and best local approximation.* Num. Funct. Anal. Optim., **26** (2005), 813–827.

[6] S. Favier. *Convergence of function averages in Orlicz spaces.* Num. Funct. Anal. Optim., **15**(3–4) (1994), 263–278.

[7] S. Favier and C. Ridolfi. *Weighted best local approximation in Orlicz spaces.* Anal. Theory. Appl., **24**(3) (2008), 225–236.

[8] M. Marano. Mejor aproximación local, Ph. D. Dissertation, Universidad Nacional de San Luis. Argentina (1986).

[9] J. Rice. *The Approximation of Functions.* Vol. I. Addison-Wesley (1964).

[10] N. Yanzón and F. Zó. *Best local approximations by abstract norms with non-homogeneous dilations.* Rev. Un. Mat. Argentina, **49**(2) (2008), 81–96.

[11] F. Zó and H. Cuenya. *Best approximations on small regions. A general approach.* Advanced courses of mathematical analysis. II, 193–213, World Sci. Publ. (2007).

Héctor H. Cuenya
Departamento de Matemática, Universidad Nacional de Río Cuarto
5800 Río Cuarto, Argentina
hcuenya@exa.unrc.edu.ar

Claudia N. Rodriguez
Departamento de Matemática, Universidad Nacional de Río Cuarto
5800 Río Cuarto, Argentina
crodriguez@exa.unrc.edu.ar

Received: May 10, 2011
Accepted: July 23, 2012