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ON MIXED BRIGHTNESS-INTEGRALS

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ABSTRACT. We establish the greatest upper bound for the product of the i-th brightness-integrals of a convex body and its polar dual. Further, the greatest upper bound for the product of the brightness-integrals of order p of a convex body and its polar dual is also given.

1. Introduction

Polar dual convex bodies are useful in geometry of numbers [1], Minkowski geometry [2, 3] and differential equations [4]. Chakerian [5] uses polar duals to discuss self-circumference of unit circles in a Minkowski plane. The upper bound for the product of volumes of a convex body and its polar dual is the well-known *Blaschke-Santaló inequality*, as follows.

If K is a convex body with centroid at the origin, then

$$V(K)\,V(K^*) \leq \omega_n^2, \tag{1.1}$$

with equality if and only if K is an ellipsoid, where K^* is the polar dual of K and ω_n is the volume of the unit ball.

The Blaschke-Santaló inequality is due to Blaschke [6] for $n = 2, 3$ and Santaló [7] for $n \geq 2$ (see also the comments of Schneider [8]). For a good discussion of the Blaschke-Santaló inequality and a further list of references, see Lutwak [9].

On the lower bound, Steinhardt [10] showed that for plane convex bodies,

$$W_1(K)\,W_1(K^*) \geq \omega_2^2 \qquad \text{or} \qquad S(K)\,S(K^*) \geq 4\omega_2^2,$$

where K is a plane convex body in 2-dimension and $S(K)$ is the surface area of K. Chai and Lee [11] also found a lower bound of $W_1(K)\,W_1(K^*)$ for all convex bodies K. On the other hand, Lutwak [12] (also see Ghandehari [13]) found a lower bound of $W_{n-1}(K)\,W_{n-1}(K^*)$ for all convex bodies K as follows:

$$W_{n-1}(K)\,W_{n-1}(K^*) \geq \omega_n^2,$$

with equality if and only if K is a ball (centered at the origin). This was obtained by Firey [14] for dimensions 2 and 3.

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However, the problem of finding the lower bound of the product $W_i(K)\,W_i(K^*)$ for all convex bodies, for each i, is not solved completely yet. This is a open problem in Lutwak [12] and Ghandehari [13]. Also see Bambah [15], Dvoretzky and Rogers [16], Firey [17], Guggenheimer [18, 19], Heil [20], and Steinhardt [10] for partial results.

For convex bodies K_i $(i = 1,\dots,n)$, Lutwak [21] defined the mixed width-integral

$$A(K_1,\dots,K_n) = \frac{1}{n}\int_{S^{n-1}} b(K_1,u)\cdots b(K_n,u)\,dS(u),$$

where $b(K,u) = \frac{1}{2}(h(K,u) + h(K,-u))$ is half the width of convex body K in the direction u and $h(K,u)$ is the support function.

Similarly, for convex bodies K_i $(i = 1,\dots,n)$, the mixed brightness-integral was defined by (see [22])

$$C(K_1,\dots,K_n) = \frac{1}{n}\int_{S^{n-1}} \delta(K_1,u)\cdots\delta(K_n,u)\,dS(u), \tag{1.2}$$

where $\delta(K,\cdot) = \frac{1}{2}h(\mathbf{\Pi}K,u)$ is half the brightness of convex body K in the direction u, $\mathbf{\Pi}K$ is the projection body of K and $\rho(K,u)$ is the radial function.

Just as the i-th width-integrals, $B_i(K)$, are defined to be the special mixed width-integrals $A(\underbrace{K,\dots,K}_{n-i},\underbrace{B,\dots,B}_{i})$, the i-th brightness-integrals $C_i(K)$ can be defined as the special mixed brightness-integrals $C(\underbrace{K,\dots,K}_{n-i},\underbrace{B,\dots,B}_{i})$.

In the paper, we discuss a similar question about the product $C(K)\,C(K^*)$ by using the idea of [23]. We establish the greatest upper bound of the product $C_i(K)\,C_i(K^*)$ as follows.

If K is a convex body with centroid at the origin and K^* its polar dual, $1/p + 1/q = 1$, and $0 < p < 1$, then

$$C_i(K)^{1/q}\,C_i(K^*)^{1/p} \le \frac{\omega_n}{2^n}\prod_{j=1}^{n} R^{1/q}(R^*)^{1/p}, \tag{1.3}$$

with equality if and only if K is a n-ball. Here, R and R^* are the out-radius of $\mathbf{\Pi}(K)$ and $\mathbf{\Pi}^*(K)$, respectively.

For a real number, Lutwak [21] also defined the mixed width-integral of order p $(p \ne 0)$ by

$$A_p(K_1,\dots,K_n) = \omega_n\left[\frac{1}{n\omega_n}\int_{S^{n-1}} b(K_1,u)^p\cdots b(K_n,u)^p\,dS(u)\right]^{1/p},$$

where K_i $(i = 1,\dots,n)$ are convex bodies.

For p equal to $-\infty$, 0 or ∞ the mixed width-integral of order p was defined by

$$A_p(K_1,\dots,K_n) = \lim_{s\to p} A_s(K_1,\dots,K_n).$$

Similarly, the mixed brightness–integral of p ($p \neq 0$) order was defined by (see [22])

$$C_p(K_1,\ldots,K_n) = \omega_n \left[\frac{1}{n\omega_n}\int_{S^{n-1}} \delta(K_1,u)^p \cdots \delta(K_n,u)^p \, dS(u)\right]^{1/p}, \tag{1.4}$$

where K_i ($i = 1,\ldots,n$) are convex bodies.

The mixed brightness-integrals of order p, $C_p(\underbrace{K,\ldots,K}_{n-i},\underbrace{B,\ldots,B}_{i})$, will be written as $C_{p,i}(K)$ and called as i-th brightness-integrals of order p.

Another aim of the paper is to establish the greatest upper bound of the product $C_{p,i}(K)\, C_{p,i}(K^*)$ by using the idea of [23].

If K is a convex body with centroid at the origin and K^* its polar dual, for $s \neq 0$, $1/p + 1/q = 1$, and $0 < p < 1$, then

$$C_{s,i}(K)^{1/q}\, C_{s,i}(K^*)^{1/p} \leq \frac{\omega_n}{2^n} \prod_{j=1}^{n} R^{1/q}(R^*)^{1/p}, \tag{1.5}$$

with equality if and only if K is a n-ball. Here, R and R^* are the out-radius of $\mathbf{\Pi}K$ and $\mathbf{\Pi}K^*$, respectively.

This is just special case of Theorem 3.2 stated in Section 3.

2. Preliminaries

The setting for this paper is n-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$). Let $\mathcal{C}^n$ denote the set of non-empty convex figures (compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathcal{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors and 0 belongs to the interior) in $\mathbb{R}^n$. We reserve the letter u for unit vectors, and the letter B is reserved for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. We will use K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u.

By a convex body in $\mathbb{R}^n$, $n \geq 2$, we mean a compact convex subset of $\mathbb{R}^n$ with nonempty interior. A set R is said to be centered if $-x \in R$ whenever $x \in R$, and centrally symmetric if there is a vector c such that the translate $R - c$ of R by $-c$ is centered. The in-radius and out-radius of a convex body K with respect to B are defined to be the largest scalar for which a homothet of B is contained in K, and the smallest scalar for which a homothet of B contains K, respectively. For each direction $u \in S^{n-1}$, we define the support function $h(K,u)$ on S^{n-1} of the convex body K by

$$h(K,u) = \max\{u \cdot x \mid x \in K\},$$

and the radial function $\rho(K,u)$ on S^{n-1} of the convex body K is

$$\rho(K,u) = \max\{\lambda > 0 \mid \lambda u \in K\}.$$

Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$,

$$\delta(K,L) = |h_K - h_L|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

The polar dual of a convex body K that contains the origin in its interior, denoted by K^*, is another convex body defined by

$$K^* = \{y \mid x \cdot y \leq 1, \text{ for all } x \in K\}.$$

The polar dual has the following well known property:

$$h(K^*, u) = \frac{1}{\rho(K, u)} \qquad \text{and} \qquad \rho(K^*, u) = \frac{1}{h(K, u)}. \tag{2.1}$$

The outer parallel set of K at the distance $\lambda > 0$, K_λ, is given by

$$K_\lambda = K + \lambda B.$$

Then the volume $V(K_\lambda)$ is a polynomial in λ whose coefficients $W_i(K)$ are geometric invariants of K:

$$V(K + \lambda B) = \sum_{i=1}^{n} \binom{n}{i} W_i(K)\, \lambda^i. \tag{2.2}$$

The functionals $W_i(K)$ $(i = 0, \ldots, n)$ are called the i-th quermassintegrals of K. The following are true:

$$W_0(K) = V(K); \qquad nW_1(K) = S(K); \qquad W_n(K) = \omega_n,$$

where $V(K)$ and $S(K)$ are the volume and surface area of K, respectively, and ω_n is the volume of the unit ball B in $\mathbb{R}^n$. If $K_1, \ldots, K_r$ are convex bodies in $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_r$ range over the positive real numbers, then the volume of $\lambda_1 K_1 + \cdots + \lambda_r K_r$ is a homogeneous polynomial, of degree n, in $\lambda_1, \ldots, \lambda_r$. That is,

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum V(K_{i_1}, \ldots, K_{i_n})\, \lambda_{i_1} \cdot \cdots \cdot \lambda_{i_n}, \tag{2.3}$$

where $i_1, \ldots, i_n$ range independently over $1, \ldots, r$. The coefficients $V(K_{i_1}, \ldots, K_{i_n})$, depending on $K_1, \ldots, K_n$, are symmetric in their variables. This coefficient is called mixed volumes of $K_{i_1}, \ldots, K_{i_n}$. It follows from (2.2) and (2.3) that

$$W_i(K) = V(\underbrace{K, \ldots, K}_{n-i}, \underbrace{B, \ldots, B}_{i}).$$

If $K_i \in \mathcal{K}^n$ $(i = 1, 2, \ldots, n-1)$, then the mixed projection body of K_i $(i = 1, 2, \ldots, n-1)$ is denoted by $\mathbf{\Pi}(K_1, \ldots, K_{n-1})$, and its support function is given, for $u \in S^{n-1}$, by

$$h(\mathbf{\Pi}(K_1, \ldots, K_{n-1}), u) = v(K_1^u, \ldots, K_{n-1}^u). \tag{2.4}$$

The mixed projection body $\mathbf{\Pi}(K_1, \ldots, K_{n-1})$ is centered.

We use $\mathbf{\Pi}^*(K_1, \ldots, K_{n-1})$ to denote the polar dual of $\mathbf{\Pi}(K_1, \ldots, K_{n-1})$, and call it polar body of mixed projection body of K_i $(i = 1, 2, \ldots, n-1)$. If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\mathbf{\Pi}^*(K_1, \ldots, K_{n-1})$ will be written as $\mathbf{\Pi}_i^*(K, L)$. If $L = B$, then $\mathbf{\Pi}_i^*(K, B)$ is denoted by $\mathbf{\Pi}_i^* K$. We write $\mathbf{\Pi}_0^* K$ as $\mathbf{\Pi}^* K$. We will simply write $\mathbf{\Pi}_i^* K$ and $\mathbf{\Pi}^* K$ rather than $(\mathbf{\Pi}_i K)^*$ and $(\mathbf{\Pi} K)^*$, respectively.

3. Main results

Theorem 3.1 *If K_i $(i = 1, \dots, n)$ are convex bodies with centroid at the origin, $1/p + 1/q = 1$, and $0 < p < 1$, then*

$$C(K_1, \dots, K_n)^{1/q}\, C(K_1^*, \dots, K_n^*)^{1/p} \le \frac{\omega_n}{2^n} \prod_{j=1}^{n} R_j^{1/q}\, (R_j^*)^{1/p}, \tag{3.1}$$

with equality if and only if K_j $(j = 1, \dots, n)$ are n-balls centered at the origin. Here, R_j and R_j^* are the out-radius of $\mathbf{\Pi} K_j$ and $\mathbf{\Pi} K_j^*$ $(j = 1, \dots, n)$, respectively.

Proof. From (1.2) we have

$$C(K_1, \dots, K_n) = \frac{1}{n} \int_{S^{n-1}} \delta(K_1, u) \cdots \delta(K_n, u)\, dS(u) \tag{3.2}$$

and

$$C(K_1^*, \dots, K_n^*) = \frac{1}{n} \int_{S^{n-1}} \delta(K_1^*, u) \cdots \delta(K_n^*, u)\, dS(u). \tag{3.3}$$

From (3.2), (3.3) and by using Hölder's inequality for integral, and in view of the following fact:

$$h(\mathbf{\Pi} K_j^*, u) \le R_j^*,\ \ h(\mathbf{\Pi} K_j, u) \le R_j,\ \ j = 1, \dots, n, \tag{3.4}$$

with equality if and only if $\mathbf{\Pi} K_j$ and $\mathbf{\Pi} K_j^*$ $(j = 1, \dots, n)$ are n-balls centered at the origin, respectively, then

$$\begin{aligned}
&C(K_1, \dots, K_n)^{1/q}\, C(K_1^*, \dots, K_n^*)^{1/p} \\
&\quad = \frac{1}{n} \left(\int_{S^{n-1}} \frac{h(\mathbf{\Pi} K_1, u)}{2} \cdots \frac{h(\mathbf{\Pi} K_n, u)}{2}\, dS(u) \right)^{1/q} \\
&\quad\quad \times \left(\int_{S^{n-1}} \frac{h(\mathbf{\Pi} K_1^*, u)}{2} \cdots \frac{h(\mathbf{\Pi} K_n^*, u)}{2}\, dS(u) \right)^{1/p} \\
&\quad \le \frac{1}{n 2^n} \int_{S^{n-1}} \left(\prod_{j=1}^{n} (h(\mathbf{\Pi} K_i, u))^{1/q} (h(\mathbf{\Pi} K_i^*, u))^{1/p} \right) dS(u) \qquad (3.5) \\
&\quad \le \frac{1}{n 2^n} \prod_{j=1}^{n} R_j^{1/q} (R_j^*)^{1/p} \int_{S^{n-1}} dS(u) \\
&\quad = \frac{\omega_n}{2^n} \prod_{j=1}^{n} R_j^{1/q} (R_j^*)^{1/p}.
\end{aligned}$$

In view of the equality conditions of (3.4) and the equality condition of Hölder's inequality, it follows that the equality holds if and only if K_i $(i = 1, \dots, n)$ are n-balls centered at the origin.

Remark 3.1 Taking for $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$ in (3.1), we obtain the following result: If K is a convex body with centroid at the origin, $1/p + 1/q = 1$, and $0 < p < 1$, then

$$C_i(K)^{1/q}\, C_i(K^*)^{1/p} \le \frac{\omega_n}{2^n} \prod_{j=1}^{n} R^{1/q} (R^*)^{1/p}, \tag{3.6}$$

with equality if and only if K is a n-ball. Here, R and R^* are the out-radius of $\mathbf{\Pi} K$ and $\mathbf{\Pi} K^*$, respectively.

This is just inequality (1.3) stated in the Introduction.

Theorem 3.2 *If K_i $(i = 1, \dots, n)$ are convex bodies with centroid at the origin, $s \neq 0$, $1/p + 1/q = 1$, and $0 < p < 1$, then*

$$C_s(K_1, \dots, K_n)^{1/q}\, C_s(K_1^*, \dots, K_n^*)^{1/p} \le \omega_n \prod_{j=1}^{n} R_j^{1/q} (R_j^*)^{1/p}, \tag{3.7}$$

with equality if and only if K_j $(j = 1, \dots, n)$ are n-balls centered at the origin. Here, R_j and R_j^* are the out-radius of $\mathbf{\Pi} K_j$ and $\mathbf{\Pi} K_j^*$ $(j = 1, \dots, n)$ respectively.

Proof. From (1.4), we have

$$C_s(K_1, \dots, K_n) = \omega_n \left[\frac{1}{n\omega_n} \int_{S^{n-1}} \delta(K_1, u)^s \cdots \delta(K_n, u)^s \, dS(u) \right]^{1/s}, \tag{3.8}$$

and

$$C_s(K_1^*, \dots, K_n^*) = \omega_n \left[\frac{1}{n\omega_n} \int_{S^{n-1}} \delta(K_1^*, u)^s \cdots \delta(K_n^*, u)^s \, dS(u) \right]^{1/s}. \tag{3.9}$$

Hence

$$\begin{aligned} &C_s(K_1, \dots, K_n)^{1/q}\, C_s(K_1^*, \dots, K_n^*)^{1/p} \\ &\quad = \omega_n \left(\frac{1}{n\omega_n} \right)^{1/s} \left\{ \left(\int_{S^{n-1}} \delta(K_1, u)^s \cdots \delta(K_n, u)^s \, dS(u) \right)^{1/q} \right. \\ &\qquad \left. \times \left(\int_{S^{n-1}} \delta(K_1^*, u)^s \cdots \delta(K_n^*, u)^s \, dS(u) \right)^{1/p} \right\}^{1/s}. \end{aligned} \tag{3.10}$$

By using the Hölder inequality on the right side of (3.10), we have

$$\begin{aligned} &C_s(K_1, \dots, K_n)^{1/q}\, C_s(K_1^*, \dots, K_n^*)^{1/p} \\ &\qquad \le \omega_n \left(\frac{1}{n\omega_n} \right)^{1/s} \left(\int_{S^{n-1}} \delta(K_1, u)^{s/q} \cdots \delta(K_n, u)^{s/q} \right. \\ &\qquad\quad \left. \times\, \delta(K_1^*, u)^{s/p} \cdots \delta(K_n^*, u)^{s/p} \, dS(u) \right)^{1/s}. \end{aligned} \tag{3.11}$$

On the other hand, by using the definition of $\delta(K, u)$ and in view of the following fact:

$$h(\mathbf{\Pi} K_j^*, u) \le R_j^*, \quad h(\mathbf{\Pi} K_j, u) \le R_j, \quad j = 1, \dots, n,$$

with equality if and only if $\mathbf{\Pi}K_j$ and $\mathbf{\Pi}K_j^*$ $(j = 1, \ldots, n)$ are n-balls centered at the origin, respectively, then

$$\begin{aligned}
\int_{S^{n-1}} &\delta(K_1, u)^{s/q} \cdots \delta(K_n, u)^{s/q}\, \delta(K_1^*, u)^{s/p} \cdots \delta(K_n^*, u)^{s/p}\, dS(u) \\
&= \int_{S^{n-1}} \prod_{j=1}^{n} \left[\frac{1}{2} h(\mathbf{\Pi}K_j, u)\right]^{s/q} \prod_{j=1}^{n} \left[\frac{1}{2} h(\mathbf{\Pi}K_j^*, u)\right]^{s/p} dS(u) \\
&= \frac{1}{2^{ns}} \int_{S^{n-1}} \prod_{j=1}^{n} h(\mathbf{\Pi}K_j, u)^{s/q}\, h(\mathbf{\Pi}K_j^*, u)^{s/p}\, dS(u) \qquad (3.12) \\
&\leq \frac{n\omega_n}{2^{ns}} \prod_{j=1}^{n} R_j^{s/q} (R_j^*)^{s/p},
\end{aligned}$$

with equality if and only if K_j $(j = 1, \ldots, n)$ are n-balls centered at the origin.

Taking (3.12) to (3.11), we obtain

$$\begin{aligned}
C_s(K_1, \ldots, K_n)^{1/q} C_s(K_1^*, \ldots, K_n^*)^{1/p} &\leq \frac{\omega_n}{2^n} \left(\prod_{j=1}^{n} R_j^{s/q} (R_j)^{s/p} \right)^{1/s} \\
&= \frac{\omega_n}{2^n} \prod_{j=1}^{n} R_j^{1/q} (R_j)^{1/p}.
\end{aligned}$$

with equality if and only if K_j $(j = 1, \ldots, n)$ are n-balls centered at the origin.

Remark 3.2 Taking for $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = B$ in (3.7), we obtain the following result: If K is a convex body with centroid at the origin, $s \neq 0$, $1/p + 1/q = 1$ and $0 < p < 1$, then

$$C_{s,i}(K)^{1/q} C_{s,i}(K^*)^{1/p} \leq \frac{\omega_n}{2^n} \prod_{j=1}^{n} R^{1/q} (R^*)^{1/p},$$

with equality if and only if K is a n-ball. Here, R and R^* are the out-radius of $\mathbf{\Pi}K$ and $\mathbf{\Pi}K^*$, respectively.

This is just inequality (1.5) stated in the Introduction.

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