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A COMBINATORIAL IDENTITY AND APPLICATIONS

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Abstract. An identity for the finite sum $\sum_1^N \frac{z^n}{q^n-r}$ is given. Related sums (or series) were studied by Scherk, Clausen, Ramanujan, Shanks, Andrews, and others. We use such identity to give new formulas for $\sum_1^\infty \frac{z^n}{q^n-r}$, the Riemann zeta function and the Euler–Mascheroni constant. An irrationality result is also proved.

1. Introduction and results

The aim of this paper is to prove a finite sum identity for the sum $\sum_{n=1}^N \frac{z^n}{q^n-r}$, which seems to be new, and to give some applications of it. Related sums have been investigated by several authors. A related accelerated series has been given by Clausen (proved in print by Scherk): If $|q|>1$,

$$\sum_{k=1}^\infty \frac{1}{q^k-1} = \sum_{k=1}^\infty \frac{q^k+1}{q^{k^2}(q^k-1)}.$$

Ramanujan has discovered this and other related ones, see [4], pp. 147–149. D. Shanks developed an acceleration method in [9] in which he considered sums of the above type. In [2] G. E. Andrews used the little q-Jacobi polynomials to explain and extend Shanks' observations.

In another direction, P. Erdös proved in 1948 the irrationality of $\sum_{n=1}^\infty 1/(q^n-1)$, $1<q\in\mathbb{N}$. In [5, 6] P. Borwein proved irrationality results for $\sum_{n=1}^\infty 1/(q^n-r)$ using the Padé approximants and later this was extended by D. Duverney [7] and M. Prevost [8] to other similar series.

The following identity, which we believe is new, is the key to obtain all the results in this paper.

Theorem 1. *Define, for $1\le k\le n$,*

$$\epsilon_{n,k}=\epsilon_{n,k}(q,r,z) := \frac{(-1)^k z^{n+1} r^{k-1}}{(q^n-r)\cdots(q^{n-k+1}-r)}.\frac{(q^{k-1}-1)\cdots(q-1)}{(q^k-z)\cdots(q-z)}.$$

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Then, if $2 < N$*, one has*

$$-\sum_{n=2}^{N-1}\frac{z^n}{q^n-r}+\frac{z^N q}{(q^N-r)(z-q)}+\frac{z^2}{(q-z)(q-r)}$$
$$=\sum_{n=2}^{N-1}\frac{(-z)^n r^{n-1}q^n}{(q^n-z)\cdots(q-z)}.\frac{(q^{n-1}-1)\cdots(q-1)}{(q^n-r)\cdots(q-r)}-\sum_{k=2}^{N-1}\epsilon_{N,k}$$
$$+\frac{(-z)^N r^{N-1}}{(q^N-r)\cdots(q-r)}.\frac{(q^{N-1}-1)\cdots(q-1)}{(q^{N-1}-z)\cdots(q-z)}.$$

For the next corollary we write, as usual, γ for the Euler–Mascheroni constant and $\zeta(s)$ for the Riemann zeta function. We give some easy applications of this theorem. In what follows, as usual, $(q^{n-1}-1)\cdots(q-1)$ is equal to 1 if $n=1$.

Corollary 1. *i) Let* $1<|q|$, $|z|<|q|$ *and* $q^n\neq r$, $q^n\neq z$ *for all* $n=1,2,3,\ldots$. *Then*

$$\sum_{n=1}^{\infty}\frac{z^n}{q^n-r}=-\sum_{n=1}^{\infty}\frac{(-z)^n r^{n-1}q^n}{(q^n-z)\cdots(q-z)}.\frac{(q^{n-1}-1)\cdots(q-1)}{(q^n-r)\cdots(q-r)}.$$

ii)

$$1-\gamma=\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{2^n}{(2^n-1)}.\frac{(-1)^{j+1}j^{n-1}}{(2^n+j)(2^{n-1}+j)\cdots(2+j)}.$$

iii) If $\operatorname{Re}s>1$ *and* $0<r<1$ *then*

$$\Gamma(s)\zeta(s)\sum_{n=1}^{\infty}\frac{r^{n-1}}{n^s}=\int_0^{\infty}\Big\{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}r^{n-1}}{(1-e^{-nt})}.\frac{1}{(e^{nt}-r)\cdots(e^t-r)}\Big\}t^{s-1}dt.$$

It is immediate using the series in the right hand side of i) that, with fixed q, $1<|q|$, the sum $\sum_{n=1}^{\infty}\frac{z^n}{q^n-r}$ can be continued analytically for $z\in\mathbb{C}$; $z,r\neq q^n$. This is a well-known fact.

Also from the above formula i) one has the following corollary.

Corollary 2. *Let* f *be a real or complex function defined at natural numbers. Let* $1<|q|$ *and* $q^n\neq r,z$ *for all* $n=1,2,3,\ldots$. *Also let* $|f(n)|=O(\beta^n)$*, for some* $0<\beta$. *If*

$$g(n):=g(q,r,n)=\frac{r^{n-1}q^n(q^{n-1}-1)\cdots(q-1)}{(q^n-r)\cdots(q-r)},$$

then

$$\sum_{n=1}^{\infty}\frac{z^n}{q^n-r}\Big\{\sum_{m=1}^{\infty}\frac{f(m)z^m}{q^{nm}(q^m-z)\cdots(q-z)}\Big\}$$
$$=\sum_{k=2}^{\infty}\frac{(-z)^k}{(q^k-z)\cdots(q-z)}\Big\{\sum_{m=1}^{k-1}(-1)^{m+1}f(m)g(k-m)\Big\}.$$

Finally we apply Theorem 1 to prove an irrationality result. We define for $2 < N < M$:

$$\alpha_{N,M} = \alpha_{N,M}(q,r,z)$$
$$:= \sum_{n=N}^{M-1} \frac{z^n}{q^n - r} + \frac{q}{q-z}\Big\{ -\frac{z^N}{q^N - r} + \frac{z^M}{q^M - r}\Big\} - \sum_{k=2}^{M-1} \epsilon_{M,k} + \sum_{k=2}^{N-1} \epsilon_{N,k},$$

where $\epsilon_{n,k}$ is defined in Theorem 1.

Theorem 2. *Let r, q, z be non zero integers such that $1 < |q|$, $q^n \neq r, z$ for all $n = 1, 2, 3, \ldots$. Assume that:*

i) there exists a fixed number $\delta > 0$ such that $2 < N_{j-1} < M_{j-1} < N_j$ and $(\sqrt{2} + \delta)M_{j-1} \leq N_j$, for $j \geq j_0 \geq 2$.

ii) $N_j = o(M_{j-1}^2)$ as $j \to \infty$.

iii) $f(i) : \mathbb{N} \to \mathbb{Z}$ is a bounded function such that there exist infinitely many indices i such that $f(i) \neq 0$.

Then

$$\sum_{i=1}^{\infty} f(i)\, \alpha_{N_i,M_i}$$

is an irrational number.

Example. Let $\delta > 0$ be a fixed number. Let $[\cdot], \mu(\cdot)$ denote the nearest integer function and the Möbius function, respectively. Set

$$M_j := [(\sqrt{2} + 3\delta)^j],$$
$$N_j := [M_{j-1}(\sqrt{2} + 2\delta)],$$
$$f(i) := \mu(i).$$

Then it easy to see that the hypotheses of Theorem 2 hold. Thus if r, q, z are non zero integers such that $1 < |q|$, $q^n \neq r, z$ for all $n = 1, 2, 3, \ldots$, then

$$\sum_{i=1}^{\infty} \mu(i)\, \alpha_{N_i,M_i}$$

is an irrational number.

2. Proofs

Proof of Theorem 1. One has

$$\frac{1}{x} - \frac{b_1 \cdots b_K}{x(x+a_1)\cdots(x+a_K)} = \sum_{k=1}^{K} \frac{b_1 \cdots b_{k-1}\{x + a_k - b_k\}}{x(x+a_1)\cdots(x+a_k)}.$$

This follows by writing $A_k = \frac{b_1 \cdots b_k}{x(x+a_1)\cdots(x+a_k)}$; $A_0 = 1/x$, where the left hand side ot this formula is $A_0 - A_K$ and each summand on the right is $A_{k-1} - A_k$.

In the above formula set $x := q^n - r$, $a_k := r - rq^k$, $b_k := b_k(q, r, z) = \frac{rq^k(q^k-1)}{z-q^k}$ with the convention that $b_1 \cdots b_{k-1} = 1$ if $k = 1$ and $K := n - 1$. Multiply by z^n and add from $n = 2$ to N to give

$$\sum_{n=2}^{N} \frac{z^n}{q^n - r} - \sum_{n=2}^{N} \frac{z^n b_1 \cdots b_{n-1}}{(q^n - r)(q^n - qr) \cdots (q^n - q^{n-1} r)} = \sum_{n=2}^{N} \sum_{k=1}^{n-1} \frac{z^n b_1 \cdots b_{k-1}(q^n - q^k r - b_k)}{(q^n - r)(q^n - qr) \cdots (q^n - q^k r)}. \tag{1.1}$$

But $\frac{z^n b_1 \cdots b_{k-1}(q^n - q^k r - b_k)}{(q^n-r)(q^n-qr)\cdots(q^n-q^k r)} = \epsilon_{n,k} - \epsilon_{n-1,k}$, $(k > 1)$, if $\epsilon_{n,k}$ is defined as in the theorem. Thus

$$\begin{aligned}\sum_{n=2}^{N} \sum_{k=1}^{n-1} \frac{z^n b_1 \cdots b_{k-1}(q^n - q^k r - b_k)}{(q^n - r)(q^n - qr) \cdots (q^n - q^k r)} &= \sum_{n=3}^{N} \sum_{k=2}^{n-1} \epsilon_{n,k} - \epsilon_{n-1,k} + \sum_{n=2}^{N} \frac{z^n (q^n - qr - b_1)}{(q^n - r)(q^n - qr)} \\ &= \sum_{k=2}^{N-1} \epsilon_{N,k} - \epsilon_{k,k} + \sum_{n=2}^{N} \frac{z^n (q^n - qr - b_1)}{(q^n - r)(q^n - qr)}.\end{aligned}$$

And (1.1) is equal to

$$\sum_{n=2}^{N} \frac{z^n}{q^n - r} - \sum_{n=2}^{N} \frac{z^n b_1 \cdots b_{n-1}}{(q^n - r)(q^n - qr) \cdots (q^n - q^{n-1} r)} = \sum_{k=2}^{N-1} \epsilon_{N,k} - \epsilon_{k,k} + \sum_{n=2}^{N} \frac{z^n (q^n - qr - b_1)}{(q^n - r)(q^n - qr)}.$$

This gives, after some slight simplification, the identity:

$$\begin{aligned}\frac{qr(1-q)}{(z-q)} \sum_{n=2}^{N} \frac{z^n}{(q^n - qr)(q^n - r)} &= \sum_{n=2}^{N-1} \frac{(-z)^n r^{n-1} q^n}{(q^n - z) \cdots (q - z)} . \frac{(q^{n-1} - 1) \cdots (q - 1)}{(q^n - r) \cdots (q - r)} - \sum_{k=2}^{N-1} \epsilon_{N,k} \\ &+ \frac{(-z)^N r^{N-1}}{(q^N - r) \cdots (q - r)} . \frac{(q^{N-1} - 1) \cdots (q - 1)}{(q^{N-1} - z) \cdots (q - z)}.\end{aligned}$$

But the left hand side formula is

$$\frac{qr(1-q)}{(z-q)}\sum_{n=2}^{N}\frac{z^n}{(q^n-qr)(q^n-r)}$$
$$=\frac{q}{(z-q)}\sum_{n=2}^{N}\Big(\frac{z^n}{(q^n-r)}-\frac{z}{q}\,\frac{z^{n-1}}{(q^{n-1}-r)}\Big)$$
$$=-\sum_{n=2}^{N-1}\frac{z^n}{q^n-r}+\frac{z^Nq}{(q^N-r)(z-q)}+\frac{z^2}{(q-z)(q-r)}.$$

This ends our proof. □

Proof of Corollary 1. (i) Our identity follows from Theorem 1 letting $N\to\infty$ and noticing that $\sum_{k=2}^{N-1}\epsilon_{N,k}\to 0$, $\frac{(-z)^Nr^{N-1}}{(q^N-r)\cdots(q-r)}\cdot\frac{(q^{N-1}-1)\cdots(q-1)}{(q^{N-1}-z)\cdots(q-z)}\to 0$, if $N\to\infty$.

(ii) We use the well-known formula:

$$\gamma=\int_0^1\frac{\sum_1^\infty x^{2^n-1}}{1+x}dx=\sum_{j=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^j}{2^n+j}=1+\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^j}{2^n+j}.$$

In the last double sum use (i) of this corollary, with $q=2$, $r=-j$, $z=1$.

(iii) Expanding $\frac{1}{e^{nt}-r}$ into a geometric series and integrating with respect to t yields:

$$\Gamma(s)\zeta(s)\sum_{n=1}^{\infty}\frac{r^{n-1}}{n^s}=\int_0^\infty\sum_{n=1}^{\infty}\frac{t^{s-1}}{e^{nt}-r}dt.$$

In the last sum use (i) of this corollary, with $q=e^t$, $z=1$, $r=r$.

This ends our proof. □

Proof of Theorem 2. Write

$$\beta_n=\beta_n(q,r,z):=\frac{(-z)^nr^{n-1}q^n}{(q^n-z)\cdots(q-z)}.\frac{(q^{n-1}-1)\cdots(q-1)}{(q^n-r)\cdots(q-r)},$$

and

$$\theta:=\sum_{i=1}^{\infty}f(i)\,\alpha_{N_i,M_i}.$$

If one subtracts the identity in Theorem 1 from the same identity with the parameter N replaced by M one gets

$$\alpha_{N,M}=-\sum_{n=N}^{M-1}\beta_n+\frac{(q^N-z)}{q^N}\beta_N-\frac{(q^M-z)}{q^M}\beta_M.$$

Therefore,

$$\theta=\sum_{i=1}^{\infty}f(i)\Big\{-\sum_{n=N_i}^{M_i-1}\beta_n+\frac{(q^{N_i}-z)}{q^{N_i}}\beta_{N_i}-\frac{(q^{M_i}-z)}{q^{M_i}}\beta_{M_i}\Big\}.$$

Next we show that for each natural number $j \geq j_0$ such that $f(j) \neq 0$, there exist two integers A_j, B_j such that $0 < |B_j\theta - A_j| = o(1)$ as $j \to \infty$. This implies the irrationality of θ.

Take

$$B_j := (q^{M_{j-1}} - z) \cdots (q - z)(q^{M_{j-1}} - r) \cdots (q - r).$$

Thus

$$\begin{aligned} B_j\theta = B_j \sum_{i=1}^{j-1} f(i) \Big\{ -\sum_{n=N_i}^{M_i-1} \beta_n + \frac{(q^{N_i} - z)}{q^{N_i}} \beta_{N_i} - \frac{(q^{M_i} - z)}{q^{M_i}} \beta_{M_i} \Big\} \\ + B_j \sum_{i=j}^{\infty} f(i) \Big\{ -\sum_{n=N_i}^{M_i-1} \beta_n + \frac{(q^{N_i} - z)}{q^{N_i}} \beta_{N_i} - \frac{(q^{M_i} - z)}{q^{M_i}} \beta_{M_i} \Big\} =: A_j + \epsilon_j. \end{aligned}$$

From the definitions of B_j, β_n it is immediate that A_j, B_j are integer numbers.

To show that $0 \neq \epsilon_j = B_j\theta - A_j = o(1)$ we need first to observe that

$$\begin{aligned} \beta_n &= (-z)^n r^{n-1} q^{-\frac{n^2}{2} - \frac{n}{2}} \{c_0 + o(1)\}, \\ B_j &= q^{M_{j-1}(M_{j-1}+1)} \{c_1 + o(1)\}, \end{aligned} \tag{1.2}$$

where c_0, c_1 are non-zero constants. Also

$$\begin{aligned} \frac{\beta_{n+1}}{\beta_n} &= -zrq^{-n-1}\{1 + O(q^{-n})\}, \\ \frac{\beta_{n+m}}{\beta_n} &= (-zr)^m q^{-(nm + \frac{m(m+1)}{2})}\{1 + O(q^{-n})\}. \end{aligned} \tag{1.3}$$

Here the O symbol depends on q, r, z but does not depend on n or m.

Therefore using (1.3) one sees that

$$-\sum_{n=N_i}^{M_i-1} \beta_n + \frac{(q^{N_i} - z)}{q^{N_i}} \beta_{N_i} - \frac{(q^{M_i} - z)}{q^{M_i}} \beta_{M_i} = \beta_{N_i} \frac{z}{q^{N_i}} \Big\{ \frac{r}{q} - 1 + O(q^{-N_i}) \Big\},$$

(consider the terms β_{N_i}, β_{N_i+1}; other terms go into the O symbol) and therefore

$$\epsilon_j = B_j \sum_{i=j}^{\infty} f(i)\, \beta_{N_i} \frac{z}{q^{N_i}} \Big\{ \frac{r}{q} - 1 + O(q^{-N_i}) \Big\}.$$

Recall that f is a bounded function; using again (1.3) yields

$$\begin{aligned} \sum_{i=j}^{\infty} f(i) \frac{\beta_{N_i}}{q^{N_i}} &= f(j) \frac{\beta_{N_j}}{q^{N_j}} \Big(1 + \frac{f(j+1)\beta_{N_{j+1}}}{f(j)\beta_{N_j} q^{N_{j+1} - N_j}} + \frac{f(j+2)\beta_{N_{j+2}}}{f(j)\beta_{N_j} q^{N_{j+2} - N_j}} + \dots \Big) \\ &= f(j) \frac{\beta_{N_j}}{q^{N_j}} (1 + o(1)), \end{aligned}$$

and in a similar way

$$\sum_{i=j}^{\infty} f(i)\, \frac{\beta_{N_i}}{q^{N_i}} O(q^{-N_i}) = f(j)\, \frac{\beta_{N_j}}{q^{N_j}} o(1).$$

Using these expressions and (1.2), our term ϵ_j is

$$\begin{aligned}\epsilon_j &= B_j f(j)\,\frac{\beta_{N_j}}{q^{N_j}} z\Big\{\frac{r}{q} - 1 + o(1)\Big\}\\ &= f(j)(-z)^{N_j} r^{N_j-1} q^{-\frac{{N_j}^2}{2}-\frac{3N_j}{2}+M_{j-1}(M_{j-1}+1)} z\Big\{c_o c_1\Big(\frac{r}{q}-1\Big)+o(1)\Big\}.\end{aligned}$$

From the hypothesis one has that $(\sqrt{2}+\delta)M_{j-1} < N_j$, which gives

$$M_{j-1}^2 - \frac{N_j^2}{2} < -\delta\Big(\sqrt{2}+\frac{\delta}{2}\Big)M_{j-1}^2.$$

Therefore the hypothesis $N_j = o(M_{j-1}^2)$ yields

$$(-z)^{N_j} r^{N_j-1} q^{-\frac{{N_j}^2}{2}-\frac{3N_j}{2}+M_{j-1}(M_{j-1}+1)} = O(q^{-\delta' M_{j-1}^2}),$$

for some positive δ'. Therefore $0 \neq \epsilon_j = o(1)$ as $j \to \infty$, if $f(j) \neq 0$. This ends the proof. □

Proof of Corollary 2. Using Corollary 1 (i) one can prove that, for $m = 0, 1, 2, 3, \ldots$ one has

$$\sum_{n=1}^{\infty} \frac{z^{n+m}}{q^{nm}(q^n - r)} \cdot \frac{1}{(q^m - z)\cdots(q-z)} = \sum_{n=1}^{\infty} \frac{(-z)^{n+m}(-1)^{m+1}}{(q^{n+m}-z)\cdots(q-z)} g(n).$$

To prove this, notice that for $m = 0$ this equality is Corollary 1 (i). By induction assume that this is true for m, the inductive parameter. Make the change $z \to z/q$ and multiply everything by $\frac{z}{q-z}$. This gives the above formula for m replaced by $m+1$.

Finally, to prove our formula, multiply the above by $f(m)$ and add from m equal to 1 to infinity. This yields the formula summing over $k = n + m$ in the second sum, that is,

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-z)^{n+m}(-1)^{m+1}}{(q^{n+m}-z)\cdots(q-z)} g(n) f(m)\\ &\qquad = \sum_{k=2}^{\infty} \frac{(-z)^k}{(q^k - z)\cdots(q-z)} \Big\{\sum_{m=1}^{k-1} (-1)^{m+1} f(m) g(k-m)\Big\}.\end{aligned}$$ □

References

[1] G. E. Andrews, *The theory of partitions*, Encyclopedia of Mathematics and its applications, vol. 2, Addison-Wesley, Reading, Mass., 1976.

[2] G. Andrews, I.P. Goulden and D. Jackson, *Shanks' convergence acceleration transform, Padé approximants and partitions*, J. Combin. Theory, Ser. A **43** (1986), 70–84.

[3] G. E. Andrews, R. Askey and R. Roy, *Special functions*. Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999.

[4] B. Berndt, *Ramanujan Notebooks Part I*, Springer-Verlag, Birkhäuser, 1990.

[5] Peter B. Borwein, *On the irrationality of* $\sum(1/(q^n + r))$, J. Number Theory **37** (1991), 253–259.

[6] Peter B. Borwein, *On the irrationality of certain series*, Math. Proc. Camb. Phil. Soc. **112** (1992), 141–146.
[7] D. Duverney, *Sur l'irrationalité de* $\sum_{n=1}^{\infty} r^n/(q^n - r)$, C. R. Acad. Sci. Paris Sér. I Math. **320** (1995), 1–4.
[8] M. Prevost, *On the irrationality of* $\sum \frac{t^n}{A\alpha^n+B\beta^n}$, J. Number Theory **73** (1998) 139–161.
[9] D. Shanks, *Non-linear transformations of divergent and slowly convergent sequences*, J. Math. and Phys. **34** (1955), 1–42.

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