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# GLOBAL CONTROLLABILITY OF THE 1D SCHRÖDINGER–POISSON EQUATION

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Abstract. This paper is concerned with both the local and global internal controllability of the 1D Schrödinger–Poisson equation $i\,u_t(x,t) = -u_{xx} + V(u)\,u$, which arises in quantum semiconductor models. Here $V(u)$ is a Hartree-type nonlinearity stemming from the coupling with the 1D Poisson equation, which includes the so-called doping profile or *impurities*. More precisely, it is shown that for both attractive and repulsive self-consistent potentials —depending on the balance between the total charge and the impurities— this problem is globally internal controllable in a suitable Sobolev space.

## 1. Introduction

We are mainly concerned with the controllability of the 1-D self-consistent Schrödinger–Poisson equation

$$iu_t = -u_{xx} + 2^{-1}\Big(|x| * \big(\mathcal{D}(x) - |u|^2\big)\Big)u, \quad x \in \mathbb{R} \tag{1.1}$$

$$u(x,t_0) = u_0(x) \tag{1.2}$$

posed in the Sobolev space $\mathcal{H} = \{\varphi \in H^1 : \int \sqrt{1+x^2}|\varphi|^2 < \infty\}$. We refer to [2] for the well-posedness of the problem (1.1)–(1.2). Here, $\mathcal{D}(x)$ denotes the fixed positively charged background or *impurities*, which will be referred to as the *doping profile*, and it is assumed to be a positive regular function with compact support.

The problem of exact internal controllability of equation (1.1)–(1.2) can be described as the question of finding a control function $h \in PC(t_0,T,\mathcal{H})$ (where $PC(t_0,T,\mathcal{H})$ means piecewise continuous with respect to the time variable) and its associated state function $u \in C(t_0,T,\mathcal{H})$, such that

$$iu_t = -u_{xx} + 2^{-1}\Big(|x| * \big(\mathcal{D}(x) - |u|^2\big)\Big)u + h(x,t), \quad x \in \mathbb{R},\ t \in (t_0,T) \tag{1.3}$$

$$u(x,t_0) = u_0(x), \quad u(x,T) = u_T(x), \tag{1.4}$$

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where $T > t_0$ are given target times and $u_0$ and $u_T$ are the given initial and target states respectively. Equation (1.1) stems from quantum semiconductor models where the main difficulty of the 1D situation is the treatment of low frequencies; see [6] and references therein for semiconductor models. The problem of controllability for Schrödinger equations of nonlinear type appears often in nonlinear optics; see [7, 3]. There are several results on controllability of the Schrödinger equation; see [10] for a review on this topic. Among them we can cite a work of Illner, Lange and Teismann [4], who considered exact internal distributed controllability of the nonlinear Schrödinger equation posed on a finite interval with periodic boundary conditions, i.e. they work in the function space $H^1_{\mathrm{per}}(0,1)$:

$$\begin{aligned} &iu_t = -u_{xx} - \alpha|u|^2u + h(x,t), \quad x \in \Omega \subset \mathbb{R}^n, t \in (0,T) \\ &u(x,0) = u_0(x), \quad u(x,T) = u_T(x). \end{aligned}$$

The same three authors proved in 2006 [5] the noncontrollability for the following nonlinear Hartree equation posed in $H^2(\mathbb{R}^3)$:

$$\begin{aligned} &iu_t = -u_{xx} - \frac{1}{|x|}u + \left(|u|^2 * \frac{1}{|x|}\right)u + (E(t)\cdot x)u, \\ &u(x,0) = u_0(x), \end{aligned}$$

where $x \in \mathbb{R}^3$, $t \in [0,T]$ and $E(t) \in \mathbb{R}^3$ is the control function.

In 2009, L. Rosier and B.Y. Zhang [9] proved a local exact boundary controllability for the nonlinear Schrödinger equation

$$iu_t = -u_{xx} - \alpha|u|^2u$$

posed on $H^s(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, with either the Dirichlet boundary conditions

$$u(x,t) = h(x,t),\ x \in \partial\Omega$$

or the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu}(x,t) = h(x,t),\ x \in \partial\Omega.$$

In this paper we present an internal global exact controllability result for the problem

$$\begin{aligned} &iu_t = -u_{xx} + 2^{-1}\left(|x| * \left(\mathcal{D}(x) - |u|^2\right)\right)u, \quad x \in \mathbb{R} \\ &u(x,t_0) = u_0(x); \end{aligned}$$

that is, given $t_0 < T$ fixed, then for every $u_0, u_T \in \mathcal{H}$ there exists a piecewise continuous control $h(x,t)$ such that the nonlinear problem (1.3)–(1.4) has a unique solution $u \in C(t_0,T,\mathcal{H})$.

The rest of the paper is organized as follows. In Section 2, we prove the existence of dynamics and establish useful estimates for the related evolution. In Section 3, we first study the linear system and prove global controllability in the space $\mathcal{H}$. Then, we prove the local controllability for the nonlinear case. The nonlinear system (1.3) is proved to be globally exact controllable in $\mathcal{H}$ by an inductive argument.

## 2. Preliminaries

2.1. **Existence of dynamics.** This subsection is devoted to show that the equation

$$iu_t = -u_{xx} + 2^{-1}\Big(|x| * \big(\mathcal{D}(x) - |u|^2\big)\Big)u \tag{2.1}$$

could be split in such a way that the resultant (related) linear operator generates a semigroup. We then introduce the functions $F(x) := 2^{-1}\int \big(|x-y| - \mu(x)\big)\mathcal{D}(y)\,dy$ and $\mu(x) := \sqrt{1+x^2}$, and the operator

$$m(\varphi) := 2^{-1}\int \big(|x-y| - \mu(x)\big)|\varphi(y,t)|^2\,dy. \tag{2.2}$$

Using that

$$\begin{aligned}
&|x| * \big(\mathcal{D}(x) - |u|^2\big)\\
&\quad= \int (|x-y| - \mu(x))(\mathcal{D}(y) - |u(y,t)|^2)\,dy + \mu(x)\int(\mathcal{D}(y) - |u(y,t)|^2)\,dy\\
&\quad= \int (|x-y| - \mu(x))\mathcal{D}(y)\,dy - \int (|x-y| - \mu(x))|u(y,t)|^2\,dy\\
&\qquad+ \mu(x)\big(\|\mathcal{D}\|_{L^1(\mathbb{R})} - \|u(\cdot,t)\|^2_{L^2(\mathbb{R})}\big),
\end{aligned}$$

after a suitable rearrangement of terms, equation (2.1) becomes

$$\begin{aligned}
iu_t(x,t) = -u_{xx}(x,t) &+ 2^{-1}\mu(x)\big(\|\mathcal{D}\|_{L^1(\mathbb{R})} - \|u(\cdot,t)\|^2_{L^2(\mathbb{R})}\big)u(x,t)\\
&+ F(x)u(x,t) - m(u(x,t))u(x,t).
\end{aligned}$$

Since the total charge is constant along the trajectory, we may set the parameter $a := 2^{-1}\big(\|\mathcal{D}\|_{L^1(\mathbb{R})} - \|u(\cdot,0)\|^2_{L^2(\mathbb{R})}\big)$, which only depends on the size of the initial datum. Setting the linear operator $\widetilde{L_a}(\varphi) := -\varphi_{xx} + a\mu(x)\varphi$, equation (2.1) reads

$$iu_t(x,t) = L_a u - m(u(x,t))u(x,t), \tag{2.3}$$

where the linear operator is given by

$$L_a(\varphi) := \widetilde{L_a}(\varphi) + F(x)\varphi. \tag{2.4}$$

**Lemma 2.1** (Existence of dynamics)**.** *With the notation previously introduced, the linear operator $L_a : D(L_a) \to \mathcal{H}$ with $\overline{D(L_a)} = \mathcal{H}$ is self adjoint.*

*Proof.* It is a direct consequence of the following two claims:

(a) $\widetilde{L_a}$ is a self adjoint operator in $\mathcal{H}$.
(b) $L_a$ is a bounded perturbation of $\widetilde{L_a}$.

Claim (a) is handled as follows. For $a \geq 0$ (focusing case) we have $\{\widetilde{L_a}\varphi, \varphi\} = \|\varphi_x\|^2_{L^2} + a\|\varphi\|^2_{L^2_\mu} \geq 0$, thus $\widetilde{L_a}$ is bounded from below in $D(\widetilde{L_a}) \subseteq \mathcal{H}$ and hence $(\widetilde{L_a}, D(\widetilde{L_a}))$ is a self adjoint operator in $\mathcal{H}$.

In the defocusing situation (i.e. for $a < 0$) we start showing that $\widetilde{L_a}$ is a closed operator. Let $\varphi \in C_0^\infty(\mathbb{R})$ and $(\phi_n; \widetilde{L_a}(\phi_n)) \in D(\widetilde{L_a}) \times \mathcal{H}$ a sequence such that $(\phi_n; \widetilde{L_a}(\phi_n)) \to (\phi; \psi)$ in $\mathcal{H} \times \mathcal{H}$; since $\langle \varphi; \phi'' - \phi''_n \rangle = \langle \varphi''; \phi - \phi_n \rangle \to 0$ and $\langle \varphi; \mu(\phi - \phi_n) \rangle = \langle \mu^{1/2}\varphi; \mu^{1/2}(\phi - \phi_n) \rangle \to 0$ we thus have $\langle \varphi; \widetilde{L_a}(\phi - \phi_n) \rangle \to 0$, and consequently we conclude $\langle \varphi; \psi - \widetilde{L_a}\phi \rangle = \langle \varphi; \psi - \widetilde{L_a}\phi_n \rangle + \langle \varphi; \widetilde{L_a}(\phi_n - \phi) \rangle \to 0$. This shows that $\widetilde{L_a} : D(\widetilde{L_a}) \to \mathcal{H}$ is a closed operator.

We now set the operator $M(\varphi) := -\varphi_{xx} + \mu(x)\varphi$ acting on $D(M) \subseteq \mathcal{H}$. Since $\mu(x) \geq 1$ we deduce that $M \geq I$ (the identity operator). For $\varphi, \psi \in D(M)$ we set the (well defined) quadratic form $Q_a(\phi, \psi) := \langle \phi_x; \psi_x \rangle + a\langle \phi; \mu\psi \rangle$. We now establish two useful estimates:

$$\begin{aligned}
|Q_a(\phi; \psi)| &\leq |\langle \phi_x; \psi_x \rangle| + |a||\langle \phi; \mu\psi \rangle| \\
&\leq (1 + |a|)\langle \phi; M\psi \rangle \\
&\leq (1 + |a|)\|M^{1/2}\phi\|_{L^2}\, \|M^{1/2}\psi\|_{L^2}
\end{aligned}$$

and

$$\begin{aligned}
|Q_a(M\phi; \psi) - Q_a(\phi; M\psi)| &= |\langle \phi; [M : \widetilde{L_a}]\psi \rangle \\
&\leq (1 + |a|)|\langle \phi; \mu''\psi \rangle| + 2(1 + |a|)|\langle \mu'\phi; \psi_x \rangle| \\
&\leq 3(1 + |a|)\|M^{1/2}\phi\|_{L^2}\, \|M^{1/2}\psi\|_{L^2},
\end{aligned}$$

where we have used the identity $\|M^{1/2}\varphi\|^2 = \|\varphi_x\|^2 + \|\varphi\|^2_{L^2_\mu}$. Applying Theorem X.36' in [8] we obtain that $\widetilde{L_a}$ is an essentially self–adjoint operator in $\mathcal{H}$; since it is closed, it follows that $(\widetilde{L_a}, D(\widetilde{L_a}))$ is a self adjoint operator in $\mathcal{H}$.

We now turn to the next claim. Since

$$\begin{aligned}
\|L_a(\varphi) - \widetilde{L_a}(\varphi)\|^2_{\mathcal{H}} &= \|(F(x)\varphi)_x\|^2_{L^2} + \|F(x)\varphi\|^2_{L^2_\mu} \\
&\leq \|F'\|^2_{L^\infty}\|\varphi\|^2_{L^2} + \|F\|^2_{L^\infty}\|\varphi\|^2_{\mathcal{H}}
\end{aligned}$$

and using that $\|F'\|_{L^\infty} \leq \|\mathcal{D}\|_{L^1}$, $\|F\|_{L^\infty} \leq \|\mathcal{D}\|_{L^1_\mu}$, we get the final estimate

$$\|L_a(\varphi) - \widetilde{L_a}(\varphi)\|_{\mathcal{H}} \leq \left(2\|\mathcal{D}\|^2_{L^1_\mu} + \|\mathcal{D}\|^2_{L^1}\right)^{1/2}\|\varphi\|_{\mathcal{H}}.$$

We recall below a result concerning semi–bounded perturbations of self–adjoint operators.

**Theorem 2.1** (Kato–Rellich Theorem; see [8], Th. X.12)**.** *Suppose that $L$ is a self-adjoint operator and $B$ is a symmetric operator satisfying, for all $\phi \in D(L)$, the estimate $\|B\phi\| \leq a\|L\phi\| + b\|\phi\|$, with $0 < a < 1$ and $0 < b$. (In such a case $B$ is said to be $L$-bounded with relative bound $a$.) Then $L + B$ is self-adjoint on $D(L)$ and essentially self-adjoint on any core of $L$.*

In our case, $B = L_a - \widetilde{L_a}$, which, from previous estimates, is a bounded operator in $\mathcal{H}$. This means that $B$ is $\widetilde{L_a}$-bounded with relative bound $a = 0$. Applying the Kato–Rellich Theorem we conclude that $(L_a, D(L_a))$ is a self adjoint operator in $\mathcal{H}$. □

Since no further distinction will be made in the sequel concerning the sign of the parameter $a$, we will omit the subscript of the linear operator $L$ given by (2.4).

2.2. **Estimates.** The previous lemma guarantees that $L$ generates a group. In the sequel we will exhibit useful bounds for the related evolution.

**Lemma 2.2.** *Let $U(t)$ be the group generated by $L$ in $\mathcal{H}$. Then*

$$\|U(t)\varphi\|_{\mathcal{H}} \leq \|\varphi\|_{\mathcal{H}}\Big(1 + |t|\big(\|\mathcal{D}\|_{L^1} + 1\big)\Big).$$

*Proof.* Let $u(t) = e^{-iLt}\varphi$. We start computing $\frac{d}{dt}\big(\|u_x\|^2_{L^2} + \|u\|^2_{L^2_\mu}\big) = 2\,\mathrm{Re}\,\big\langle u_t; \mu u - u_{xx}\big\rangle_{L^2}$. Using the self-adjointness of $\widetilde{L}$ we get $\frac{d}{dt}\big(\|u_x\|^2_{L^2} + \|u\|^2_{L^2_\mu}\big) = 2\,\mathrm{Re}\,\big\langle u_x; (\mu' + iF')u\big\rangle_{L^2}$, from where we deduce the estimate

$$\begin{aligned}
\|u\|^2_{\mathcal{H}}(t) &\leq \|\varphi\|^2_{\mathcal{H}} + 2\big(\|F'\|_{L^\infty} + \|\mu'\|_{L^\infty}\big)\|\varphi\|_{L^2}\int_0^{|t|}\|u_x\|_{L^2}(s) \\
&\leq \|\varphi\|^2_{\mathcal{H}} + 2\big(\|\mathcal{D}\|_{L^1} + 1\big)\|\varphi\|_{L^2}\int_0^{|t|}\|u\|_{\mathcal{H}}(s) \\
&\leq \|\varphi\|^2_{\mathcal{H}} + 2\big(\|\mathcal{D}\|_{L^1} + 1\big)\|\varphi\|_{\mathcal{H}}\int_0^{|t|}\|u\|_{\mathcal{H}}(s).
\end{aligned}$$

The inequality is obtained by means of a standard ODE argument given by the following lemma. Details are given due to the lack of a suitable reference.

**Lemma 2.3.** *Let $y : [0,T] \to [0,+\infty)$ be an $L^1$ function satisfying the inequality $y^2(t) \leq y^2(0) + C\int_0^t y(s)\,ds$ for some constant $C > 0$. Then $y(t) \leq y(0) + Ct/2$.*

*Proof.* Let $w(t) := \int_0^t y(s)\,ds$ and $z(t) := \sqrt{y^2(0) + Cw(t)}$. Then $\dot z(t) \leq C/2$ and therefore $y(t) \leq z(t) \leq z(0) + Ct/2$. □

We then take $y(t) = \|u\|_{\mathcal{H}}(t)$ and we get the result, where $C = 2\big(\|\mathcal{D}\|_{L^1} + 1\big)\|\varphi\|_{\mathcal{H}}$. □

We now turn our attention to the non linear term in equation (2.3), and give the following estimates.

**Lemma 2.4.** *Let $\varphi, \phi \in \mathcal{H}$ and let $m(\cdot)$ be given by identity (2.2). Then the following estimates hold.*

(1) *(a)* $\|m(\varphi)\|_{L^\infty} \leq 2^{-1}\|\varphi\|^2_{L^2_\mu}$
*(b)* $\|m(\varphi) - m(\phi)\|_{L^\infty} \leq 2^{-1}\big(\|\phi\|_{L^2_\mu} + \|\varphi\|_{L^2_\mu}\big)\|\varphi - \phi\|_{L^2_\mu}$

(2) *(a)* $\|\big(m(\varphi)\big)_x\|_{L^\infty} \leq \|\varphi\|^2_{L^2}$
*(b)* $\|\big(m(\varphi) - m(\phi)\big)_x\|_{L^\infty} \leq \big(\|\phi\|_{L^2} + \|\varphi\|_{L^2}\big)\|\varphi - \phi\|_{L^2}$

(3) $\|\big(m(\varphi)-m(\phi)\big)\big(\varphi_1-\phi_1\big)\|_{\mathcal{H}} \leq \frac{3}{2}\|\varphi_1-\phi_1\|_{\mathcal{H}}\big(\|\varphi\|_{\mathcal{H}}+\|\phi\|_{\mathcal{H}}\big)\|\varphi-\phi\|_{\mathcal{H}}$

(4) $\|m(\varphi)\varphi-m(\phi)\phi\|_{\mathcal{H}} \leq \frac{3}{2}\left(\|\varphi\|_{\mathcal{H}}^2+\|\varphi\|_{\mathcal{H}}\|\phi\|_{\mathcal{H}}+\|\phi\|_{\mathcal{H}}^2\right)\|\varphi-\phi\|_{\mathcal{H}}$

*Proof.* The first and second assertions follow directly from the three estimates $||x-y|-\mu(x)| \leq \mu(y)$, $|\varphi|^2-|\phi|^2 = \varphi(\varphi^*-\phi^*)-\phi^*(\varphi-\phi)$, and $\left|\operatorname{sg}(x-y)-\mu'(x)\right| \leq 2$, where $z^*$ is the complex conjugate of $z$.

The third assertion is handled as follows:

$$\begin{aligned}
&\|\big(m(\varphi)-m(\phi)\big)\big(\varphi_1-\phi_1\big)\|_{\mathcal{H}} \\
&\quad= \Bigg(\Big(\|\big(m(\varphi)-m(\phi)\big)_x\big(\varphi_1-\phi_1\big)\|_{L^2}+\|\big(m(\varphi)-m(\phi)\big)\big(\varphi_1-\phi_1\big)_x\|_{L^2}\Big)^2 \\
&\qquad+\|\big(m(\varphi)-m(\phi)\big)\big(\varphi_1-\phi_1\big)\|_{L^2_\mu}^2\Bigg)^{1/2} \\
&\quad\leq \frac{3}{2}(\|\varphi\|_{\mathcal{H}}+\|\phi\|_{\mathcal{H}})\|\varphi-\phi\|_{\mathcal{H}}\|\varphi_1-\phi_1\|_{\mathcal{H}}.
\end{aligned}$$

A straightforward computation yields the last inequality. □

**Remark 2.1.** *Local interactions of type $|u|^{2\sigma}u$ with $0<\sigma<2$ satisfy an estimate similar to that in assertion 4 of Lemma (2.4), and therefore we can prove (3.3) for $\tilde{m}(u) := m(u)+|u|^{2\sigma}$.*

## 3. Controllability

We start this section taking into consideration the controllability of the linear problem, which in the present article means the existence of a control $h(x,t)$ such that the unique solution of the related non homogeneous linear equation

$$iu_t(x,t) = Lu(x,t)+h(x,t) \tag{3.1}$$

$$u(x,t_j) = u_j(x), \quad x\in\mathbb{R} \tag{3.2}$$

satisfies $u(x,t_k)=u_k(x)$, for given $t_k>t_j$ and $u_k(x)\in\mathcal{H}$, where $L$ is given by (2.4). Moreover, the control is given explicitly by

$$h_{\text{lin}}^{jk}(x,s) := \frac{i}{t_k-t_j}e^{-iLs}\big(e^{iLt_k}u_k(x)-e^{iLt_j}u_j(x)\big). \tag{3.3}$$

**Lemma 3.1** (Global controllability: linear case)**.** *Let $t_k>t_j$ and $u_j,u_k\in\mathcal{H}$; let the control function $h$ be given by (3.3), and let $\widetilde{w}(x,t)$ be the unique solution of the related system (3.1)–(3.2). Then $\widetilde{w}(x,t_k)=u_k(x)$.*

*Proof.* Since we have no restrictions for the control $h$ and $L$ generates a group, it is easily proved that the control (3.3) makes the solution of (3.1)–(3.2) to satisfy the required final condition.

In fact, if we use the Hilbert Uniqueness Method, we define the application $S$ as follows: let $v_0\in\mathcal{H}$ and let $v$ denote the solution of the linear and homogenous equation (3.1) with $h=0$ and initial condition $u(x,t_j)=v_0(x)$. Now, let $w$ denote the backward solution of the linear equation (3.1) with $h=v$ and final condition $w(x,t_k)=u_k(x)$, and define $S(v_0)=w(x,t_j)=e^{iL(t_k-t_j)}u_k+i(t_k-t_j)v_0$. $S$ is clearly an isomorphism, and therefore we have no need to prove an observability inequality.

We now make use of the following lemma (known as variation of parameters formula or Duhamel's formula); see [1, Lemma 4.1.1].

**Lemma 3.2.** *Let $X$ be a Banach space, $A$ an m-dissipative operator with a dense domain $D(A)$, and $U_A(t)$ the contraction semigroup generated by $A$. Let $\phi \in D(A)$ and $f \in C([0,T],X)$; we consider a solution $u \in C([0,T],D(A)) \cap C^1([0,T],X)$ of the problem $u_t(t) = Au(t) + f(t)$ for all $t \in [0,T]$, $u(0) = \phi$. Then we have the identity, valid for $t \in [0,T]$,*

$$u(t) = U_A(t)\phi + \int_0^t U_A(t-s)f(s)\,ds.$$

Using the previous identity with $X = \mathcal{H}$, $A = -iL$, and $f = -ih$, we get

$$\begin{aligned}\widetilde{w}(x,t) &= e^{-iL(t-t_j)}u_j - i\int_{t_j}^t e^{iL(s-t)}h(x,s)\,ds \\ &= \left(1 - \frac{t-t_j}{t_k-t_j}\right)e^{-iL(t-t_j)}u_j(x) + \frac{t-t_j}{t_k-t_j}e^{iL(t_k-t)}u_k(x). \end{aligned} \tag{3.4}$$

Evaluating in $t = t_k$ we obtain $\widetilde{w}(x,t_k) = u_k$. This finishes the proof. □

We thus turn to the non linear situation:

$$\begin{aligned} iu_t(x,t) &= Lu - m(u)u + h(x,t) \\ u(x,t_j) &= u_j(x), \quad x \in \mathbb{R}, \end{aligned}$$

which shall be written as the integral equation,

$$\begin{aligned} u(x,t) = e^{-iL(t-t_j)}u_j(x) &+ i\int_{t_j}^t e^{iL(s-t)}m(u(x,s))u(x,s)\,ds \\ &- i\int_{t_j}^t e^{iL(s-t)}h(x,s)\,ds \end{aligned}$$

We then set, for $t_k > t_j$ and $v \in C(t_j,t_k,\mathcal{H})$, the mappings

$$\mathcal{N}(v,t_j,r) := i\int_{t_j}^r e^{iL(s-r)}\big(m(v(s))\,v(s)\big)\,ds \tag{3.5}$$

$$h^{jk}_{\text{lin}}(x,s) := \frac{i}{t_k-t_j}e^{-iLs}\big(e^{iLt_k}u_k(x) - e^{iLt_j}u_j(x)\big) \qquad \text{(control: linear case)}$$

$$h^{jk}(x,s) := \frac{-i}{t_k-t_j}e^{-iLs}e^{iLt_k}\mathcal{N}(v,t_j,t_k) + h^{jk}_{\text{lin}}(x,s) \qquad \text{(control: non linear case)}$$

We next define $\Gamma : C(t_j,t_k,\mathcal{H}) \to C(t_j,t_k,\mathcal{H})$ as follows:

$$\Gamma(v)(t) := \mathcal{N}(v,t_j,t) - \frac{t-t_j}{t_k-t_j}e^{-iL(t-t_k)}\mathcal{N}(v,t_j,t_k) + \widetilde{w}(x,t). \tag{3.6}$$

We shall remark that any fixed point of $\Gamma$ yields the function needed to build the control $h^{jk} \in C(t_j,t_k,\mathcal{H})$. Hence, it only remains to show that $\Gamma$ has a fixed point. Let $\delta > 0$ and set $K_\delta := \{v \in C(t_j, t_k, \mathcal{H}) : v(t_j) = u_j,\ v(t_k) = u_k,\ \|v - \widetilde{w}\|_{L^\infty(t_j,t_k,\mathcal{H})} < \delta\}$, where $\widetilde{w}$ is given by (3.4). As usual, we must show that $K_\delta$ is left invariant by $\Gamma$, and also that this is a contractive mapping. With this in mind we list below some useful estimates.

**Lemma 3.3.** *Let $R > 0$ and let $u_j, u_k \in \mathcal{H}$ be such that $\max\{\|u_j\|_{\mathcal{H}}; \|u_k\|_{\mathcal{H}}\} < R$; let also $\delta > 0$ and take $v, u \in K_\delta$. Thus the following estimates hold, where $C_{jk} := 1 + \frac{t_k - t_j}{2}\big(1 + \|\mathcal{D}\|_{L^1}\big)$:*

- $\|\widetilde{w}\|_{L^\infty(t_j,t_k,\mathcal{H})} < C_{jk} R$
- $\|\Gamma(v) - \widetilde{w}\|_{L^\infty(t_j,t_k,\mathcal{H})} < 3\,|t_k - t_j|\,C_{jk}\,(C_{jk}R + \delta)^3$
- $\|\Gamma(v) - \Gamma(u)\|_{L^\infty(t_j,t_k,\mathcal{H})} < 9\,|t_k - t_j|\,C_{jk}\,(C_{jk}R + \delta)^2\,\|v - u\|_{L^\infty(t_j,t_k,\mathcal{H})}$.

*Proof.* Let $t \in [t_j, t_k]$. From the identity (3.4), taking $\mathcal{H}-$norm, and using Lemma 2.2 we get the estimate

$$\begin{aligned}
\|\widetilde{w}(t)\|_{\mathcal{H}} &\le \left(1 - \frac{t - t_j}{t_k - t_j}\right) \|e^{iL(t_j - t)} u_j\|_{\mathcal{H}} + \frac{t - t_j}{t_k - t_j} \|e^{iL(t_k - t)} u_k\|_{\mathcal{H}} \\
&\le \left(1 - \frac{t - t_j}{t_k - t_j}\right) \|u_j\|_{\mathcal{H}} \Big(1 + (t - t_j)(1 + \|\mathcal{D}\|_{L^1})\Big) \\
&\quad + \frac{t_k - t}{t_k - t_j} \|u_k\|_{\mathcal{H}} \Big(1 + (t_k - t)(1 + \|\mathcal{D}\|_{L^1})\Big) \\
&\le \left(1 - \frac{t - t_j}{t_k - t_j}\right) \|u_j\|_{\mathcal{H}} + \frac{t_k - t}{t_k - t_j} \|u_k\|_{\mathcal{H}} \\
&\quad + \frac{(t_k - t)(t - t_j)}{t_k - t_j} (1 + \|\mathcal{D}\|_{L^1}) \big(\|u_k\|_{\mathcal{H}} + \|u_j\|_{\mathcal{H}}\big) \\
&\le C_{jk} \max\{\|u_j\|_{\mathcal{H}}; \|u_k\|_{\mathcal{H}}\},
\end{aligned}$$

which proves the first assertion.

The remaining estimates rely on the identities below, valid for $u, v \in \mathcal{H}$, which follow directly from identities (3.5) and (3.6):

$$\begin{aligned}
\Gamma(v)(t) - \widetilde{w}(t) &= -i \int_{t_j}^{t} e^{iL(s-t)} \big(m(v(s))\, v(s)\big)\, ds \\
&\qquad + i \frac{t - t_j}{t_k - t_j} \int_{t_j}^{t_k} e^{iL(s-t)} \big(m(v(s))\, v(s)\big)\, ds
\end{aligned} \tag{a}$$

$$\begin{aligned}
\Gamma(v)(t) - \Gamma(u)(t) &= -i \int_{t_j}^{t} e^{iL(s-t)} \big(N(v)(s) - N(u)(s)\big)\, ds \\
&\qquad + i \frac{t - t_j}{t_k - t_j} \int_{t_j}^{t_k} e^{iL(s-t)} \big(N(v)(s) - N(u)(s)\big)\, ds,
\end{aligned} \tag{b}$$

where $N(v)(s) := m(v(s))\, v(s)$.

Identity (a), together with the estimates of Lemmas (2.2)–(2.4), allows us to write the inequality, valid for $t \in [t_j, t_k]$,

$$\begin{aligned}
\|\Gamma(v)(t) - \widetilde{w}(t)\|_{\mathcal{H}} &\leq 2 \int_{t_j}^{t_k} \|e^{iL(s-t)}\big(m(v(s))\, v(s)\big)\|_{\mathcal{H}}\, ds \\
&\leq 2 \int_{t_j}^{t_k} \|m(v(s))\, v(s)\|_{\mathcal{H}} \left(1 + (t-s)\big(\|\mathcal{D}\|_{L^1} + 1\big)\right) ds \\
&\leq 3 \int_{t_j}^{t_k} \|v(s)\|_{\mathcal{H}}^3 \left(1 + (t-s)\big(\|\mathcal{D}\|_{L^1} + 1\big)\right) ds \\
&\leq 3\|v\|^3_{L^\infty(t_j,t_k,\mathcal{H})} \int_{t_j}^{t_k} 1 + (t-s)\big(\|\mathcal{D}\|_{L^1} + 1\big)\, ds \\
&\leq 3\|v\|^3_{L^\infty(t_j,t_k,\mathcal{H})}\, (t_k - t_j) C_{jk}.
\end{aligned}$$

Since $\|v\|_{L^\infty(t_j,t_k,\mathcal{H})} \leq \|v - \widetilde{w}\|_{L^\infty(t_j,t_k,\mathcal{H})} + \|\widetilde{w}\|_{L^\infty(t_j,t_k,\mathcal{H})}$, we conclude the second estimate.

A similar reasoning leads us to the inequality

$$\begin{aligned}
&\|\Gamma(v)(t) - \Gamma(u)(t)\|_{\mathcal{H}} \\
&\quad \leq 2 \int_{t_j}^{t_k} \|e^{iL(s-t)}\big(m(v(s))\, v(s) - m(u(s))u(s)\big)\|_{\mathcal{H}}\, ds \\
&\quad \leq 2 \int_{t_j}^{t_k} \|m(v(s))\, v(s) - m(u(s))u(s)\|_{\mathcal{H}} \left(1 + (t-s)\big(\|\mathcal{D}\|_{L^1} + 1\big)\right) ds \\
&\quad \leq 9(t_k - t_j)\left(C_{jk} R + \delta\right)^2 C_{jk} \|v - u\|_{L^\infty(t_j,t_k,\mathcal{H})},
\end{aligned}$$

from where the third estimate follows easily. This finishes the proof. □

We are now in a position to present both the local and global controllability of the non linear problem

$$iu_t(x,t) = Lu - m(u)u + h(x,t) \tag{3.7}$$

$$u(x,t_j) = u_j(x), \quad x \in \mathbb{R}, \tag{3.8}$$

which, as in the linear case, means the existence of a *control* $h \in C(t_j, t_k, \mathcal{H})$ such that the related solution satisfies $u(x,t_k) = u_k(x)$. In addition, the control is given by

$$\begin{aligned}
h^{jk}(x,s) := \frac{-i}{t_k - t_j} e^{-iLs} e^{iLt_k} + i \int_{t_j}^{t_k} e^{iL(s-t_k)}\big(m(v(s))\, v(s)\big) \\
+ \frac{i}{t_k - t_j} e^{-iLs}\left(e^{iLt_k} u_k(x) - e^{iLt_j} u_j(x)\right).
\end{aligned} \tag{3.9}$$

**Theorem 3.1** (Local controllability: non linear case)**.** *Let $t_k > t_j$ be fixed, let the control be given by $h^{jk}$ as in (3.9); then there exists $\varepsilon > 0$ such that for every*

$u_j, u_k \in \mathcal{H}$ *with* $\max\{\|u_0\|_{\mathcal{H}}; \|u_T\|_{\mathcal{H}}\} < \varepsilon$ *the unique solution of (3.7)–(3.8) satisfies* $u(x, t_k) = u_k(x)$.

*Proof.* As we have stated above, the proof relies on a fixed point argument. Set $K_\delta := \{v \in C(t_j, t_k, \mathcal{H}) : v(t_j) = u_j,\ v(t_k) = u_k,\ \|v - \widetilde{w}\|_{L^\infty(t_j,t_k,\mathcal{H})} < \delta\}$, where $\widetilde{w}$ is given by (3.4), and let $v \in K_\delta$. Using the estimates given by Lemma 3.3, we get the the following sufficient conditions, where $0 < \gamma < 1$:

$$3\,|t_k - t_j|\,C_{jk}\Big(C_{jk}\varepsilon + \delta\Big)^3 < \delta$$

$$9\,|t_k - t_j|\,C_{jk}\Big(C_{jk}\varepsilon + \delta\Big)^2 \leq \gamma,$$

which are easily satisfied taking $\varepsilon \leq \delta$ and $\delta < \gamma^{1/2} 3^{-1} |t_k - t_j|^{-1/2} C_{jk}^{-1/2} \big(1 + C_{jk})\big)^{-1}$. □

Before giving the global result, we shall write down the explicit control. Let $t_0 < T$, $u_0, u_T \in \mathcal{H}$ and let $t_0 < t_1 < \cdots < t_N = T$ be a partition of $[t_0, T]$; we set, for $j = 0, \ldots, N$, the family $u_j = \widetilde{w}(t_j)$, where $\widetilde{w}(x, t_j)$ is given by (3.4), and we define, for $s \in [t_j, t_k]$, the piecewise continuous function

$$h(x, s) := h^{jk}(x, s), \tag{3.10}$$

where $h^{jk}$ is the control given by (3.3). In the next theorem we show that $h$ yields a control for the non linear problem

$$iu_t(x, t) = Lu - m(u)u + h(x, t) \tag{3.11}$$

$$u(x, t_0) = u_0(x), \quad x \in \mathbb{R} \tag{3.12}$$

$$u(x, T) = u_T(x). \tag{3.13}$$

**Theorem 3.2** (Global controllability: non linear case)**.** *Let $t_0 < T$ be fixed; then for every $u_0, u_T \in \mathcal{H}$ the piecewise continuous control $h(x,t)$ given by (3.10) is such that the nonlinear problem (3.11),(3.12),(3.13) has a unique solution $u \in C(t_0, T, \mathcal{H})$.*

*Proof.* It relies on an inductive argument. Let $N$ be an integer to be fixed, and let $\{t_0, \ldots, t_N := T\}$ be a (regular) mesh. Let also, for $j = 0, \ldots, N$, $u_j(x) = \widetilde{w}(x, t_j)$, where $\widetilde{w}(x, t_j)$ is given by (3.4); notice that $\widetilde{w}(x, t_0) = u_0$ and $\widetilde{w}(x, t_N) = u_T$. Henceforth, we shall focus in the inductive step. Let $[t_j, t_{j+1}]$ be a subinterval, let $\delta > 0$ to be fixed, and set $K_\delta^j := \{v \in C(t_j, t_{j+1}, \mathcal{H}) : v(t_j) = u_j,\ v(t_{j+1}) = u_{j+1},\ \|v - \widetilde{w}\|_{L^\infty(t_j,t_{j+1},\mathcal{H})} < \delta\}$, and $\Gamma_j : K_\delta^j \to C(t_j, t_{j+1}, \mathcal{H})$ as in (3.6). Applying inequalities of Lemma 3.3 we get the the following sufficient conditions required for the existence of a fixed point, where $\eta := |T - t_0|/N$, $R := \max\{\|u_0\|_{\mathcal{H}}, \|u_T\|_{\mathcal{H}}\}$, and $C(|T-t_0|, \|\mathcal{D}\|_{L^1}) := 1 + 2^{-1}|T-t_0|(1+\|\mathcal{D}\|_{L^1})$:

$$3\,\eta\, C(|T-t_0|, \|\mathcal{D}\|_{L^1})\Big(C(|T-t_0|, \|\mathcal{D}\|_{L^1})R+\delta\Big)^3 < \delta$$

$$9\,\eta\, C(|T-t_0|, \|\mathcal{D}\|_{L^1})\Big(C(|T-t_0|, \|\mathcal{D}\|_{L^1})R+\delta\Big)^2 \leq \gamma,$$

which are easily satisfied taking $\delta = R$ and $\eta < \gamma(3R)^{-2}\big(1+C(|T-t_0|, \|\mathcal{D}\|_{L^1})\big)^{-4}$. □

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