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UNIFORM DISTRIBUTION MODULO ONE OF SOME SEQUENCES CONCERNING THE EULER FUNCTION

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Abstract. In this paper, we follow the recent method in the theory of uniform distribution, developed by J.-M. Deshouillers and H. Iwaniec, to prove uniform distribution modulo one of various sequences involving the Euler function, together with some generalizations.

1. Introduction

A real sequence $(a_n)_{n\geqslant 1}$ is said to be uniformly distributed modulo one if for all real numbers a, b with $0 \leqslant a < b \leqslant 1$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n\leqslant N\\ \{a_n\}\in[a,b]}} 1 = b-a,$$

where as usual, by $\{x\}$ we denote the fractional part of x. At the Czech–Slovak Number Theory Conference in Smolenice in August 2007, F. Luca asked whether the sequences of general term

$$a_n = \frac{1}{n}\sum_{m\leqslant n}\varphi(m), \qquad \text{and} \qquad g_n = \Big(\prod_{m\leqslant n}\varphi(m)\Big)^{\frac{1}{n}}, \tag{1.1}$$

where φ is the Euler function, are uniformly distributed modulo one [3]. About one year later, J.-M. Deshouillers and H. Iwaniec [2] developed a method to attack this problem. Their method implies that a_n is uniformly distributed modulo one, and also g_n is uniformly distributed modulo one if and only if the number

$$c_g = \frac{1}{e}\prod_p\Big(1-\frac{1}{p}\Big)^{\frac{1}{p}} \tag{1.2}$$

is irrational, which of course seems very likely to be, but there is no known proof as we know. Indeed, Deshouillers and Iwaniec proved uniform distribution modulo one of families of arithmetical functions including the arithmetic and the geometric means of the first n values of the Euler function.

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Our aim in writing this paper is to obtain some results related to the main results of the work of Deshouillers and Iwaniec. These include uniform distribution modulo one of several sequences consisting of arithmetical functions defined by them, as well as some sequences related by the Euler function. In Section 2, we recall main results from the work of Deshouillers and Iwaniec [2], which is necessary to introduce our results, and in Section 3 and Section 4 we prove them. Before going to the next sections and introducing results and their proofs, we mention some notes.

1. First, we note that all sequences in the work of Deshouillers and Iwaniec [2], and in the present work, have the linear form $cn+o(n)$, where the leading coefficient c is irrational, or at least is assumed to be irrational. This seems to be the reason for the fact that the method of Deshouillers and Iwaniec is not applicable for the sequence $(h_n)_{n\geqslant 1}$ of the harmonic mean of the first n values of the Euler function, which is defined by

$$h_n = \frac{n}{\sum_{m\leqslant n} \frac{1}{\varphi(m)}}.$$

Indeed, Landau [7] showed the validity of the approximation

$$\sum_{m\leqslant n} \frac{1}{\varphi(m)} = A\log n + B + O\left(\frac{\log n}{n}\right),$$

with constants A and B defined by

$$A = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \approx 1.94 \qquad \text{and} \qquad B = A\Big(\gamma - \sum_p \frac{\log p}{p^2-p+1}\Big) \approx -0.06,$$

where γ is Euler's constant. This approximation implies that $h_n \asymp n/\log n$.

2. Let us explain why uniform distribution modulo one of the sequence a_n defined in (1.1), and similarly other sequences under study, is not trivial. If we write $a_n = 3n/\pi^2 + R(n)$, then $R(n) \ll (\log n)^{\frac{2}{3}}(\log\log n)^{\frac{4}{3}}$ is the best known approximation, due to Walfisz [8]. Now, we note that if we can reduce $R(n)$ up to $o(1)$, then uniform distribution modulo one of a_n becomes trivial by applying the Weyl criterion [9], which asserts that the sequence $(a_n)_{n\geqslant 1}$ is uniformly distributed modulo one if and only if, for every integer $h \geqslant 1$ we have

$$\lim_{X\to\infty} \frac{1}{X}\sum_{n\leqslant X} e(ha_n) = 0,$$

where, for the whole text, we set

$$e(x) = e^{2\pi i x}.$$

Despite the above observation, Erdös and Shapiro [4] proved that the inequalities

$$R(n) > c\log\log\log\log n \qquad \text{and} \qquad R(n) < -c\log\log\log\log n,$$

each are valid for infinitely many integers n for some $c > 0$. So, in practice we can not expect reducing $R(n)$ up to $o(1)$. In fact, the essence of the method developed by Deshouillers and Iwaniec is to cope with this problem. Their main idea is splitting $R(n)$ into some parts, such that the Weyl sum of each part admits suitable conditions of the Weyl criterion.

3. We mention that the method of Deshouillers and Iwaniec is applicable for the sequence with general term

$$a_n = \frac{1}{n} \sum_{m \leqslant n} \sigma(m), \tag{1.3}$$

where $\sigma(m) = \sum_{d|m} d$ is the sum of the positive divisors of m. In [6] we prove that the sequence $(a_n)_{n\geqslant 1}$ defined by (1.3) is uniformly distributed modulo one.

4. We propose some problems, which seem to be open as far as we know. The first one is to prove that the sequence with general term

$$\Big(\prod_{m\leqslant n} \sigma(m)\Big)^{\frac{1}{n}},$$

as well as the sequences with general terms

$$\frac{n}{\sum_{m\leqslant n} \frac{1}{\varphi(m)}}, \qquad \text{and} \qquad \frac{n}{\sum_{m\leqslant n} \frac{1}{\sigma(m)}},$$

are uniformly distributed modulo one. To formulate the second question, we recall that a sequence of real numbers $(a_n)_{n\geqslant 1}$ is said to be dense modulo one if the sequence of its fractional parts $(\{a_n\})_{n\geqslant 1}$ is dense in the interval $[0,1)$. In [1] and [5] we proved that sequences with general terms

$$\sum_{m\leqslant n} \frac{\varphi(m^2+1)}{m^2+1}, \qquad \text{and} \qquad \sum_{m\leqslant n} \frac{m^2+1}{\sigma(m^2+1)},$$

are dense modulo one. Now, the problem is to prove that both of the above sequences are uniformly distributed modulo one, too.

2. Main results

2.1. **Sequences consisting of arithmetic mean.** The following result is Theorem 1 of [2].

Theorem 2.1. *Assume that $\nu(n)$ is a completely multiplicative function which satisfies the conditions*

$$|\nu(p)| \leqslant \nu, \quad \textit{for some positive number } \nu \textit{ and every prime } p, \tag{2.1}$$

and

$$\sum_{d\leqslant x} \mu(d)\nu(d) \ll_{\nu,A} x(\log x)^{-A}, \quad \textit{for every positive } A, \tag{2.2}$$

where by $\ll_{\nu,A}$ we mean that the implied constant depends only on ν and A. We define the arithmetic function ϕ by

$$\phi(m) = m \prod_{p|m} \Big(1 - \frac{\nu(p)}{p}\Big). \tag{2.3}$$

Then, the sequence $\mathcal{A} = (a_n)_{n\geqslant 1}$ *defined by*

$$a_n = \frac{1}{n}\sum_{m\leqslant n}\phi(m) \tag{2.4}$$

is uniformly distributed modulo one, provided the number

$$c_a = \frac{1}{2}\prod_p\Big(1-\frac{\nu(p)}{p^2}\Big)$$

is irrational.

We generalize the above theorem by introducing the following result, which covers the truth of Theorem 2.1 by letting $\lambda = 1$.

Proposition 2.2. *Assume that* $\lambda \neq 0$ *is an arbitrary real number. Then, under the assumptions of Theorem 2.1, the sequence* $\mathcal{A} = (a_\lambda(n))_{n\geqslant 1}$ *defined by*

$$a_\lambda(n) = n^{1-2\lambda}\Big(\sum_{m\leqslant n}\phi(m)\Big)^\lambda$$

is uniformly distributed modulo one, provided the number c_a^λ *is irrational.*

If in the above proposition we let $\nu(p) = 1$, and $\lambda = \frac{1}{2}, -\frac{1}{2}, -1$, then we obtain some interesting results concerning the Euler function φ, as follows.

Corollary 2.3. *Let* $a_n = \frac{1}{n}\sum_{m\leqslant n}\varphi(m)$. *The sequences with general terms*

$$s_n = \sqrt{\sum_{m\leqslant n}\varphi(m)}, \qquad w_n = \frac{n^2}{s_n}, \qquad r_n = \frac{n^2}{a_n},$$

all are uniformly distributed modulo one.

2.2. **Sequences consisting of geometric mean.** The following result is Theorem 2 of [2].

Theorem 2.4. *Let* $\nu(n)$ *be a completely multiplicative function with*

$$-\nu \leqslant \nu(p) < \min\{p, \nu\} \tag{2.5}$$

for some positive number ν *and every prime* p, *and*

$$\prod_{p\leqslant n}\Big(1-\frac{\nu(p)}{p}\Big) = \beta(\log n)^{-\lambda}\Big(1+O\big(\frac{1}{\log n}\big)\Big) \tag{2.6}$$

for some positive real numbers β *and* λ, *where the implied constant depends only on* ν. *Recall the arithmetic function* $\phi(m)$ *defined by (2.3). Then, the sequence* $\mathcal{G} = (g_n)_{n\geqslant 1}$ *defined by*

$$g_n = \Big(\prod_{m\leqslant n}\phi(m)\Big)^{\frac{1}{n}} \tag{2.7}$$

is uniformly distributed modulo one, provided the number

$$c_g = \frac{1}{e} \prod_p \left(1 - \frac{\nu(p)}{p}\right)^{\frac{1}{p}} \tag{2.8}$$

is irrational.

Our first observation on the geometric mean implies an approximation for the sequence with general term g_n, defined in (1.1). Regarding this sequence, Deshouillers and Luca [3] proved that

$$g_n = c_g n + O(\log n),$$

where c_g is defined by (1.2). We improve this approximation as follows.

Corollary 2.5. *The sequence $\mathcal{G} = (g_n)_{n\geqslant 1}$ defined in* (1.1) *has the approximate expansion*

$$g_n = c_g n + \frac{c_g}{2} \log n + O(\log\log n),$$

where c_g is defined by (1.2).

Then, we prove the following results.

Proposition 2.6. *Recall the assumptions of Theorem 2.4. For any real number $\eta \neq 0$, the sequence $\mathcal{G} = (g_\eta(n))_{n\geqslant 1}$ defined by*

$$g_\eta(n) = n^{1-\eta} \left(\prod_{m\leqslant n} \phi(m)\right)^{\frac{\eta}{n}}$$

is uniformly distributed modulo one, provided the number c_g^η is irrational.

If in the above proposition we take $\eta = 1/2, -1/2$ and -1, then we get the following corollary.

Corollary 2.7. *Recall the assumptions of Theorem 2.4. The sequences with general terms*

$$s_n = \sqrt{n g_n}, \qquad w_n = \frac{n^2}{s_n}, \qquad r_n = \frac{n^2}{g_n},$$

are uniformly distributed modulo one, provided the number c_g is irrational.

Proposition 2.8. *Recall the assumptions of Theorem 2.4. For any real number $\eta \neq 0$, the sequence $\mathcal{G} = (g_\eta(n))_{n\geqslant 1}$ defined by*

$$g_\eta(n) = n^{-\eta} \left(\prod_{m\leqslant n} m^\eta \phi(m)\right)^{\frac{1}{n}}$$

is uniformly distributed modulo one, provided the number $e^{-\eta} c_g$ is irrational.

We take $\eta = -1$ and 1 in the above proposition to get the following.

Corollary 2.9. *Recall the assumptions of Theorem 2.4. The sequences with general terms*

$$h_n = \frac{n}{\left(\prod_{m\leqslant n}\frac{\phi(m)}{m}\right)^{\frac{1}{n}}}, \qquad r_n = \frac{\left(\prod_{m\leqslant n} m\phi(m)\right)^{\frac{1}{n}}}{n}$$

are uniformly distributed modulo one, provided the numbers ec_g *and* $e^{-1}c_g$ *are irrational, respectively.*

Proposition 2.10. *Recall the assumptions of Theorem 2.4. Let*

$$f(n) = an^d + O\left(n^{d-2}\log^s n\right),$$

where $a \neq 0$, d *and* s *are some real numbers. Then, for any real number* $\eta \neq 0$, *the sequence* $\mathcal{G} = (g_\eta(n))_{n\geqslant 1}$ *defined by*

$$g_\eta(n) = f(n)^{-\eta}\left(\prod_{m\leqslant n} m^{\eta d}\phi(m)\right)^{\frac{1}{n}}$$

is uniformly distributed modulo one, provided the number $a^{-\eta}e^{-\eta d}c_g$ *is irrational.*

Proposition 2.11. *Recall the assumptions of Theorem 2.4, and assume that* $\eta \neq -1$ *is an arbitrary real number. Then, the sequence* $\mathcal{G} = (g_\eta(n))_{n\geqslant 1}$ *with general term*

$$g_\eta(n) = \left(\prod_{m\leqslant n} m^{\eta}\phi(m)\right)^{\frac{1}{n(\eta+1)}}$$

is uniformly distributed modulo one, provided the number $(e^{-\eta}c_g)^{\frac{1}{\eta+1}}$ *is irrational.*

In the above proposition we let $\eta = 1$ to get the following.

Corollary 2.12. *Recall the assumptions of Theorem 2.4. Then, the sequence with general term*

$$s_n = \sqrt{\left(\prod_{m\leqslant n} m\phi(m)\right)^{\frac{1}{n}}}$$

is uniformly distributed modulo one, provided the number $e^{-1}c_g$ *is irrational.*

3. Proofs: sequences consisting of arithmetic mean

We recall some notations. Let ψ be the saw function, defined by $\psi(x) = \{x\} - 1/2$. For a real $z \geqslant 2$ we denote by $P(z)$, or simply P, the product $\prod_{p<z} p$. For real numbers $t, z \geqslant 2$ and D satisfying $P < D$ we define $\rho_t(z)$ and $\rho_t(D,z)$ by

$$\rho_t(z) = \sum_{d\mid P}\frac{\mu(d)\nu(d)}{d}\psi\Big(\frac{t}{d}\Big), \tag{3.1}$$

and

$$\rho_t(D,z) = \sum_{d\leqslant D} \frac{\mu(d)\nu(d)}{d}\psi\Big(\frac{t}{d}\Big) - \rho_t(z), \tag{3.2}$$

where ν is defined as in Theorem 2.1. The following result, which evaluates the arithmetic mean a_n defined by (2.4), is Lemma 4 of [2]. For the whole text, the symbol $B(\nu)$ denotes a constant which depends only on ν, and its value may change from one occurrence to another.

Lemma 3.1. *Recall the assumptions of Theorem 2.1. Let $2 \leqslant z \leqslant D < n$ be such that $P < D$ holds, too. Then, for any $A > 0$, we have*

$$a_n = c_a n - \rho_n(z) - \rho_n(D,z) + O_{\nu,A}\Big(\frac{D}{n}(\log D)^{B(\nu)} + \frac{n}{D}(\log D)^{-A}\Big), \tag{3.3}$$

where by $O_{\nu,A}$ we mean that the constant implied in the O term depends only on ν and A.

The following technical result gives a mean-square bound for $\rho_n(D,z)$. It is Lemma 5 and Lemma 6 of [2]. Here $\tau_\nu(d)$ denotes the generalized divisor function.

Lemma 3.2. *For any complex numbers $c(d)$ with $|c(d)| \leqslant \tau_\nu(d)$ we have*

$$\int_{-T}^{T} \Big| \sum_{z\leqslant d\leqslant D} \frac{c(d)}{d}\psi\Big(\frac{t}{d}\Big)\Big|^2 dt \ll_\nu Tz^{-1}(\log z)^{B(\nu)} + D(\log D)^{B(\nu)}, \tag{3.4}$$

and

$$\sum_{|n|\leqslant T} \Big| \sum_{z\leqslant d\leqslant D} \frac{c(d)}{d}\psi\Big(\frac{n}{d}\Big)\Big|^2 \ll_\nu Tz^{-1}(\log z)^{B(\nu)} + D(\log D)^{B(\nu)}. \tag{3.5}$$

As an immediate and useful consequence of this lemma, we can get the following corollary.

Corollary 3.3. *For any complex numbers $c(d)$ with $|c(d)| \leqslant \tau_\nu(d)$, we have*

$$\sum_{|n|\leqslant T} \Big| \sum_{z\leqslant d\leqslant D} \frac{c(d)}{d}\psi\Big(\frac{n}{d}\Big)\Big| \ll_\nu Tz^{-\frac12}(\log z)^{B(\nu)} + \sqrt{TD}(\log D)^{B(\nu)}. \tag{3.6}$$

In particular, we have

$$\sum_{n\leqslant X} |\rho_n(D,z)| \ll_\nu Xz^{-\frac12}(\log z)^{B(\nu)} + \sqrt{XD}(\log X)^{B(\nu)}. \tag{3.7}$$

Proof. We simply denote the inner sum by $\mathfrak{S}$, and then we use Cauchy's inequality, $(\sum xy)^2 \leqslant (\sum x^2)(\sum y^2)$, to write

$$\frac{1}{2T+1}\sum_{|n|\leqslant T}|\mathfrak{S}|^2 = \sum_{|n|\leqslant T}\frac{1}{(2T+1)^2}\sum_{|n|\leqslant T}|\mathfrak{S}|^2 \geqslant \Big(\sum_{|n|\leqslant T}\frac{|\mathfrak{S}|}{2T+1}\Big)^2.$$

This implies that

$$\sum_{|n|\leqslant T}|\mathfrak{S}|\leqslant\sqrt{(2T+1)\sum_{|n|\leqslant T}|\mathfrak{S}|^2}.$$

Now, we use (3.5) with the inequality $\sqrt{a+b}\leqslant\sqrt{a}+\sqrt{b}$ to get (3.6). Also, we obtain (3.7) from (3.6) by taking

$$c(d)=\begin{cases}0 & \text{if } d\mid P(z),\\ \mu(d)\nu(d) & \text{if } d\nmid P(z).\end{cases}\tag{3.8}$$

This completes the proof. □

Proof of Proposition 2.2. We consider the truth of Lemma 3.1 in the form $a_n = c_a n + R(n)$, say. Then, we have

$$a_\lambda(n)=n^{1-\lambda}a_n^\lambda=c_a^\lambda n\Big(1+\frac{R(n)}{c_a n}\Big)^\lambda=c_a^\lambda n\left(1+\frac{\lambda R(n)}{c_a n}+O\Big(\frac{\log^2 n}{n^2}\Big)\right).$$

Thus, we obtain

$$a_\lambda(n)=c_a^\lambda n+\lambda c_a^{\lambda-1}R(n)+O\Big(\frac{\log^2 n}{n}\Big).$$

Now, for any integer $h\geqslant 1$, we have

$$\sum_{n\leqslant X}e\left(ha_\lambda(n)\right)=\sum_{n\leqslant X}e\left(h\Big(c_a^\lambda n-\lambda c_a^{\lambda-1}\rho_n(z)\Big)\right)+E,$$

where

$$\begin{aligned}E&\ll\sum_{n\leqslant X}\Big\{|\rho_n(D,z)|+\frac{D}{n}(\log D)^{B(\nu)}+\frac{n}{D}(\log D)^{-A}+\frac{\log^2 n}{n}\Big\}\\&\ll\sum_{n\leqslant X}|\rho_n(D,z)|+D(\log X)^{B(\nu)}+X^2D^{-1}(\log D)^{-A}+\log^3 X,\end{aligned}$$

and the implied constant depends on λ, ν, A and h. We set $D=X(\log X)^{-c}$ for some constant $c>0$, and we use the approximation (3.7), by taking suitable values for A and c, to obtain

$$E\ll_{\nu,\lambda,h}Xz^{-\frac{1}{3}},\tag{3.9}$$

for $2\leqslant z\leqslant\log X$. On the other hand, we note that $-\lambda c_a^{\lambda-1}\rho_n(z)$ is periodic in n with period P. This fact allows us to write

$$\begin{aligned}\left|\sum_{n\leqslant X}e\left(h\Big(c_a^\lambda n-\lambda c_a^{\lambda-1}\rho_n(z)\Big)\right)\right|&\leqslant\left|\sum_{b=0}^{P-1}\sum_{\substack{n\leqslant X\\ n\equiv b\ [P]}}e\left(h\Big(c_a^\lambda n-\lambda c_a^{\lambda-1}\rho_n(z)\Big)\right)\right|\\&\leqslant\sum_{b=0}^{P-1}\left|\sum_{\substack{n\leqslant X\\ n\equiv b\ [P]}}e\left(hc_a^\lambda n\right)\right|\leqslant\frac{P}{|\sin(hc_a^\lambda P\pi)|},\end{aligned}$$

where by $n \equiv b\ [P]$ we mean $n \equiv b \pmod{P}$. Thus, by using (3.9), we obtain

$$\left|\sum_{n\leqslant X} e(ha_\lambda(n))\right| \leqslant \frac{P(z)}{|\sin(hc_a^\lambda P(z)\pi)|} + O_{\nu,\lambda,h}(Xz^{-\frac{1}{3}}).$$

By considering the assumption of irrationality of the number c_a^λ, we can find a function $\varepsilon(z,X)$ with the property that $\varepsilon(z,X)\to 0$ as $X\to\infty$, such that

$$\left|\sum_{n\leqslant X} e(ha_\lambda(n))\right| \leqslant \varepsilon(z,X)XP(z) + O_{\nu,\lambda,h}(Xz^{-\frac{1}{3}}).$$

This yields that

$$\sum_{n\leqslant X} e(ha_\lambda(n)) = o(X),$$

as $X\to\infty$. Now, the proof is completed. ☐

Proof of Corollary 2.3. We take $\nu(n)=1$ in Proposition 2.2. The sequences under study are the sequence $a_\lambda(n)$ with $\lambda = 1/2, -1/2$ and -1, respectively. Also, the corresponding characteristic constants c_a^λ are respectively $\sqrt{3}/\pi$, $\pi/\sqrt{3}$ and $3/\pi^2$. Since the set of algebraic numbers with ordinary $+$ and $\times$ forms a field, thus we imply that $\sqrt{3}/\pi$ can not be rational. This gives the required assumption of irrationality of the leading coefficients, and completes the proof. ☐

4. Proofs: sequences consisting of geometric mean

To prove Theorem 2.4, J.-M. Deshoiullers and H. Iwaniec [2] follow the same method as applied for the arithmetic mean case. We recall its main points, and then give our modified proof of the remarks. Suppose that $2\leqslant z<D\leqslant n$. We define the functions

$$\Psi_n(z) = \sum_{p<z} \psi\Big(\frac{n}{p}\Big)\log\Big(1-\frac{\nu(p)}{p}\Big),$$

and

$$\Psi_n(z,D) = \sum_{z\leqslant p<D} \psi\Big(\frac{n}{p}\Big)\log\Big(1-\frac{\nu(p)}{p}\Big).$$

We recall that $P=P(z)=\prod_{p<z} p$. The following lemma is the relation (38) of [2].

Lemma 4.1. *Recall the assumptions of Theorem 2.4. Then, for any positive integer n we have*

$$\begin{aligned} g_n &= c_g\Big(n+\frac{1}{2}\log\big(\frac{2\pi n}{\beta}\big)+\frac{\lambda}{2}\log\log n\Big) \\ &\quad - c_g\sum_{p\leqslant n}\psi\Big(\frac{n}{p}\Big)\log\Big(1-\frac{\nu(p)}{p}\Big)+O\Big(\frac{1}{\log n}\Big). \end{aligned} \tag{4.1}$$

Corollary 4.2. *Recall the assumptions of Theorem 2.4. Then, for $2 \leqslant z < D \leqslant n$, we have*

$$\begin{aligned} g_n = c_g\Big(n + \frac{1}{2}\log\big(\frac{2\pi n}{\beta}\big) + \frac{\lambda}{2}\log\log n\Big) \\ - c_g\Psi_n(z) - c_g\Psi_n(z,D) + E(n,D), \end{aligned} \tag{4.2}$$

where

$$E(n,D) \ll_\nu \log\frac{\log n}{\log D} + \frac{1}{\log n}.$$

Proof. By considering the relation (4.1), for the error term $E(n,D)$ in (4.2) we have

$$E(n,D) = -c_g \sum_{D\leqslant p\leqslant n} \psi\Big(\frac{n}{p}\Big)\log\Big(1 - \frac{\nu(p)}{p}\Big) + O\Big(\frac{1}{\log n}\Big).$$

The assumption (2.5) implies that

$$\sum_{D\leqslant p\leqslant n} \psi\Big(\frac{n}{p}\Big)\log\Big(1 - \frac{\nu(p)}{p}\Big) \ll_\nu \sum_{D\leqslant p\leqslant n}\frac{1}{p} \ll_\nu \log\frac{\log n}{\log D}.$$

This completes the proof. □

Proof of Corollary 2.5. We use the relation (4.1) by putting $\nu(n) = 1$, and we note that

$$\sum_{p\leqslant n}\psi\Big(\frac{n}{p}\Big)\log\Big(1 - \frac{1}{p}\Big) \ll \sum_{p\leqslant n}\frac{1}{p} = O(\log\log n).$$

This gives desired approximation. □

Proof of Proposition 2.6. We use the relation (4.2) in the form $g_n = c_g n + R(n)$, say. We have

$$g_\eta(n) = n^{1-\eta}g_n^\eta = c_g^\eta n\Big(1 + \frac{R(n)}{c_g n}\Big)^\eta = c_g^\eta n\left(1 + \frac{\eta R(n)}{c_g n} + O\Big(\frac{\log^2 n}{n^2}\Big)\right).$$

Thus, we obtain

$$g_\eta(n) = c_g^\eta n + \eta c_g^\eta\left(\frac{1}{2}\log\big(\frac{2\pi n}{\beta}\big) + \frac{\lambda}{2}\log\log n - \Psi_n(z) - \Psi_n(z,D)\right) + E_1,$$

where

$$E_1 = E_1(n,D) \ll_\eta E(n,D) + O\Big(\frac{\log^2 n}{n}\Big) \ll_{\nu,\eta} \log\frac{\log n}{\log D} + \frac{1}{\log n}.$$

To prove that the sequence $\mathcal{G} = (g_\eta(n))_{n\geqslant 1}$ is uniformly distributed modulo one, it is enough to show that this is the case for the sequence $\mathcal{G}' = \big(g'_\eta(n)\big)_{n\geqslant 1}$ defined by

$$g'_\eta(n) = g_\eta(n) - \frac{\eta c_g^\eta}{2}(\log n + \lambda\log\log n).$$

For any integer $h \geqslant 1$, the Weyl sum associated to the sequence $\mathcal{G}'$ is

$$\sum_{n\leqslant X} e(hg'_\eta(n)) = S(z,X) + E,$$

where

$$S(z,X)=\sum_{n\leqslant X} e\Big(c_g^\eta h\big(n+\frac{\eta}{2}\log(\frac{2\pi}{\beta})-\eta\Psi_n(z)\big)\Big),$$

and

$$E\ll_{\nu,h}\sum_{n\leqslant X}\Big\{|\Psi_n(z,D)|+E_1(n,D)\Big\}.$$

We have

$$\sum_{n\leqslant X}E_1(n,D)\ll_{\nu,\eta}\int_2^X\Big(\log\frac{\log t}{\log D}+\frac{1}{\log t}\Big)\,dt=X\log\frac{\log X}{\log D}+O_{\nu,\eta}(1).$$

To approximate $\sum_{n\leqslant X}|\Psi_n(z,D)|$, we write

$$\Psi_n(z,D)=\sum_{z\leqslant d\leqslant D}\frac{c(d)}{d}\psi(\frac{n}{d}),$$

with

$$c(d)=\begin{cases}0, & \text{if } d \text{ is not prime},\\ d\log\big(1-\frac{\nu(d)}{d}\big), & \text{if } d \text{ is prime}.\end{cases}$$

For $d=p$, since $-\nu\leqslant\nu(p)$, we obtain

$$c(p)=p\log\Big(1-\frac{\nu(p)}{p}\Big)\leqslant p\log\Big(1+\frac{\nu}{p}\Big)\leqslant\nu=\tau_\nu(p).$$

Hence, by using the statement of Corollary 3.3, we imply that

$$\sum_{n\leqslant X}|\Psi_n(z,D)|\ll_\nu Xz^{-\frac12}(\log z)^{B(\nu)}+\sqrt{XD}(\log D)^{B(\nu)},$$

and consequently, we get

$$E\ll_{\nu,\eta,h}Xz^{-\frac12}(\log z)^{B(\nu)}+\sqrt{XD}(\log D)^{B(\nu)}+X\log\frac{\log X}{\log D}.$$

Now, we let $2\leqslant z\leqslant\log X$, and we take $D=X(\log X)^{-c}$ for some positive c with $-\frac{c}{2}+B(\nu)<-\frac12$, which yields that

$$E\ll_{\nu,\eta,h}Xz^{-\frac12}(\log z)^{B(\nu)}+\frac{X\log\log X}{\log X}\ll_{\nu,\eta,h}Xz^{-\frac13}+o(X).$$

To approximate $S(z,X)$ we use the fact that the function $\Psi_n(z)$ is periodic in n with period P. We have

$$S(z,X)=e\Big(\frac{\eta c_g^\eta h}{2}\log(\frac{2\pi}{\beta})\Big)\sum_{n\leqslant X}e\Big(c_g^\eta h\big(n-\eta\Psi_n(z)\big)\Big).$$

Thus, we get

$$\begin{aligned}|S(z,X)| &= \left|\sum_{b=0}^{P-1}\sum_{\substack{n\leqslant X\\ n\equiv b\ [P]}} e\Big(c_g^\eta h\big(n-\eta\Psi_n(z)\big)\Big)\right|\\ &\leqslant \sum_{b=0}^{P-1}\left|\sum_{\substack{n\leqslant X\\ n\equiv b\ [P]}} e\left(c_g^\eta hn\right)\right| \leqslant \frac{P(z)}{|\sin(c_g^\eta hP(z)\pi)|}.\end{aligned}$$

Hence, we obtain

$$\left|\sum_{n\leqslant X} e(hg_\eta'(n))\right| \leqslant \frac{P(z)}{|\sin(c_g^\eta hP(z)\pi)|} + O_{\nu,\eta,h}(Xz^{-\frac{1}{3}}) + o(X).$$

Now, we end the implication similar to the arithmetic mean case. Indeed, by using the assumption of irrationality of the number c_g^η, we find a function $\varepsilon(z,X)$ with $\varepsilon(z,X)\to 0$ as $X\to\infty$, such that

$$\left|\sum_{n\leqslant X} e(hg_\eta'(n))\right| \leqslant \varepsilon(z,X)XP(z) + O_{\nu,\eta,h}(Xz^{-\frac{1}{3}}) + o(X).$$

Therefore, we obtain $\sum_{n\leqslant X} e(hg_\eta'(n)) = o(X)$ as $X\to\infty$, and this completes the proof. □

Proof of Proposition 2.8. We note that

$$g_\eta(n) = n^{-\eta} n!^{\frac{\eta}{n}} g_n.$$

By using Stirling's formula, we have

$$\begin{aligned}\left(\frac{n!^{\frac{1}{n}}}{n}\right)^\eta &= e^{-\eta}\left(1+\frac{\log\sqrt{2\pi n}}{n} + O\Big(\frac{\log^2 n}{n^2}\Big)\right)^\eta\\ &= e^{-\eta} + \frac{\eta e^{-\eta}\log\sqrt{2\pi n}}{n} + O\Big(\frac{\log^2 n}{n^2}\Big).\end{aligned} \tag{4.3}$$

We use this relation and the relation (4.2) in the form $g_n = c_g n + R(n)$, say. Thus, we obtain

$$g_\eta(n) = e^{-\eta}c_g n + \eta e^{-\eta}c_g\log\sqrt{2\pi n} + e^{-\eta}R(n) + O\Big(\frac{\log^2 n}{n}\Big). \tag{4.4}$$

We put the expression of $R(n)$ in the above approximation to get

$$\begin{aligned}g_\eta(n) = e^{-\eta}c_g n + c + \frac{e^{-\eta}c_g}{2}\left((\eta+1)\log n + \lambda\log\log n\right)&\\ - e^{-\eta}c_g\Psi_n(z) - e^{-\eta}c_g\Psi_n(z,D) + E_2,&\end{aligned}$$

where $c = \frac{e^{-\eta}c_g}{2}\log(\frac{(2\pi)^{\eta+1}}{\beta})$ is an absolute constant, and we have

$$E_2 = E_2(n,D) \ll_\eta E(n,D) + O\Big(\frac{\log^2 n}{n}\Big) \ll_{\nu,\eta} \log\frac{\log n}{\log D} + \frac{1}{\log n}.$$

We consider the sequence $\mathcal{G}' = (g'_\eta(n))_{n\geqslant 1}$ defined by

$$g'_\eta(n) = g_\eta(n) - \frac{e^{-\eta}c_g}{2}\left((\eta+1)\log n + \lambda\log\log n\right).$$

Approximation of the Weyl sum related to the sequence $\mathcal{G}'$ is exactly the same as what we have done in the proof of Proposition 2.6. Following a similar argument, we get finally

$$\sum_{n\leqslant X} e(hg'_\eta(n)) \ll_{\nu,\eta,h} Xz^{-\frac{1}{3}} + \frac{X\log\log X}{\log X} + \frac{P(z)}{|\sin(e^{-\eta}c_g hP(z)\pi)|},$$

for any integer $h\geqslant 1$ and for $2\leqslant z\leqslant \log X$. The assumption of the irrationality of $e^{-\eta}c_g$ implies $\sum_{n\leqslant X} e(hg'_\eta(n)) = o(X)$, which takes care of our assertion for the sequence $\mathcal{G}' = \big(g'_\eta(n)\big)_{n\geqslant 1}$, and consequently for the sequence $\mathcal{G} = (g_\eta(n))_{n\geqslant 1}$. This completes the proof. □

Proof of Proposition 2.10. We have $g_\eta(n) = f(n)^{-\eta} n!^{\frac{\eta d}{n}} g_n$. By using the approximation (4.3), we obtain

$$\begin{aligned} f(n)^{-\eta} n!^{\frac{\eta d}{n}} &= a^{-\eta}e^{-\eta d}\left(1 + \frac{\eta d\log\sqrt{2\pi n}}{n} + O\Big(\frac{\log^2 n}{n^2}\Big)\right)\left(1 + O\Big(\frac{\log^s n}{n^2}\Big)\right) \\ &= a^{-\eta}e^{-\eta d}\left(1 + \frac{\eta d\log\sqrt{2\pi n}}{n} + O\Big(\frac{\log^v n}{n^2}\Big)\right), \end{aligned}$$

where $v = \max\{2, s\}$. We consider the relation (4.2) in the form $g_n = c_g n + R(n)$, say, to get

$$\begin{aligned} g_\eta(n) &= a^{-\eta}e^{-\eta d}\left(1 + \frac{\eta d\log\sqrt{2\pi n}}{n} + O\Big(\frac{\log^v n}{n^2}\Big)\right)(c_g n + R(n)) \\ &= a^{-\eta}e^{-\eta d}c_g n + a^{-\eta}e^{-\eta d}\eta d c_g\log\sqrt{2\pi n} + a^{-\eta}e^{-\eta d}R(n) + O\Big(\frac{\log^v n}{n}\Big). \end{aligned}$$

This relation has similar structure as (4.4), and the continuation of the proof is similar to the proofs of Proposition 2.6 and Proposition 2.8. □

Proof of Proposition 2.11. We note that $g_\eta(n) = \big(n!^{\frac{\eta}{n}} g_n\big)^{\frac{1}{\eta+1}}$. We use the relation (4.3), and we consider the relation (4.2) in the form $g_n = c_g n + R(n)$, say, to write

$$
\begin{aligned}
g_\eta(n) &= \left(e^{-\eta} n^\eta \Big(1 + \frac{\eta \log \sqrt{2\pi n}}{n} + O\Big(\frac{\log^2 n}{n^2}\Big)\Big) \left(c_g n + R(n)\right) \right)^{\frac{1}{\eta+1}} \\
&= \left(e^{-\eta} c_g n^{\eta+1} \left(1 + \frac{\eta \log \sqrt{2\pi n}}{n} + \frac{R(n)}{c_g n} + O\Big(\frac{\log^2 n}{n^2}\Big)\right)\right)^{\frac{1}{\eta+1}}.
\end{aligned}
$$

Thus, we obtain

$$
g_\eta(n) = \left(e^{-\eta} c_g\right)^{\frac{1}{\eta+1}} n + \frac{\eta \left(e^{-\eta} c_g\right)^{\frac{1}{\eta+1}}}{\eta+1} \log \sqrt{2\pi n} + \frac{\left(e^{-\eta} c_g\right)^{\frac{1}{\eta+1}}}{c_g(\eta+1)} R(n) + O\Big(\frac{\log^2 n}{n}\Big).
$$

Similar to the proof of Proposition 2.10, this relation concludes the proof. □

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