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TRANSLATIONS, NORM-ATTAINING FUNCTIONALS, AND ELEMENTS OF MINIMUM NORM

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ABSTRACT. In this paper we continue a work that James started in 1971 about norm-attaining functionals on non-complete normed spaces by proving that every functional on a normed space is norm-attaining if and only if every proper, closed, convex subset with non-empty interior can be translated to have a non-zero, minimum-norm element. We also study this type of spaces when they are non-complete. Finally, we consider translations and elements of maximum norm.

1. Introduction

In the year 1964 James proved a characterization of reflexivity in the class of Banach spaces in terms of norm-attaining functionals (see [3]).

Theorem 1.1 (James, 1964)**.** *A Banach space X is reflexive if and only if every functional on X is norm-attaining.*

Afterwards, James was asked for the possibility of removing the completeness hypothesis. As a negative answer, he came up with the following counterexample (see [4]).

Theorem 1.2 (James, 1971)**.** *There exists a non-complete normed space on which every functional is norm-attaining.*

This result of James motivated Blatter to characterize reflexivity in the class of all normed spaces (see [2]).

Theorem 1.3 (Blatter, 1976)**.** *A normed space X is reflexive if and only if every closed, convex subset of X has a minimum-norm element.*

In 2005 Blatter's Theorem 1.3 was slightly improved (see [1]).

Theorem 1.4 (Aizpuru and García-Pacheco, 2005)**.** *A normed space X is reflexive if and only if every bounded, closed, convex subset of X with non-empty interior has a minimum-norm element.*

The next step is to provide a characterization of normed spaces on which every functional is norm-attaining.

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2. A geometric characterization of normed spaces on which every functional is norm-attaining

We intend to characterize all those normed spaces on which every functional attains its norm. Given a normed space X and a subset M of X, we will let $\mathsf{SA}(M)$ denote the set of functionals on X whose real part attains its supremum on M. Sometimes, $\mathsf{NA}(X)$ is used to denote $\mathsf{SA}(\mathsf{B}_X)$.

Theorem 2.1. *Let X be a normed space. Let M be a closed, convex subset of X and let $x \in \operatorname{bd}(M)$ (the boundary of M). The following conditions are equivalent:*

1. *There is a translate of M with a non-zero element of minimum-norm; in other words, there exists $a \in X$ such that $x+a$ is a non-zero minimum-norm element of $M+a$.*
2. *There exists $f \in \mathsf{S}_{X^*} \cap \mathsf{NA}(X) \cap \mathsf{SA}(M)$ such that $\operatorname{Re} f$ attains its supremum on M at x.*
3. *There exists a closed ball B with non-empty interior such that $x \in M \cap B$ and $M \cap \operatorname{int}(B) = \varnothing$.*

Proof. Assume that there exists a closed ball B with non-empty interior such that $x \in M \cap B$ and $M \cap \operatorname{int}(B) = \varnothing$. The Hahn-Banach Theorem allows us to deduce the existence of an element $f \in \mathsf{S}_{X^*}$ such that $\operatorname{Re} f(u) > \operatorname{Re} f(m)$ for every $u \in \operatorname{int}(B)$ and every $m \in M$. Since $\operatorname{cl}(\operatorname{int}(B)) = B$,

$$x \in \{m \in M : \operatorname{Re} f(m) = \sup \operatorname{Re}(f)(M)\}.$$

We will show that $-f$ is norm-attaining. Let b be the center of B. Since $x \in \operatorname{bd}(B)$ we have that the radius of B is $\|x-b\|$. We will show that

$$\operatorname{Re}(-f)\left(\frac{x-b}{\|x-b\|}\right) = 1.$$

For all $z \in \mathsf{U}_X(0, \|x-b\|)$ (the open ball of center 0 and radius $\|x-b\|$) we have that $\operatorname{Re}(-f)(z+b) < \operatorname{Re}(-f)(x)$, therefore $\operatorname{Re}(-f)(z) < \operatorname{Re}(-f)(x-b)$. Thus,

$$\begin{aligned}\|x-b\| &= \sup \operatorname{Re}(-f)\, \mathsf{U}_X(0, \|x-b\|) \\ &\leq \operatorname{Re}(-f)(x-b) \\ &\leq \|x-b\|.\end{aligned}$$

Assume now that $f \in \mathsf{S}_{X^*} \cap \mathsf{NA}(X) \cap \mathsf{SA}(M)$ is such that $\operatorname{Re} f$ attains its supremum on M at x. Let $y \in f^{-1}(1) \cap \mathsf{B}_X$ and consider $a = -y - x$. We will prove that $x + a$ is a minimum-norm element of $M + a$. Let $m \in M$. Then

$$\begin{aligned}\|m+a\| &\geq |f(m+a)| \\ &= |f(m) - 1 - f(x)| \\ &\geq 1 + \operatorname{Re} f(x) - \operatorname{Re} f(m) \\ &\geq 1 \\ &= \|x+a\|.\end{aligned}$$

Finally, if there exists $a \in X$ such that $x+a$ is a non-zero minimum-norm element of $M+a$, then $B := \mathsf{B}_X(-a, \|x+a\|)$ verifies that $x \in M \cap B$ and $M \cap \operatorname{int}(B) = \varnothing$. $\square$

We are in the right position to state and prove a characterization of normed spaces on which every functional is norm-attaining.

Theorem 2.2. *Let X be a normed space. The following conditions are equivalent:*

(1) *Every functional on X is norm-attaining.*
(2) *If M is a proper, closed, convex subset of X with non-empty interior, then for every $x \in \mathrm{bd}(M)$ there exists $a \in X$ such that $x+a$ is a non-zero minimum-norm element of $M+a$.*
(3) *If M is a proper, closed, convex subset of X with non-empty interior, then there exist $x \in \mathrm{bd}(M)$ and $a \in X$ such that $x+a$ is a non-zero minimum-norm element of $M+a$.*

Proof. Suppose first that $\mathsf{NA}(X) = X^*$. Let M be a proper, closed, convex subset of X with non-empty interior. Let x be any point in the boundary of M. The Hahn-Banach Theorem allows us to deduce the existence of a functional $f \in \mathsf{S}_{X^*}$ such that $\mathrm{Re} f(u) < \mathrm{Re} f(x)$ for every $u \in \mathrm{int}(M)$. Note that $\mathrm{Re} f$ attains its supremum on M at x since $\mathrm{cl}(\mathrm{int}(M)) = M$. In accordance with Theorem 2.1 there exists $a \in X$ such that $x+a$ is a non-zero minimum-norm element of $M+a$. Conversely, assume that (3) holds. Let $f \in \mathsf{S}_{X^*}$ and consider the proper, closed, convex set $M = \mathrm{Re} f^{-1}([1,\infty))$ that has non-empty interior. By hypothesis, there exist $x \in \mathrm{bd}(M)$ and $a \in X$ such that $x+a \neq 0$ and $x+a$ is a minimum-norm element of $M+a$. Observe that $M+a = \mathrm{Re} f^{-1}([1+\mathrm{Re} f(a),\infty))$. Since $0 \notin M+a$ we have that $1+\mathrm{Re} f(a) > 0$. Therefore,

$$\|x+a\| = \mathrm{dist}(0, M+a) = 1 + \mathrm{Re} f(a)$$

and $\mathrm{Re} f(x+a) = 1 + \mathrm{Re} f(a)$, which means that

$$\mathrm{Re} f\left(\frac{x+a}{\|x+a\|}\right) = 1. \qquad \square$$

Corollary 2.3. *Let X be a Banach space. The following conditions are equivalent:*

(1) *X is reflexive.*
(2) *Every proper, closed, convex subset of X with non-empty interior can be translated to have a non-zero minimum-norm element.*

The end of this section is aimed at showing that if every functional on a normed space X is norm-attaining, then a bigger class of proper, closed, and convex subsets of X (containing those which have non-empty interior) can be found so that every element of it can be translated to have a non-zero minimum-norm element. For this we will strongly rely on the Bishop-Phelps Support Point Theorem (see [5, Theorem 2.11.9]).

Lemma 2.4. *Let X be a Banach space. Let M be a proper, closed, convex subset of X. Assume either one of the following conditions holds:*

(1) *There are $a \in X$ and $\delta > 0$ so that $\sup(\mathrm{Re} f(M+a)) \geq \delta$ for all $f \in \mathsf{S}_{X^*}$.*
(2) *There are $a \in X$ and $f \in \mathsf{S}_{X^*}$ such that $\mathrm{Re} f(M+a) = \{0\}$.*

Then every $x \in \mathrm{bd}(M)$ is a support point of M; in other words, there is a non-zero real functional on X attaining its supremum on M at x.

Proof. Notice that we may assume that $a = 0$. Let us suppose first that condition (1) above holds. In accordance with the Bishop-Phelps Support Point Theorem there exist a sequence $(x_n)_{n\in\mathbb{N}} \subset \mathrm{bd}\,(M)$ converging to x and a sequence $(f_n)_{n\in\mathbb{N}} \subset \mathsf{S}_{X^*}$ so that $\mathrm{Re}\, f_n\,(x_n) = \sup\,(\mathrm{Re}\, f_n\,(M))$. By the ω^*-compactness of B_{X^*}, there exists a subnet $(f_{n_i})_{i\in I}$ that is ω^*-convergent to some $f \in \mathsf{B}_{X^*}$. Next, $(f_{n_i}\,(x_{n_i}))_{i\in I}$ converges to $f\,(x)$ and thus $\mathrm{Re}\, f\,(x) = \sup\,(\mathrm{Re}\, f\,(M))$. In order to see that $f \neq 0$ it suffices to realize that $\mathrm{Re}\, f\,(x) \geq \delta$, because $\mathrm{Re}\, f_{n_i}\,(x_{n_i}) \geq \delta$ for all $n \in \mathbb{N}$. Finally, assume that condition (2) above holds. We trivially have that $M = \{m \in M : \mathrm{Re}\, f\,(m) = \sup\,(\mathrm{Re}\, f\,(M))\}$. □

Remark 2.5. *If M is a proper, closed, convex subset of a normed space X with non-empty interior, then there are $a \in X$ and $\delta > 0$ so that* $\sup\,(\mathrm{Re}\, f\,(M + a)) \geq \delta$ *for all $f \in \mathsf{S}_{X^*}$. Indeed, it suffices to take a to be the opposite of the center of a closed ball contained in M and δ the radius of this ball.*

Lemma 2.4 together with Theorem 2.1 afford the following result.

Theorem 2.6. *Let X be a normed space. Assume that every functional on X is norm-attaining. Let M be a proper, closed, convex subset of X verifying (1) or (2) in Lemma 2.4. Then M can be translated to have a non-zero minimum-norm element.*

Proof. Observe that in order to be able to use Theorem 2.1 it is sufficient to apply Lemma 2.4 to the completion of X. □

3. Non-complete normed spaces on which every functional is norm-attaining

As we mentioned earlier at the beggining of this chapter, in 1972 James gave an example of a non-complete normed space on which every functional is norm-attaining (see [4]).

Example 3.1 (James, 1971)**.** Consider the infinite dimensional, separable, reflexive real Banach space

$$Y := \ell_\infty^1 \oplus_2 \ell_\infty^2 \oplus_2 \ell_\infty^3 \oplus_2 \cdots \oplus_2 \ell_\infty^n \oplus_2 \cdots .$$

The subspace of Y given by

$$X := \mathrm{span}\left\{\left(x_1^1; x_1^2, x_2^2; x_1^3, x_2^3, x_3^3; \dots\right) \in Y : |x_1^n| = \cdots = |x_n^n| \text{ for all } n \in \mathbb{N}\right\}$$

is non-complete and verifies that $\mathsf{NA}\,(X) = X^*$.

In the same paper (see [4]) James also noticed the following property verified by non-complete normed spaces on which every functional is norm-attaining. We remind the reader that a normed space is said to be rotund when its unit sphere is free of non-trivial segments (see [5]).

Theorem 3.2 (James, 1971)**.** *If X is a non-complete normed space on which every functional is norm-attaining, then the completion of X is reflexive but not rotund.*

The previous result motivates the following definition.

Definition 3.3. Let X be a normed space.

1. We say that X is almost-reflexive if $\mathsf{NA}(X) = X^*$.
2. We say that X is dense-reflexive if the completion of X is a reflexive Banach space.

The reformulation of James' Theorem 3.2 in the terms of the previous definition follows.

Remark 3.4 (James, 1971)**.** *Let X be a normed space. If X is almost-reflexive, then X is dense-reflexive. However, the converse is not true. Indeed, if X is an infinite dimensional, rotund, reflexive Banach space, then every non-complete subspace of X is dense-reflexive but not almost-reflexive.*

From James' Example 3.1 more examples of almost-reflexive normed spaces can be constructed. Indeed, let X be a non-complete almost-reflexive normed space. Take Z to be any reflexive Banach space. It is obvious that $X \oplus_2 Z$ is non-complete and almost-reflexive. On the other hand, observe that both reflexivity and dense-reflexivity are isomorphic properties in the class of normed spaces. By taking into consideration James' Theorem 3.2 one can realize that almost-reflexivity is not an isomorphic property in that class. Indeed, let X be a non-complete, almost-reflexive normed space. Let Y denote the completion of X. Observe that Y admits an equivalent rotund norm because it is reflexive. Hence, X cannot be almost-reflexive endowed with this new norm. We will show that non-complete, almost-reflexive normed spaces can actually be equivalently renormed to be non-almost-reflexive and non-rotund. We remind the reader that $\exp(\mathsf{B}_X)$ stands for the set of exposed points of the unit ball of a normed space X, that is, the points $x \in \mathsf{S}_X$ such that there exists $f \in \mathsf{S}_{X^*}$ verifying that $\{y \in \mathsf{S}_X : f(y) = 1\} = \{x\}$ (see [5]).

Lemma 3.5. *Let X and Y be Banach spaces. Then:*

1. $\mathrm{co}(\mathsf{S}_X \times \mathsf{S}_Y) = \mathsf{B}_{X\oplus_\infty Y}$.
2. $\exp(\mathsf{B}_{X\oplus_\infty Y}) = \exp(\mathsf{B}_X) \times \exp(\mathsf{B}_Y)$.

Proof.

1. It is sufficient to show that
$$\mathrm{co}(\mathsf{S}_X \times \mathsf{S}_Y) \supseteq \mathsf{S}_{X\oplus_\infty Y} = (\mathsf{S}_X \times \mathsf{B}_Y) \cup (\mathsf{B}_X \times \mathsf{S}_Y).$$
Let $(x, y) \in \mathsf{S}_X \times \mathsf{B}_Y$. There are $y_1, y_2 \in \mathsf{S}_Y$ and $\alpha \in [0, 1]$ such that $y = \alpha y_1 + (1-\alpha) y_2$. Therefore,
$$(x, y) = \alpha (x, y_1) + (1-\alpha)(x, y_2),$$
and $(x, y) \in \mathrm{co}(\mathsf{S}_X \times \mathsf{S}_Y)$. Likewise, it can be proved that if $(x, y) \in \mathsf{B}_X \times \mathsf{S}_Y$ then $(x, y) \in \mathrm{co}(\mathsf{S}_X \times \mathsf{S}_Y)$.
2. Let $(x, y) \in \exp(\mathsf{B}_{X\oplus_\infty Y})$. There exists $(f, g) \in \mathsf{S}_{X^*\oplus_1 Y^*}$ such that $\mathrm{Re}\, f(x) + \mathrm{Re}\, g(y) = 1$ and $\mathrm{Re}\, f(a) + \mathrm{Re}\, g(b) < 1$ for all $(a, b) \in \mathsf{S}_{X\oplus_\infty Y} \setminus \{(x, y)\}$. Then
$$1 = \mathrm{Re}\, f(x) + \mathrm{Re}\, g(y) \leq \|f\| \|x\| + \|g\| \|y\| \leq \|f\| + \|g\| = 1.$$

Therefore, $\operatorname{Re} f(x) = \|f\| \|x\| = \|f\|$ and $\operatorname{Re} g(y) = \|g\| \|y\| = \|g\|$. From these two equalities we deduce that $(x, y) \in \exp(\mathsf{B}_X) \times \exp(\mathsf{B}_Y)$. Conversely, let $(x, y) \in \exp(\mathsf{B}_X) \times \exp(\mathsf{B}_Y)$. Let $f \in \mathsf{S}_{X^*}$ and $g \in \mathsf{S}_{Y^*}$ denote functionals that characterize x and y as exposed points of B_X and B_Y, respectively. Then, $\left(\frac{f}{2}, \frac{g}{2}\right) \in \mathsf{S}_{X^* \oplus_1 Y^*}$ characterizes (x, y) as an exposed point of $\mathsf{B}_{X \oplus_\infty Y}$. □

Theorem 3.6. *Let X be an infinite dimensional reflexive Banach space. There exists an equivalent norm $\|\cdot\|'$ on X such that $(X, \|\cdot\|')$ is not rotund and has no dense proper almost-reflexive subspaces.*

Proof. In the first place, every reflexive Banach space can be equivalently renormed to be rotund, therefore we can suppose that X is already rotund. Let $f \in \mathsf{S}_{X^*}$ and $x \in \mathsf{S}_X$ such that $f(x) = 1$. Consider the non-rotund Banach space $Y = \mathbb{K}x \oplus_\infty \ker(f)$. By Lemma 3.5, we have that

$$\begin{aligned} \operatorname{co}(\exp(\mathsf{B}_Y)) &= \operatorname{co}\left(\exp(\mathsf{B}_{\mathbb{K}x}) \times \exp\left(\mathsf{B}_{\ker(f)}\right)\right) \\ &= \operatorname{co}\left(\mathsf{S}_{\mathbb{K}x} \times \mathsf{S}_{\ker(f)}\right) \\ &= \mathsf{B}_Y. \end{aligned}$$

Finally, observe that if Z is a dense, almost-reflexive subspace of Y, then $\exp(\mathsf{B}_Y) \subseteq \mathsf{B}_Z$, which implies that $Z = Y$. □

We would like to finish this section by showing our interest in finding dense-reflexive normed spaces which are not isomorphic to any almost-reflexive normed space. The candidate we have in mind is the following:

$$W := \bigcap \left\{\operatorname{span}\left(\exp\left(\mathsf{B}_{\|\cdot\|}\right)\right) : \|\cdot\| \text{ is an equivalent norm on } X\right\},$$

where X is any infinite dimensional, reflexive, Banach space. We believe that W is dense in X. If so, then any proper dense subspace of W is dense-reflexive and can never be almost-reflexive under any equivalent renorming of X.

4. Translations and elements of minimum norm

In the second section of this paper we characterized the normed spaces on which every functional is norm-attaining as those normed spaces in which every proper, closed, convex subset with non-empty interior can be translated to have a non-zero minimum-norm element. We will show now the existence of a certain type of non-complete normed spaces containing bounded, closed, convex subsets with non-empty interior which cannot be translated to have a non-zero minimum-norm element.

Lemma 4.1. *Let X be a normed space. If $x \in X \setminus \{0\}$ and $0 < r < \|x\|$, then*

$$\left(1 - \frac{r}{\|x\|}\right)x \quad \text{and} \quad \left(1 + \frac{r}{\|x\|}\right)x$$

are a minimum-norm element and a maximum-norm element of $\mathsf{B}_X(x, r)$, respectively.

Proof. Let $y \in \mathsf{B}_X(x, r)$. Then
$$\|x\| \leq \|x - y\| + \|y\| \leq r + \|y\|,$$
and
$$\|y\| \leq \|y - x\| + \|x\| \leq r + \|x\|,$$
therefore
$$\left\| \left(1 - \frac{r}{\|x\|}\right) x \right\| \leq \|y\| \leq \left\| \left(1 + \frac{r}{\|x\|}\right) x \right\|.$$
□

Theorem 4.2. *Let X be a non-complete normed space whose completion Y is rotund. There exists a bounded, closed, convex subset M of X with non-empty interior that cannot be translated to have a non-zero minimum-norm element.*

Proof. Let $y \in \mathsf{S}_Y \setminus \mathsf{S}_X$. Let $M := \mathsf{B}_Y\left(y, \frac{1}{2}\right) \cap X$. Assume that there exists $a \in X$ such that $M + a$ has a minimum-norm element $m + a \neq 0$ with $m \in M$. We have that $M + a = \mathsf{B}_Y\left(y + a, \frac{1}{2}\right) \cap X$. Since $0 \notin M + a$, we deduce that $\|y + a\| > \frac{1}{2}$. On the other hand, $M + a$ is dense in $\mathsf{B}_Y\left(y + a, \frac{1}{2}\right)$. Thus,
$$\operatorname{dist}(0, M + a) = \operatorname{dist}\left(0, \mathsf{B}_Y\left(y + a, \frac{1}{2}\right)\right).$$
The rotundity of y allows us to deduce that $\mathsf{B}_Y\left(y + a, \frac{1}{2}\right)$ has a unique element of minimum-norm. Therefore, by Lemma 4.1
$$m + a = \frac{\|y + a\| - \frac{1}{2}}{\|y + a\|}(y + a),$$
which implies that $y \in X$. This is a contradiction. □

In certain types of complete spaces a totally different situation occurs.

Theorem 4.3. *If X is a Banach space such that $\mathsf{NA}(X)$ has non-empty interior in X^*, then every bounded, closed, convex subset of X can be translated to have a non-zero minimum-norm element.*

Proof. Let M be a bounded, closed, convex subset of X. The completeness of X places us in the right position to apply the Bishop-Phelps Theorem to deduce that $\mathsf{SA}(M)$ is dense in X^*. Since $\mathsf{NA}(X)$ has non-empty interior, we must have that $\mathsf{S}_{X^*} \cap \mathsf{NA}(X) \cap \mathsf{SA}(M) \neq \varnothing$. Finally, apply Theorem 2.1. □

5. Translations and elements of maximum-norm

This section is the continuation of a series of results that appear in [1]. Everything starts with the following result.

Theorem 5.1. *Let X be a normed space. If M is a closed, convex subset of X with a non-zero maximum-norm element m, then there exists $a \in X$ such that $m + a \neq 0$ and $m + a$ is a minimum-norm element of $M + a$.*

Proof. Consider

$$a = -\frac{m}{\|m\|} - m.$$

If $x \in M$, then

$$\begin{aligned}
\|x + a\| &= \left\| x - \frac{1 + \|m\|}{\|m\|} m \right\| \\
&\geq \frac{1}{\|m\|} \left| \|m\| \|x\| - (1 + \|m\|) \|m\| \right| \\
&= \left| \|x\| - (1 + \|m\|) \right| \\
&= 1 + \|m\| - \|x\| \\
&\geq 1 \\
&= \|m + a\| .
\end{aligned}$$

As a consequence, $m + a$ is a minimum-norm element of $M + a$. □

The point of this section is to show that the reverse situation does not hold in general; in other words, the existence of a non-zero minimum-norm element does not imply the existence of a translation mapping the non-zero minimum-norm element to a maximum-norm element.

Theorem 5.2. *Let X be a normed space with the Radon-Riesz property that fails to have the Schur property. There exists a bounded, closed, convex subset M of X with a non-zero minimum-norm element $m \in M$ such that no translation exists which maps m to a non-zero maximum-norm element.*

Proof. Since X fails to have the Schur property, we can pick a sequence $(y_n)_{n\in\mathbb{N}}$ in S_X which is ω-convergent to 0. By passing to a subsequence and by considering $(-y_n)_{n\in\mathbb{N}}$ if necessary, we can assume without loss of generality that there exist $f \in \mathsf{S}_{X^*}$ and $x \in \mathsf{S}_X$ such that $f(x) = 1$ and $\operatorname{Re} f(y_n) \geq 0$ for every $n \in \mathbb{N}$. Next, denote $x_n := y_n + x$ for every $n \in \mathbb{N}$. Then $(x_n)_{n\in\mathbb{N}}$ is ω-convergent to x but none of its subsequences converges to x. We will show that x is a minimum-norm element of

$$M := \overline{\text{co}} \left(\{x_n : n \in \mathbb{N}\} \cup \{x\} \right).$$

Indeed, let $\lambda x + \lambda_1 x_{n_1} + \cdots + \lambda_k x_{n_k} \in \text{co}\left(\{x_n : n \in \mathbb{N}\} \cup \{x\} \right)$. Then

$$\begin{aligned}
\|\lambda x + \lambda_1 x_{n_1} + \cdots + \lambda_k x_{n_k}\| &\geq \operatorname{Re} f \left(\lambda x + \lambda_1 x_{n_1} + \cdots + \lambda_k x_{n_k} \right) \\
&= 1 + \lambda_1 \operatorname{Re} f(y_1) + \cdots + \lambda_k \operatorname{Re} f(y_k) \\
&\geq 1 \\
&= \|x\| .
\end{aligned}$$

Suppose we could find $a \in X$ so that $x + a$ is a maximum-norm element of $M + a$. Then $(x_n + a)_{n\in\mathbb{N}}$ is ω-convergent to $x + a$ and there exists a subsequence of $(\|x_n + a\|)_{n\in\mathbb{N}}$ which converges to $\|x + a\|$. Since X has the Radon-Riesz property, there exists a subsequence of $(x_n + a)_{n\in\mathbb{N}}$ converging to $x + a$; in other words, there exists a subsequence of $(x_n)_{n\in\mathbb{N}}$ converging to x. This is a contradiction. □

Corollary 5.3. *If X is an infinite dimensional almost-reflexive normed space, then X can be equivalently renormed to have a bounded, closed, convex subset M with a non-zero minimum-norm element $x \in M$ such that no translation exists which maps x to a maximum-norm element.*

Proof. It suffices to observe that every reflexive space can be equivalently renormed to be locally uniformly rotund, and in particular to have the Radon-Riesz property. □

Lemma 5.4. *Let X be a real normed space. Assume that $x \in X$ and $f \in X^*$ are so that $\delta := f(x) > 0$ and $t := \|x\| > 0$. Then:*

1. *If there exists $r > 0$ such that $\mathsf{B}_X(x,r) \cap f^{-1}(\delta) \subseteq \mathsf{S}_X(0,t)$, then $\mathsf{B}_X\left(x,\frac{r}{4}\right) \cap \mathsf{S}_X(0,t) \subseteq f^{-1}(\delta)$.*
2. *If there exists $r > 0$ such that $\mathsf{B}_X(x,r) \cap \mathsf{S}_X(0,t) \subseteq f^{-1}(\delta)$, then $\mathsf{B}_X\left(x,\frac{r}{2}\right) \cap f^{-1}([0,\delta]) \subseteq \mathsf{B}_X(0,t)$ and $\mathsf{B}_X\left(x,\frac{r}{2}\right) \cap f^{-1}(\delta) \subseteq \mathsf{S}_X(0,t)$.*

Proof.

(1) In the first place, we will show that $\|f\| = \frac{\delta}{t}$. Obviously,

$$f\left(\frac{x}{t}\right) = \frac{\delta}{t},$$

therefore $\|f\| \geq \frac{\delta}{t}$. If $y \in \mathsf{B}_X$ and $f(y) > \frac{\delta}{t}$ then we can take $0 < \alpha < 1$ small enough to assure that $f(\alpha(ty) + (1-\alpha)x) > 0$ and

$$\delta \frac{\alpha(ty) + (1-\alpha)x}{f(\alpha(ty) + (1-\alpha)x)} \in \mathsf{B}_X(x,r) \cap f^{-1}(\delta),$$

which means that

$$\begin{aligned} t &= \left\| \delta \frac{\alpha(ty) + (1-\alpha)x}{f(\alpha(ty) + (1-\alpha)x)} \right\| \\ &= \frac{\delta}{f(\alpha(ty) + (1-\alpha)x)} \|\alpha(ty) + (1-\alpha)x\| \\ &\leq \frac{\delta}{f(\alpha(ty) + (1-\alpha)x)} t, \end{aligned}$$

and $\delta < f(\alpha(ty) + (1-\alpha)x) \leq \delta$, which is impossible. In the second place, we will show that $r \leq t$. Let $y \in \mathsf{S}_X(x,r) \cap f^{-1}(\delta)$. Then $2x - y \in \mathsf{S}_X(x,r) \cap f^{-1}(\delta)$. Therefore,

$$2r = \|y - (2x - y)\| \leq 2t.$$

Finally, if $y \in \mathsf{B}_X\left(x,\frac{r}{4}\right)$, then

$$f(y) \geq f(x) - |f(y) - f(x)| \geq \delta - \frac{\delta}{t}\frac{r}{4} \geq \frac{\delta}{2}.$$

Now, take any $y \in \mathsf{B}_X\left(x, \frac{r}{4}\right) \cap \mathsf{S}_X(0,t)$. Then

$$\begin{aligned}
\left\| \frac{\delta y}{f(y)} - x \right\| &= \frac{1}{f(y)} \|\delta y - f(y)x\| \\
&= \frac{1}{f(y)} \|\delta y - \delta x + \delta x - f(y)x\| \\
&\leq \frac{\delta}{f(y)} \|y - x\| + \frac{1}{f(y)} |f(x) - f(y)| t \\
&\leq \frac{\delta}{f(y)} \|y - x\| + \frac{t}{f(y)} \frac{\delta}{t} \|y - x\| \\
&= 2\frac{\delta}{f(y)} \|y - x\| \\
&\leq 4\|y - x\| \\
&\leq r.
\end{aligned}$$

Therefore,

$$\frac{\delta y}{f(y)} \in \mathsf{B}_X(x,r) \cap f^{-1}(\delta) \subseteq \mathsf{S}_X(0,t);$$

in other words,

$$t = \left\| \frac{\delta y}{f(y)} \right\| = \frac{\delta}{f(y)} t$$

and $f(y) = \delta$.

(2) In the first place, let us see that $\|f\| = \frac{\delta}{t}$. Obviously,

$$f\left(\frac{x}{t}\right) = \frac{\delta}{t},$$

therefore $\|f\| \geq \frac{\delta}{t}$. If $y \in \mathsf{B}_X$ and $f(y) > \frac{\delta}{t}$ then we can take $0 < \alpha < 1$ small enough to assure that

$$t\frac{\alpha(ty) + (1-\alpha)x}{\|\alpha(ty) + (1-\alpha)x\|} \in \mathsf{B}_X(x,r) \cap \mathsf{S}_X(0,t),$$

which means that

$$\begin{aligned}
\delta &= f\left(t\frac{\alpha(ty) + (1-\alpha)x}{\|\alpha(ty) + (1-\alpha)x\|}\right) \\
&= \frac{t}{\|\alpha(ty) + (1-\alpha)x\|} f(\alpha(ty) + (1-\alpha)x) \\
&> \frac{t}{\|\alpha(ty) + (1-\alpha)x\|} \delta,
\end{aligned}$$

and $t < \|\alpha(ty) + (1-\alpha)x\| \leq t$, which is impossible. Next, let $y \in \mathsf{B}_X\left(x, \frac{r}{2}\right) \cap f^{-1}([0,\delta])$ with $\|y\| > t$. Then

$$\begin{aligned}
\left\| \frac{ty}{\|y\|} - x \right\| &= \frac{1}{\|y\|} \|ty - \|y\| x\| \\
&= \frac{1}{\|y\|} \|ty - tx + tx - \|y\| x\| \\
&\leq \frac{t}{\|y\|} \|y - x\| + \frac{1}{\|y\|} |\|x\| - \|y\|| \, t \\
&\leq 2 \frac{t}{\|y\|} \|y - x\| \\
&< 2 \|y - x\| \\
&\leq r.
\end{aligned}$$

Therefore,

$$\frac{ty}{\|y\|} \in \mathsf{B}_X(x, r) \cap \mathsf{S}_X(0, t) \subseteq f^{-1}(\delta);$$

in other words,

$$\delta = f\left(\frac{ty}{\|y\|}\right) \leq \frac{t}{\|y\|}\delta < \delta,$$

which is a contradiction. Finally, since $\|f\| = \frac{\delta}{t}$, we have that $\mathsf{B}_X\left(x, \frac{r}{2}\right) \cap f^{-1}(\delta) \subseteq \mathsf{S}_X(0, t)$. □

Theorem 5.5. *Let X be a real normed space. The following conditions are equivalent:*

1. *There exists a norm-attaining $f \in \mathsf{S}_{X^*}$ such that $f^{-1}(1) \cap \mathsf{B}_X$ has empty interior relative to S_X.*
2. *There exists a bounded, closed, convex subset M of X with non-empty interior and with a non-zero minimum-norm element x such that there is no translation mapping x to a maximum-norm element.*

Proof. Assume that (1) holds. Let us pick $x \in \mathsf{S}_X$ such that $f(x) = 1$. Let $M := \mathsf{B}_X(x, 1) \cap f^{-1}([1, \infty))$ and $C := \mathsf{B}_X(x, 1) \cap f^{-1}(1)$. Suppose that there is $a \in X$ such that $x+a$ is a maximum-norm element of $M+a$. Then, $\|x+a\| \neq 0$ and $M+a \subseteq \mathsf{B}_X(0, \|x+a\|)$. Let us show that $C+a \subset \mathsf{S}_X(0, \|x+a\|)$. If $c+a \in C+a$ with $c \in C$, then, by assuming that $c \neq x$, we can find $d \in C$ such that $x \in (c, d)$. Now, $x+a \in (c+a, d+a)$, which means that $\|c+a\| = \|x+a\| = \|d+a\|$, because $x+a$ is a maximum-norm element of $M+a$. On the other hand, $C+a = \mathsf{B}_X(x+a, 1) \cap f^{-1}(1 + f(a))$. Next, we will show that $1 + f(a) < 0$. Otherwise, pick $y \in M$ such that $f(y) > 1$. Now, $\|y+a\| \geq f(y+a) > 1 + f(a) = \|x+a\|$, which contradicts the fact that $x+a$ is a maximum-norm element of $M+a$. Finally, since

$$\mathsf{B}_X(x+a, 1) \cap (-f)^{-1}(-1 - f(a)) \subset \mathsf{S}_X(0, \|x+a\|),$$

we deduce, according to the first paragraph of Lemma 5.4, that

$$\mathsf{B}_X\left(x+a,\frac{1}{4}\right)\cap\mathsf{S}_X\left(0,\|x+a\|\right)\subset(-f)^{-1}\left(-1-f\left(a\right)\right),$$

which is impossible since $f^{-1}(1)\cap\mathsf{B}_X$ has empty interior relative to S_X. Conversely, assume that (2) holds and consider M to be a bounded, closed, convex subset of X with non-empty interior so that M has a non-zero minimum-norm element $x\in M$ and cannot be translated mapping x into a maximum-norm element. Let $f\in\mathsf{S}_{X^*}$ verify that $f(u)<f(m)$ for all $u\in\mathsf{U}_X(0,\operatorname{dist}(0,M))$ and all $m\in M$. Clearly, $f(x)=\operatorname{dist}(0,M)=\|x\|$, that is, f is norm-attaining. Since M is bounded we can consider a number $K>\operatorname{diam}(M)$ such that $M\subseteq\mathsf{B}_X(0,K)$. Suppose that $(-f)^{-1}(K)\cap\mathsf{B}_X(0,K)$ has non-empty interior relative to $\mathsf{S}_X(0,K)$. Then, there exists $z\in(-f)^{-1}(K)\cap\mathsf{B}_X(0,K)$ and $r>0$ such that $\mathsf{B}_X(z,r)\cap\mathsf{S}_X(0,K)\subseteq(-f)^{-1}(K)$. By the second paragraph of Lemma 5.4,

$$\mathsf{B}_X\left(z,\frac{r}{2}\right)\cap(-f)^{-1}\left([0,K]\right)\subseteq\mathsf{B}_X\left(0,K\right).$$

Observe that by taking K large enough we may assume that $\frac{r}{2}\geq\operatorname{diam}(M)$. Finally, consider the translated set $M+(z-x)$. If $m\in M$, then

$$m+(z-x)\in\mathsf{B}_X\left(z,\operatorname{diam}(M)\right)\subseteq\mathsf{B}_X\left(z,\frac{r}{2}\right),$$

and

$$\begin{aligned}0&\leq K-\operatorname{diam}(M)\\&\leq K-\|m-x\|\\&\leq-\|m\|+K+\|x\|\\&=-f(m)-f(z)+f(x)\\&=(-f)(m+(z-x))\\&=-f(m)-f(z)+f(x)\\&\leq-\operatorname{dist}(0,M)+K+\operatorname{dist}(0,M)\\&=K.\end{aligned}$$

Thus, $m+(z-x)\in\mathsf{B}_X(0,K)$. And $\|x+(z-x)\|=\|z\|=K$, which means that $x+(z-x)$ is a maximum-norm element of $M+(z-x)$, reaching a contradiction. □

Corollary 5.6. *Let X be a complex normed space. There exists a bounded, closed, convex subset M with non-empty interior so that M has a non-zero minimum-norm element $x\in M$ but cannot be translated mapping x to a maximum-norm element.*

Proof. It is sufficient to observe that the unit sphere of any normed complex space is free of convex sets with non-empty interior relative to the unit sphere. □

Theorem 5.7. *Let X be a real normed space with $\dim(X)>1$. Assume that $f^{-1}(1)\cap\mathsf{B}_X$ has non-empty interior relative to S_X for every $f\in\mathsf{NA}(X)\cap\mathsf{S}_{X^*}$. Then:*

(1) *X is smooth.*
(2) $\exp(\mathsf{B}_X) = \varnothing$.
(3) $\operatorname{char}(X) = \operatorname{card}(\mathsf{NA}(X) \cap \mathsf{S}_{X^*})$.
(4) *X is not separable.*

Proof.

(1) Let $x \in \mathsf{S}_X$. We will show that x is a smooth point of B_X. If it is not, then there are $f \neq g \in \mathsf{S}_{X^*}$ such that $f(x) = g(x) = 1$. We have that $\frac{f+g}{2} \in \mathsf{S}_{X^*}$ and

$$\left(\frac{f+g}{2}\right)^{-1}(1) \cap \mathsf{B}_X = \left(f^{-1}(1) \cap \mathsf{B}_X\right) \cap \left(g^{-1}(1) \cap \mathsf{B}_X\right)$$
$$= \operatorname{bd}_{\mathsf{S}_X}\left(f^{-1}(1) \cap \mathsf{B}_X\right) \cap \operatorname{bd}_{\mathsf{S}_X}\left(g^{-1}(1) \cap \mathsf{B}_X\right).$$

Therefore, $\left(\frac{f+g}{2}\right)^{-1}(1) \cap \mathsf{B}_X$ cannot have non-empty interior with respect to S_X.

(2) If $x \in \exp(\mathsf{B}_X)$, then there exists $f \in \mathsf{S}_{X^*}$ such that $\{x\} = f^{-1}(1) \cap \mathsf{B}_X$. This contradicts the fact that $f^{-1}(1) \cap \mathsf{B}_X$ has non-empty interior with respect to S_X.

(3) We first remind the reader that $\operatorname{char}(X)$ stands for the density character of X. On the one hand, $\operatorname{char}(X) = \operatorname{char}(\mathsf{S}_X)$. On the other hand,

$$\mathsf{S}_X = \operatorname{cl}\left(\dot{\bigcup}\left\{\operatorname{int}_{\mathsf{S}_X}\left(f^{-1}(1) \cap \mathsf{B}_X\right) : f \in \mathsf{NA}(X) \cap \mathsf{S}_{X^*}\right\}\right).$$

Finally, the map

$$f \in \mathsf{NA}(X) \cap \mathsf{S}_{X^*} \longmapsto \operatorname{int}_{\mathsf{S}_X}\left(f^{-1}(1) \cap \mathsf{B}_X\right)$$

is a bijection.

(4) Assume that X is separable. Then we have that $\mathsf{NA}(X) \cap \mathsf{S}_{X^*}$ is countable. If Y is a 2-dimensional subspace of X, then S_{Y^*} is uncountable. By the Hahn-Banach Theorem, we can extend all the elements in S_{Y^*} to an uncountable set contained in $\mathsf{NA}(X) \cap \mathsf{S}_{X^*}$, which is impossible. □

At this point, we feel obligated to let the reader know that so far we have not been able to find a real normed space verifying the hypothesis of Theorem 5.7.

Corollary 5.8. *Let X be a real normed space with $\dim(X) > 1$. Then:*

(1) *If X is separable, then there exists a bounded, closed, convex subset M with non-empty interior so that M has a non-zero minimum-norm element $x \in M$ and cannot be translated mapping x to a maximum-norm element.*
(2) *If X is not separable, then it can be equivalently renormed to possess a bounded, closed, convex subset M with non-empty interior so that M has a non-zero minimum-norm element $x \in M$ but cannot be translated mapping x to a maximum-norm element.*

Proof. The results follow from Lemma 5.7 and from the easy fact that every normed space can be equivalently renormed so that its new unit ball has an exposed point. □

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References

1. A. Aizpuru and F.J. García-Pacheco, *Reflexivity, contraction functions and minimum-norm elements*, Studia Sci. Math. Hungar. **42** (2005), no. 4, 431–443.
2. J. Blatter, *Reflexivity and the existence of best approximations*, Approximation Theory, II (Proc. Internat. Sympos., Univ. Texas, Austin, Tex., 1976), pp. 299–301. Academic Press, New York, 1976.
3. R.C. James, *Characterizations of reflexivity*, Studia Math. **23** (1964), 205–216.
4. R.C. James, *A counterexample for a sup theorem in normed spaces*, Israel J. Math. **9** (1971), 511–512.
5. R. E. Megginson, *An Introduction to Banach Space Theory.* Graduate Texts in Mathematics 183, Springer-Verlag, New York, 1998.

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