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SEQUENTIAL ENTRY IN ONE-TO-ONE MATCHING MARKETS

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Abstract. We study in one-to-one matching markets a process of sequential entry, in which participants enter in the market one at a time, in some arbitrary given order. We identify a large family of orders (optimal orders) which converge to the optimal stable matching.

1. Introduction

We study sequential entry in a marriage model. The marriage model describes a two-sided, one-to-one matching market without money where the two sides of the market for instance are workers and firms (job matching), or medical students and hospitals. We use the common terminology in the literature and refer to one side of the market as men and to the other as women. An outcome of a marriage market is called a matching, which can simply be described by a collection of single agents and married pairs (consisting of one man and one woman). Loosely speaking, a matching is stable if all agents have acceptable spouses and there is no couple whose members both like each other better than their current spouses.

Originally, Gale and Shapley [4] proved that the set of stable matchings is not empty. Their constructive proof identifies an algorithm which starts with the empty matching and generates a sequence of matchings where blocking pairs are matched at each iteration. In the original Gale-Shapley algorithm men propose simultaneously at each iteration. McVitie and Wilson [8] observed that this can be modified by letting —at each iteration— only one man propose to the woman he prefers most among those who have not yet turned him down.

Knuth [6] demonstrated that starting with an arbitrary matching and iteratively satisfying blocking pairs will not necessarily lead to a stable matching. Roth and Vande Vate [11], however, showed that it is always possible to reach a stable matching from any arbitrary matching by satisfying a sequence of blocking pairs. They do this by designing an algorithm that introduces the agents successively into the system and lets them iteratively satisfy blocking pairs at each stage by a decentralized system.

Key words and phrases. Stable matching, sequential entry, optimal matching.

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Blum and Rothblum [3] consider analysis of stable matchings when a new agent joins the market. Under such changes a stable matching may become unstable, as the new agent may become part of new blocking pairs. They show that a natural procedure which reflects the greedy behavior of agents leads to a stable matching in the new (extended) market. Such process constitutes a particular case of an equilibration mechanism studied by Blum, Roth and Rothblum [2], which is applicable to situations where a group of (new) agents belonging to one side of the market joins, while another group of (veteran) agents from the other side of the market departs. They also analyze a special case of the algorithm described by Roth and Vande Vate [11], where agents join the market sequentially. They consider that at each stage, a new agent joins and the natural greedy correcting procedure is applied to regain stability for the new market, with the extended subset of agents that are present. They have shown that this procedure converges to a stable matching and that for each agent, if the order of all other agents is given, he or she weakly improves his or her final outcome by deferring his or her arrival time.

Authors such as Ma [9], Biró, Cechlárová and Fleiner [1], have also considered incremental algorithms in a similar way to Blum and Rothblum [3].

Following Roth and Vande Vate, and Blum and Rothblum [3] we suppose that agents sequentially enter the market. We study a process of sequential entry, in which participants enter the market one at a time, in some arbitrary given order. At each stage, a new agent enters and the Deferred Acceptance algorithm with input matchings is applied. Applying such algorithm re-establishes the stability, within the subset of agents that have entered the market. This iterative process generates a stable matching. The natural question is: what stable matchings can be obtained from the process of sequential entry? We partially answer it, by identifying a large family of orders (optimal orders) which converge to the optimal stable matching. We also show, as a consequence of Blum, Roth and Rothblum's results [2], that given an arbitrary order, the last agent to enter the market through the process gets his or her optimal outcome under stable matchings. Finally, we observe that not all stable matchings can be obtained through the process of sequential entry.

This article is organized as follows. In the next section, the model is formally described. Section 3 develops the relation between the cycles and the optimal matching. Finally, our iterative process is presented and we also show that an optimal order leads to the optimal stable matching in section 4.

2. Preliminaries

First we introduce the marriage market. In this market there are two finite and disjoint sets of agents, the set M of n "men" and the set $W$of p "women". We refer to $V = M \cup W$ as the set of agents and we denote a generic agent by v. Each $m \in M$ has a strict preference relation $P(m)$ over the set $W \cup \{m\}$, and each $w \in W$ has a strict preference relation $P(w)$ over the set $M \cup \{w\}$.[1] The preference relation of a man m, for instance, can be represented by an ordered

[1] $P(m)$ is a complete, antireflexive, and transitive binary relation on $W \cup \{m\}$.

list of the elements in $W \cup \{m\}$, $P(m) = w_1, w_3, m, w_2, \ldots, w_k$, which indicates that m prefers w_1 to w_3 and he prefers remaining single to any other woman. A preference profile is a $(n+p)$-tuple of preference relations and it is represented by $P = (P(m_1), \ldots, P(m_n), P(w_1), \ldots, P(w_p))$. We write $wP(m)w'$ if w is preferred to w' under the preference relation $P(m)$, and $wR(m)w'$ if $wP(m)w'$ or $w = w'$. Similarly we write $mP(w)m'$ and $mR(w)m'$. A woman w is acceptable to a man m if $wP(m)m$. Analogously, m is acceptable to w if $mP(w)w$. The marriage market is fully described by the triplet (M, W, P).

A matching μ is a one-to-one correspondence from V to itself, such that for each $m \in M$ and for each $w \in W$ we have $\mu(m) = w$ if and only if $\mu(w) = m$, $\mu(m) \notin W$ then $\mu(m) = m$, and similarly $\mu(w) = w$ if $\mu(w) \notin M$. If $\mu(m) = w$, then man m and woman w are matched to one another. We say that agent v is single if $\mu(v) = v$. Given a matching μ, we call $\mu(v)$ the outcome for v under μ.

Given two matchings μ and μ' and a set of agents $V' \subseteq V$ we denote $\mu \succeq_{V'} \mu'$ if for each $v \in V'$, $\mu(v)R(v)\mu'(v)$, and we denote $\mu \succ_{V'} \mu'$ if $\mu \succeq_{V'} \mu'$ and $\mu(v) \neq \mu'(v)$ for some $v \in V'$.

A matching μ is individually rational if $\mu(v)R(v)v$ for all $v \in V$. A blocking pair for a matching μ is a pair $(m, w) \in M \times W$ such that $mP(w)\mu(w)$ and $wP(m)\mu(m)$.

Definition 2.1. A matching μ is stable if it is individually rational and there are no blocking pairs for it.

We denote the set of stable matchings by $S(M, W, P)$. Gale and Shapley [4] proved that a stable matching must exist. They further proved the existence of an optimal stable matching μ_M for all men in the sense that no other stable matching μ exists that gives any man m an outcome $\mu(m)$ that he prefers to $\mu_M(m)$, and a (possibly different) optimal stable matching μ_W for all women.

For any two matchings μ and μ' we define the correspondence $\mu \vee_M \mu'$ on V that assigns to each man his more preferred outcome of μ and μ' and to each woman her less preferred outcome. Formally, let $\lambda = \mu \vee_M \mu'$ be defined for all $m \in M$ by $\lambda(m) = \mu(m)$ if $\mu(m)P(m)\mu'(m)$ and $\lambda(m) = \mu'(m)$ otherwise, and for all $w \in W$ by $\lambda(w) = \mu(w)$ if $\mu'(w)P(w)\mu(w)$ and $\lambda(w) = \mu'(w)$ otherwise. Similarly, we define the correspondence $\mu \wedge_M \mu'$ that gives each man his less preferred outcome of μ and μ' and to each woman her more preferred outcome. Analogously, we define the correspondences $\mu \vee_W \mu'$ and $\mu \wedge_W \mu'$. For an arbitrary pair of matchings those four correspondences might fail to be matchings. But for a pair of stable matchings, Knuth [6] credits Conway with the following theorem.

Theorem 2.1. *Let* $\mu, \mu' \in S(P)$. *Then* $\mu \vee_M \mu'$, $\mu \wedge_M \mu'$, $\mu \vee_W \mu'$, $\mu \wedge_W \mu' \in S(P)$. *Moreover,* $\mu \vee_M \mu' = \mu \wedge_W \mu'$ *and* $\mu \wedge_M \mu' = \mu \vee_W \mu'$.

The previous theorem asserts that the set of stable matchings forms a lattice under the partial order $\succeq_M$ with lattice operators $\vee_M$ and $\wedge_M$. Furthermore, this lattice is the dual of the lattice defined on $S(P)$ with the partial order $\succeq_W$ and lattice operators $\vee_W$ and $\wedge_W$. See Roth and Sotomayor [10] for a detailed discussion.

Let $V' = M' \cup W'$ be a subset of agents and P' the natural restriction of P to the members of V'. Denote by (M', W', P') a reduced market of (M, W, P). The following theorem will be useful in coming sections.

Lemma 2.1. *Let μ' be a stable matching in (M, W, P) and let $M' \subseteq M$. Also, let μ be a matching defined by $\mu(m) = \mu'(m)$ for all $m \in M'$. Then μ is a stable matching in the reduced market (M', W', P') where $W' = \mu'(M')$.*

Proof. Assume that there exists a blocking pair (m, w) for μ. Then $(m, w) \in M' \times W'$ and $wP'(m)\mu(m)$ and $mP'(w)\mu(w)$. Since (M', W', P') is a reduced market, $(m, w) \in M \times W$ and $wP(m)\mu'(m)$ and $mP(w)\mu'(w)$, contradicting that μ' is stable for (M, W, P). □

2.1. **Deferred acceptance algorithm with arbitrary input.** We will describe the Deferred Acceptance (DA) algorithm with arbitrary input matchings following Blum, Roth and Rothblum [2]. The algorithm starts with an arbitrary matching, selects a single man m and its most preferred woman w (if any) then it checks whether they form a blocking pair. If they do, this will be a maximal blocking pair (w is the most preferred woman for m among those with whom m forms a blocking pair), and when this blocking pair is satisfied a new matching is formed. This process is then iterated until there is no single man who is part of a blocking pair. Formally, the algorithm is described as follows.

Input

Let μ be an arbitrary matching.

Initial stage

(0) (i) For all $m \in M$: $A^0_m = \{w \in W : wP(m)m\} \setminus \{\mu(m)\}$.

(ii) $\mu^0 = \mu$; $i = 1$.

Iteration stage

(1) If there is no $m \in M$ such that $\mu^{i-1}(m) = m$ and $A^{i-1}_m \neq \emptyset$, stop with output μ^{i-1}.

(2) Let m be such that $\mu^{i-1}(m) = m$ and $A^{i-1}_m \neq \emptyset$ and set w be the most preferred woman for m in A^{i-1}_m.

(3) i) If $\mu^{i-1}(w)P(w)m$, then $\mu^i = \mu^{i-1}$.

ii) Otherwise,

if $\mu^{i-1}(w) = w$, then $\mu^i(w) = m$ and $\mu^i(v) = \mu^{i-1}(v)$ for all $v \in V \setminus \{w, m\}$, and

if $\mu^{i-1}(w) = m^* \in M$, then $\mu^i(w) = m$, $\mu^i(m^*) = m^*$ and $\mu^i(v) = \mu^{i-1}(v)$ for all $v \in V \setminus \{w, m, m^*\}$.

(4) $A^i_m = A^{i-1}_m \setminus \{w\}$ and for all $m' \neq m$, $A^i_{m'} = A^{i-1}_{m'}$

(5) $i = i + 1$, go to (1).

We denote by $DA_M(\mu)$ the output of the DA algorithm with input μ and the men proposing. By exchanging the roles of men and women, we get an alternative algorithm to which we refer to as the female version of the DA algorithm (DA_W). When the input is the empty matching, the above formulation corresponds to the McVitie-Wilson [8] version of the deferred acceptance algorithm where at each step at most one pair is satisfied.

Blum, Roth and Rothblum [2] showed that if the input of the DA algorithm is in a class of matchings called quasi-stables, then the outcome of the DA algorithm is always a stable matching. They provide several characterizations of such output.

Definition 2.2. A matching μ is man-quasi-stable if it is individually rational and $\mu(m) = m$ for any blocking pair (m, w). Similarly, a matching is woman-quasi-stable if it is individually rational and $\mu(w) = w$ for any blocking pair (m, w).

We denote by $Q_M(M, W, P)$ the set of man-quasi-stable matchings, and by $Q_W(M, W, P)$ the set of woman-quasi-stable matchings.

The following properties were shown by Blum, Roth and Rothblum [2].

Theorem 2.2. *Let* $\mu \in Q_M(M, W, P)$. *Then:*

1. $DA_M(\mu) \succeq_W \mu$.
2. $DA_M(\mu) = \mu \vee_W \mu_M \in S(M, W, P)$.
3. *If* $\mu(m) = m$ *then* $[DA_M(\mu)](m) = \mu_M(m)$.

Symmetrically we can obtain the dual theorem.

Theorem 2.3. *Let* $\mu \in Q_W(M, W, P)$. *Then:*

1. $DA_W(\mu) \succeq_M \mu$.
2. $DA_W(\mu) = \mu \vee_M \mu_W \in S(M, W, P)$.
3. *If* $\mu(w) = w$ *then* $[DA_W(\mu)](w) = \mu_W(w)$.

3. Cycles of $P(\mu)$

Irving and Leather [7] introduced the concept of cycle. The cycle can be used to generate all the stable matchings.[2] It is easy to describe a cycle in terms of certain "reduced" preference lists obtained from the original preference lists by eliminating some unachievable agents.

If μ is any stable matching of the model, the profile obtained through the following process will be called reduced preference profile and will be denoted by $P(\mu)$.

Step 1. Remove from m's list of acceptable women all w who are more preferred than $\mu(m)$. Remove from w's list of acceptable men all m who are more preferred than $\mu_W(w)$.

Step 2. Remove from m's list of acceptable women all w who are less preferred than $\mu_W(m)$. Remove from w's list of acceptable men all m less preferred than $\mu(w)$.

[2] Gusfield and Irving called these cycles "rotations". See Gusfield and Irving [5] for a detailed description of these structures.

Step 3. After steps 1 and 2, if m is not acceptable to w (i.e., m is not on w's preferences list as now modified), then remove w from m's list of acceptable women. Similarly, if w is not acceptable to m remove m from the w's list of acceptable men.

Remark 3.1. It is clear from the construction of $P(\mu)$ that

1. $\mu(m)$ is the first entry of $P(\mu)(m)$ and $\mu(w)$ is the last entry of $P(\mu)(w)$. $\mu_W(m)$ is the last entry of $P(\mu)(m)$ and $\mu_W(w)$ is the first entry of $P(\mu)(w)$.
2. m is acceptable to w if and only if w is acceptable to m under $P(\mu)$.

Definition 3.1. A set of men $\{a_1, \ldots, a_r\}$ is a cycle for the reduced preferences profile $P(\mu)$, if

1. The second woman in $P(\mu)(a_i)$ is $\mu(a_{i+1})$ for all $i = 1, \ldots, r-1$ (i.e., the first woman in $P(\mu)(a_{i+1})$).
2. The second woman in $P(\mu)(a_r)$ is $\mu(a_1)$ (i.e., the first woman in $P(\mu)(a_1)$).

We denote a cycle by $\sigma = (a_1, \ldots, a_r)$ and we say that a_i generates cycle σ for any $i = 1, \ldots, r$. We denote by $\Sigma(\mu)$ the set of cycles for $P(\mu)$.

Example 3.1. Let $M = \{m_1, m_2, m_3, m_4\}$ and $W = \{w_1, w_2, w_3, w_4\}$ with preferences P given by:

$$\begin{array}{ll} P(m_1) = w_3, w_1, w_2, w_4 & P(w_1) = m_2, m_1, m_3, m_4 \\ P(m_2) = w_2, w_4, w_1, w_3 & P(w_2) = m_3, m_1, m_2, m_4 \\ P(m_3) = w_3, w_4, w_1, w_2 & P(w_3) = m_4, m_3, m_1, m_2 \\ P(m_4) = w_4, w_3, w_1, w_2 & P(w_4) = m_3, m_4, m_1 \end{array}$$

Then, $\mu_M = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \\ m_1 & m_2 & m_3 & m_4 \end{pmatrix}$ and $\mu_W = \begin{pmatrix} w_1 & w_2 & w_3 & w_4 \\ m_2 & m_1 & m_4 & m_3 \end{pmatrix}$.

Step 1

$$\begin{array}{ll} P(m_1) = w_1, w_2, w_4 & P(w_1) = m_2, m_1, m_3, m_4 \\ P(m_2) = w_2, w_4, w_1, w_3 & P(w_2) = m_1, m_2, m_4 \\ P(m_3) = w_3, w_4, w_1, w_2 & P(w_3) = m_4, m_3, m_1, m_2 \\ P(m_4) = w_4, w_3, w_1, w_2 & P(w_4) = m_3, m_4, m_1 \end{array}$$

Step 2

$$\begin{array}{ll} P(m_1) = w_1, w_2 & P(w_1) = m_2, m_1 \\ P(m_2) = w_2, w_4, w_1 & P(w_2) = m_1, m_2 \\ P(m_3) = w_3, w_4 & P(w_3) = m_4, m_3 \\ P(m_4) = w_4, w_3 & P(w_4) = m_3, m_4 \end{array}$$

Step 3

$$\begin{array}{ll} P(m_1) = w_1, w_2 & P(w_1) = m_2, m_1 \\ P(m_2) = w_2, w_1 & P(w_2) = m_1, m_2 \\ P(m_3) = w_3, w_4 & P(w_3) = m_4, m_3 \\ P(m_4) = w_4, w_3 & P(w_4) = m_3, m_4 \end{array}$$

There are two cycles for $P(\mu_M)$, $\sigma_1 = (m_1, m_2)$ and $\sigma_2 = (m_3, m_4)$

Irving and Leather [7] showed the following Lemma.

Lemma 3.1. *Let μ, μ' be stable matchings such that $\mu \succ_M \mu'$ and σ be a cycle for $P(\mu)$. Then, either $\mu(m) = \mu'(m)$ for all $m \in \sigma$ or $\mu(m) \neq \mu'(m)$ for all $m \in \sigma$.*[3]

Let μ be a stable matching and $\sigma = (a_1, \ldots, a_r)$ a cycle for $P(\mu)$. Then, we can define a matching μ' as follows:

$$\begin{aligned} \mu'(a_i) &= \mu(a_{i+1}) \text{for all } i = 1, \ldots, r-1, \\ \mu'(a_r) &= \mu(a_1), \\ \mu'(m) &= \mu(m) \text{for all } m \notin \sigma. \end{aligned}$$

We refer to μ' as a cycle matching under $P(\mu)$ and we say that σ is the cycle associated with μ'.

Example 3.2. Let us recall that in Example 3.2 there are two cycles for $P(\mu_M)$, $\sigma_1 = (m_1, m_2)$ and $\sigma_2 = (m_3, m_4)$. We can obtain the cycle matchings μ_1 and μ_2 by

$$\mu_1 = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_2 & w_1 & w_3 & w_4 \end{pmatrix}, \qquad \mu_2 = \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ w_1 & w_2 & w_4 & w_3 \end{pmatrix}.$$

Two stable matchings μ and μ' are called consecutive if $\mu \succ_M \mu'$ and there is no stable matching μ'' such that $\mu \succ_M \mu'' \succ_M \mu'$. From Roth and Sotomayor [10] we can obtain the following lemma.

Lemma 3.2. *Let μ, μ' be consecutive stable matchings. Then μ' is a cycle matching under $P(\mu)$.*[4]

We can now show that if a stable matching assigns a man belonging to each cycle for $P(\mu)$ with its optimal result under stable matchings, then such stable matching is the optimal stable matching for men.

Lemma 3.3. *Let σ be a cycle for $P(\mu_M)$. If there exists $m' \in \sigma$ and a stable matching μ such that $\mu(m') = \mu_M(m')$, then $\mu(m) = \mu_M(m)$ for all $m \in \sigma$.*

Proof. If $\mu = \mu_M$ then the lemma is verified. Suppose now that $\mu_M \succ_M \mu$. Since by hypothesis σ is a cycle for $P(\mu_M)$ and $\mu(m') = \mu_M(m')$ for $m' \in \sigma$, Lemma 3.1 implies that $\mu(m) = \mu_M(m)$ for all $m \in \sigma$. □

From the preceeding lemmas, we obtain the following theorem.

Theorem 3.1. *Let μ be a stable matching such that for each cycle σ for $P(\mu_M)$ there exists $m' \in \sigma$ satisfying $\mu(m')R(m')\mu_M(m')$. Then $\mu = \mu_M$.*

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[3]See Gusfield and Irving [5, Lemma 2.5.5] for a detailed proof.

[4]See Roth and Sotomayor [10, p. 66] for a detailed proof.

Proof. Assume that $\mu_M \succ_M \mu$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be a sequence of stable matchings such that $\mu_M = \mu_1 \succ_M \mu_2 \succ_M \cdots \succ_M \mu_n = \mu$ and μ_i, μ_{i+1} are consecutive for all $i = 1, \ldots, n-1$. By Lemma 3.2 μ_2 is a cycle matching under $P(\mu_M)$. Let σ' be the cycle for $P(\mu_M)$ associated with μ_2. Then by hypothesis there exists $m' \in \sigma'$ such that $\mu(m') = \mu_M(m')$. Hence, by Lemma 3.3

$$\mu(m) = \mu_M(m) \quad \text{for all } m \in \sigma'. \tag{1}$$

On the other hand, the definition of matching cycle states that $\mu_M(m)P(m)\mu_2(m)$ $R(m)\mu(m)$ for all $m \in \sigma'$, which contradicts statement (1). □

4. Optimal sequential entry

In this section we study a process of sequential entry. We assume that participants enter the market one at a time, in some arbitrary given order. At the beginning, we assume stability and the DA algorithm with input matchings is applied after the entry of each agent. This iterative process generates a stable matching. We identify a large family of orders (optimal orders) which converges to the optimal stable matching. We also show, as a consequence of Blum, Roth and Rothblum's results [2], that given an arbitrary order, the last agent to enter the market through the process gets his or her optimal outcome under stable matchings. We finally observe that not all stable matchings can be obtained under the process of sequential entry.

An order over the agents is a bijective function from $\{n \leq |V|\}$ to V. We denote by Γ the set of all orders. For $\gamma \in \Gamma$, $\gamma(i)$ represents the ith agent that enters the market and v enters the market before v' if $\gamma^{-1}(v) < \gamma^{-1}(v')$. We consider the kth reduced market $(M(k), W(k), P_k)$ where $M(k)$ is the set of men that enters the market until the kth position, i.e. $M(k) = \{m \in M : \gamma^{-1}(m) \leq k\}$. Similarly, $W(k)$ is the set of women that enters the market until the kth position.

The process of sequential entry is described as follows:

Input

Let $\gamma \in \Gamma$.

Initial stage

(0) Let $(M(1), W(1), P_1)$ be the first reduced market and $\mu_1(\gamma(1)) = \gamma(1)$.

Iteration stage

For $k = 2$ until $|V|$ do:

(1) Let $(M(k), W(k), P_k)$ be the kth reduced market.

(2) $\mu'_{k-1}(v) = \begin{cases} \mu_{k-1}(v) & \text{if } v \neq \gamma(k) \\ v & \text{if } v = \gamma(k) \end{cases}$ for all $v \in M(k) \cup W(k)$.

(3) If $\gamma(k) \in M$, then $\mu_k = DA_{M(k)}\left(\mu'_{k-1}\right)$,

if not $\mu_k = DA_{W(k)}\left(\mu'_{k-1}\right)$.

Given any order γ we denote by μ_γ the matching that is obtained by the application of the process of sequential entry with input γ. The process previously described has a simple interpretation. Every time an agent enters, he or she proposes to his or her favorite agent in the reduced market. Each agent who receives a proposal, selects his or her most preferred agent between his present mate outcome and those who proposed to him or her. If an agent is rejected, he or she proposes to his or her next choice as long as there is an acceptable agent to whom he or she has not been proposed yet.

The two theorems below are an inmediate consequence of the results shown by Blum, Roth and Rothblum [2]. Alternative proofs have also been provided by Blum and Rothblum [3].

Theorem 4.1. *Let $\gamma \in \Gamma$. Then μ_γ is stable.*

It is clear that the matching μ_1 is stable in $(M(1), W(1), P_1)$. Then the matching μ_1' is either man-quasi-stable or women-quasi-stable in $(M(2), W(2), P_2)$. By Theorem 2.2 or Theorem 2.3, we have that μ_2 is a stable matching in $(M(2), W(2), P_2)$. Repeated applications of this procedure show the Theorem 4.1.

The following theorem shows that the last agent who enters the market receives his or her optimal outcome under stable matchings.

Theorem 4.2. *Let $\gamma \in \Gamma$ and $\gamma^{-1}(|V|) = v$. Then $\mu_\gamma(v)$ is the optimal outcome for v under stable matchings.*

Proof. Let $\mu_\gamma = \mu_k$ be the last matching defined at the application of the process of sequential entry with input γ. Assume that $v = m \in M$ (the case where $v = w \in W$ follows from symmetric arguments). As $\gamma(k) = m$ we have that $\mu_{k-1}'(m) = m$. Then Theorem 2.2 implies that $\mu_\gamma(m) = \mu_k(m) = [DA_M(\mu_{k-1}')](m) = \mu_M(m)$. □

Theorem 4.2 shows the fact that not all stable matchings can be reached by the process of sequential entry, since a matching that does not match any agent with his or her optimal outcome under stable matchings cannot be generated by the process of sequential entry. For instance, the egalitarian stable matching in example 1 of Gale-Shapley's paper [4].

The natural question is which stable matchings can we obtain from the process of sequential entry. We partially answer this question by identifying a large family of orders (optimal orders) which converges to the optimal stable matching.

For $\gamma \in \Gamma$, we define $s = \min\{n \leq |V| : \text{there exists } \sigma \in \Sigma(\mu_M) \text{ satisfying } \sigma \subseteq M(n)\}$.

Definition 4.1. Let $\gamma \in \Gamma$. γ is an optimal order for M if:

(i) $\gamma^{-1}(\mu_M(m)) < s$ for each $m \in M(s)$.
(ii) $\gamma^{-1}(\mu_M(m)) < \gamma^{-1}(m)$ for each $m \notin M(s)$.

Remark 4.1. Let σ_1 be the first cycle for $P(\mu_M)$ that is completed (i.e. the set σ_1 enters the market). Note that $\gamma(s)$ is the last man of σ_1 to enter the market.

Loosely speaking, the order γ is optimal for M if a man m enters the market before the first cycle for $P(\mu_M)$ is completed, then $\mu_M(m)$ should also enter before the first cycle is completed, and if a man m enters the market after the first cycle for $P(\mu_M)$ is completed, then $\mu_M(m)$ should enter the market before him.

The following theorem shows that an optimal order for M leads to the optimal stable matching μ_M.

Theorem 4.3. *Let γ be an optimal order for M. Then $\mu_\gamma = \mu_M$.*

The following lemma is needed to prove Theorem 4.3.

Lemma 4.1. *Let $\gamma \in \Gamma$ and $m' \in M$ which verify:*

(i) $\mu_M(m') \neq m'$.
(ii) $\gamma^{-1}(\mu_M(m)) < \gamma^{-1}(m')$ *if* $m \in M(\gamma^{-1}(m'))$.
(iii) $\gamma^{-1}(\mu_M(m)) < \gamma^{-1}(m)$ *if* $m \notin M(\gamma^{-1}(m'))$.

Then $\mu_\gamma(m')R(m')\mu_M(m')$.

Proof. For $k : 1, \ldots, |V|$, let μ_k and μ'_{k-1} be the matchings defined at the kth iteration of the application of the process of sequential entry with input γ.

Let $k = \gamma(m')$. First, we will show that $\mu_k(m')R(m')\mu_M(m')$. By Theorem 4.2, $\mu_k(m') = \mu_{M(k)}(m')$, where $\mu_{M(k)}$ is the optimal stable matching for all men in the reduced market $(M(k), W(k), P_k)$. By conditions (ii), we can define a matching μ in the reduced market $(M(k), W(k), P_k)$ by $\mu(m) = \mu_M(m)$ for all $m \in M(k)$, and $\mu(w) = w$ for all $w \in W(k) \setminus \mu_M(M(k))$. Lemma 2.1 assures that no pair that contains a non-single woman blocks μ under $(M(k), W(k), P_k)$; hence, $\mu \in Q_{W(k)}(M(k), W(k), P_k)$. Then Theorem 2.3 implies that $DA_{W(k)}(\mu) \in S(M(k), W(k), P_k)$ and since $\mu_{M(k)}$ is the optimal stable matching for all men, $\mu_{M(k)}(m')R_k(m')[DA_{W(k)}(\mu)](m')$. Furthermore, by Theorem 2.2 $[DA_{W(k)}(\mu)](m')$ $R_k(m')\mu(m')$. Hence, as P_k is the natural restriction of P, we have that

$$\mu_k(m') = \mu_{M(k)}(m')R(m')\mu(m') = \mu_M(m') \tag{2}$$

(where the last equality follows the definition of μ).

Now we will show that $\mu_{k+1}(m')R(m')\mu_M(m')$. We next consider two cases:

Case 1. $\gamma(k+1) = w \in W$: By Theorem 2.3 we have that

$$\mu_{k+1}(m') = [DA_{W(k+1)}(\mu'_k)](m')R_{k+1}(m')\mu'_k(m').$$

Furthermore, as $m' \neq \gamma(k+1)$ it follows that $\mu'_k(m') = \mu_k(m')$. By the statement (2) and as P_{k+1} is the natural restriction of P, we conclude that

$$\mu_{k+1}(m')R(m')\mu_k(m')R(m')\mu_M(m').$$

Case 2. $\gamma(k+1) = m \in M$: By conditions (ii) and (iii), we can define a matching μ in the reduced market $(M(k+1), W(k+1), P_{k+1})$ by $\mu(m) = \mu_M(m)$ for all $m \in M(k+1)$, and $\mu(w) = w$ for all $w \in W(k+1) \setminus \mu_M(M(k+1))$. Lemma 2.1 assures that no pair that contains a non-single woman blocks μ under $(M(k+1), W(k+1), P_{k+1})$. Hence, $\mu \in Q_{W(k+1)}(M(k+1), W(k+1), P_{k+1})$. Then Theorem 2.3 implies that $DA_{W(k+1)}(\mu) \in S(M(k+1), W(k+1), P_{k+1})$ and since $\mu_{M(k+1)}$ is the optimal stable matching for all men, $\mu_{M(k+1)}(m')R_{k+1}(m')[DA_{W(K+1)}(\mu)](m')$. Furthermore, by Theorem 2.3 $[DA_{W(k+1)}(\mu)](m')R_{k+1}(m')\mu(m')$. Hence, as P_{k+1} is the natural restriction of P, we have that

$$\mu_{M(k+1)}(m')R(m')\mu(m') = \mu_M(m'). \tag{3}$$

(where the last equality follows the definition of μ). From statement (3) and conditions (i), we have that $\mu_{M(k+1)}(m')P_{k+1}(m')m'$. Furthermore from claim (2) of Theorem 2.2 $\mu_{k+1}(m') = [DA_{M(k+1)}(\mu'_k)](m') = [\mu'_k \vee_{W(k+1)} \mu_{M(k+1)}](m')$. Therefore, $\mu_{k+1}(m') \in \{\mu'_k(m'), \mu_{M(k+1)}(m')\}$. If $\mu_{k+1}(m') = \mu'_k(m')$, we get from statement (2) and $\mu'_k(m') = \mu_k(m')$ (since $m' \neq \gamma(k+1)$) that $\mu_{k+1}(m')R(m')\mu_M(m')$. Now, if $\mu_{k+1}(m') = \mu_{M(k+1)}(m')$, then statement (3) implies that $\mu_{k+1}(m')R(m')$ $\mu_M(m')$.

By iterating this procedure we conclude that $\mu_\gamma(m')R(m')\mu_M(m')$. □

Proof of Theorem 4.3. Let σ_1 be the first cycle for $P(\mu_M)$ that is completed. By Remark 4.1, $\gamma(s) = m_1 \in \sigma_1$. As γ is an optimal order for M, we have that $\gamma(s) = m_1$ satisfies the conditions of Lemma 4.1. By the conclusion of lemma $\mu_\gamma(m_1)R(m_1)\mu_M(m_1)$.

Suppose that there exists $\sigma_2 \in \Sigma(\mu_M)$ such that $\sigma_2 \neq \sigma_1$. Then the assumption that σ_1 is the first cycle for $P(\mu_M)$ that is completed implies that there exists $m_2 \in \sigma_2$ such that $\gamma^{-1}(m_2) > s$. Now, let m be such that $\gamma^{-1}(m) \leq \gamma^{-1}(m_2)$. If $\gamma^{-1}(m) \leq s$ then by condition (i) of the Definition 4.1 $\gamma^{-1}(\mu_M(m)) < s < \gamma^{-1}(m_2)$, and if $\gamma^{-1}(m) > s$ then by condition (ii) from Definition 4.1 $\gamma^{-1}(\mu_M(m)) < \gamma^{-1}(m) \leq \gamma^{-1}(m_2)$. Furthermore, let m' be such that $\gamma^{-1}(m') > \gamma^{-1}(m_2)$. Then $\gamma^{-1}(m') > s$ and by condition (ii) from Definition 4.1 we have that $\gamma^{-1}(\mu_M(m')) < \gamma^{-1}(m')$. We established that three conditions of Lemma 4.1 are satisfied. The conclusion of the lemma implies that $\mu_\gamma(m_2)R(m_2)\mu_M(m_2)$.

Hence, for every cycle σ for $P(\mu_M)$ there exists $m \in \sigma$ such that $\mu_\gamma(m)R(m)$ $\mu_M(m)$. By Corollary 3.1, we conclude that $\mu_\gamma = \mu_M$. □

We conclude our paper with the following remarks.

Remark 4.2. The following example shows that the two conditions of Definition 4.1 are necessary for the optimal matching to be the outcome of the application of the process of sequential entry with an optimal order as input.

Example 4.1. Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ with preferences P given by:

$$\begin{aligned} P(m_1) &= w_1, w_2 & P(w_1) &= m_2, m_1 \\ P(m_2) &= w_2, w_1 & P(w_2) &= m_3, m_1, m_2 \\ P(m_3) &= w_3, w_2 & P(w_3) &= m_3. \end{aligned}$$

Then, $\mu_M = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ and $\mu_W = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$, and $P(\mu_M)$ is:

$$\begin{array}{ll} P(m_1) = w_1, w_2 & P(w_1) = m_2, m_1 \\ P(m_2) = w_2, w_1 & P(w_2) = m_1, m_2 \\ P(m_3) = w_3 & P(w_3) = m_3. \end{array}$$

The only cycle for $P(\mu_M)$ is $\sigma = \{m_1, m_2\}$.

The sequence $(w_1, m_1, m_2, w_2, w_3, m_3)$ does not verify the condition (i) of the Definition 4.1 and we show that it leads to μ_W. When w_1 enters she does not propose. This leaves w_1 single and the resulting matching is:

$$\mu_1 = \begin{pmatrix} w_1 \\ w_1 \end{pmatrix}.$$

When m_1 enters he proposes to w_1, who accepts. Thus, the resulting matching is:

$$\mu_2 = \begin{pmatrix} w_1 \\ m_1 \end{pmatrix}.$$

Next, when m_2 enters he proposes to w_1, who accepts. Now m_1 is single and the resulting matching is:

$$\mu_3 = \begin{pmatrix} w_1 & (m_1) \\ m_2 & m_1 \end{pmatrix}.$$

Now, when w_2 enters she proposes to the single man m_1, who accepts. Thus, the resulting matching is:

$$\mu_4 = \begin{pmatrix} w_1 & w_2 \\ m_2 & m_1 \end{pmatrix}.$$

Next, when w_3 enters she does not propose. Thus, w_3 is single and the resulting matching is:

$$\mu_5 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & w_3 \end{pmatrix}.$$

Now, when m_3 enters he proposes to the single woman w_3, who accepts. Thus, the resulting matching is:

$$\mu_6 = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \mu_W.$$

Furthermore, the sequence $(m_1, w_1, w_2, m_2, m_3, w_3)$ does not verify the condition (ii) of the Definition 4.1 and it leads to μ_W.

Remark 4.3. We obtain a necessary condition on the order of entry to converge to the optimal matching. We can think of obtaining a characterization of the orders that converge the optimal one. This objective is very complicated because this condition depends not only on the set of optimal matchings and its cycles but also on the agent's preferences. The following example shows two marriage markets with the same set of agents, the same set of cycles and different preferences, and an order that converges to the optimal matching for men in a market, and to the optimal matching for women in another one.

Example 4.2. Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ with the profile of preferences P given by:

$$\begin{aligned} P(m_1) &= w_1, w_2 & \quad P(w_1) &= m_2, m_1 \\ P(m_2) &= w_2, w_1 & \quad P(w_2) &= m_3, m_1, m_2 \\ P(m_3) &= w_3, w_2 & \quad P(w_3) &= m_3. \end{aligned}$$

Then, $\mu_M = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ and $\mu_W = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$, and $P(\mu_M)$ is:

$$\begin{aligned} P(m_1) &= w_1, w_2 & \quad P(w_1) &= m_2, m_1 \\ P(m_2) &= w_2, w_1 & \quad P(w_2) &= m_1, m_2 \\ P(m_3) &= w_3 & \quad P(w_3) &= m_3. \end{aligned}$$

The only cycle for $P(\mu_M)$ is $\sigma = \{m_1, m_2\}$. The sequence $(w_1, m_1, w_2, m_3, m_2, w_3)$ leads to μ_W in the marriage market (M, W, P).

Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$ with the profile of preferences P' given by:

$$\begin{aligned} P'(m_1) &= w_1, w_2 & \quad P'(w_1) &= m_2, m_1 \\ P'(m_2) &= w_2, w_1 & \quad P'(w_2) &= m_1, m_2 \\ P'(m_3) &= w_3, w_2 & \quad P'(w_3) &= m_3. \end{aligned}$$

Then, $\mu_M = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ and $\mu_W = \begin{pmatrix} w_1 & w_2 & w_3 \\ m_2 & m_1 & m_3 \end{pmatrix}$, and $P'(\mu_M)$ is:

$$\begin{aligned} P'(m_1) &= w_1, w_2 & \quad P'(w_1) &= m_2, m_1 \\ P'(m_2) &= w_2, w_1 & \quad P'(w_2) &= m_1, m_2 \\ P'(m_3) &= w_3 & \quad P'(w_3) &= m_3. \end{aligned}$$

The only cycle for $P'(\mu_M)$ is $\sigma = \{m_1, m_2\}$. The sequence $(w_1, m_1, w_2, m_3, m_2, w_3)$ leads to μ_M in the marriage market (M, W, P').

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