REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 54, No. 2, 2013, Pages 59–65
Published online: December 8, 2013

ON TAUBERIAN CONDITIONS FOR $(C, 1)$ SUMMABILITY OF INTEGRALS

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Abstract. We investigate some Tauberian conditions in terms of the general control modulo of the oscillatory behavior of integer order of continuous real functions on $[0, \infty)$ for $(C, 1)$ summability of integrals. Moreover, we obtain a Tauberian theorem for a real bounded function on $[0, \infty)$.

1. Introduction

Throughout this paper we assume that $f(x)$ is a real valued continuous function on $[0, \infty)$ and $s(x) = \int_0^x f(t)\,dt$. The symbols $s(x) = o(1)$ and $s(x) = O(1)$ mean that $\lim_{x\to\infty} s(x) = 0$ and $s(x)$ is bounded on $[0, \infty)$, respectively. The identity

$$s(x) - \sigma(x) = v(x), \tag{1}$$

where $v(x) = \frac{1}{x}\int_0^x tf(t)\,dt$ and $\sigma(x) = \sigma(s(x)) = \frac{1}{x}\int_0^x s(t)\,dt$, is well-known and will be used in various steps of the proofs.

The classical control modulo of the oscillatory behavior of the function $s(x)$ is denoted by $\omega_0(x) = xf(x)$, and the general control modulo of the oscillatory behavior of integer order $m \geq 1$ of the function $s(x)$ is defined in [1] by $\omega_m(x) = \omega_{m-1}(x) - \sigma(\omega_{m-1}(x))$. We note that $\omega_m(x) = v(\omega_{m-1}(x))$ for any integer $m \geq 1$. Moreover, $\omega_m(x)$ can be written as $\omega_m(x) = \underbrace{(v(v(\dots(v}_{m \text{ times}}(\omega_0(x)))\dots)$ for any integer $m \geq 1$.

For each integer $m \geq 0$, $\sigma_m(x)$ and $v_m(x)$ are defined in [2] by

$$\sigma_m(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x \sigma_{m-1}(t)\,dt & m > 1 \\ \sigma(x) & m = 1 \end{cases}$$

and

$$v_m(x) = \begin{cases} \dfrac{1}{x}\displaystyle\int_0^x v_{m-1}(t)\,dt & m \geq 1 \\ v(x) & m = 0 \end{cases},$$

2010 *Mathematics Subject Classification.* 40A10, 40E05, 40G05.

Key words and phrases. $(C, 1)$ summability of integrals, improper integrals, bounded functions, Tauberian conditions and theorems.

respectively. The relationship between the functions $\sigma_m(x)$ and $v_m(x)$ is given by the identity $\sigma_m(x) - \sigma_{m+1}(x) = v_m(x)$.

For a function $s(x)$, we define

$$\left(x\frac{d}{dx}\right)_m s(x) = \left(x\frac{d}{dx}\right)_{m-1}\left(x\frac{d}{dx}s(x)\right) = x\frac{d}{dx}\left(\left(x\frac{d}{dx}\right)_{m-1} s(x)\right)$$

for any positive integer m and nonnegative x, where $\left(x\frac{d}{dx}\right)_0 s(x) = s(x)$, and $\left(x\frac{d}{dx}\right)_1 s(x) = x\frac{d}{dx}s(x)$. A more useful identity for the general control modulo of the oscillatory behavior of integer order $m \geq 1$ of $s(x)$ is

$$\omega_m(x) = \left(x\frac{d}{dx}\right)_m v_{m-1}(x) \tag{2}$$

(see [2] for the proof of this identity).

A function $s(x)$ is said to be $(C,1)$ summable to a finite number ℓ if

$$\lim_{x\to\infty} \sigma(x) = \ell. \tag{3}$$

It is well known that if $s(x)$ is $(C,1)$ summable to ℓ, then $\sigma(x)$ is $(C,1)$ summable to ℓ. But the converse is not true. For example, let s be defined by $s(x) = \int_0^x (2\sin t + t\cos t)dt$. Then, $\sigma(x)$ is $(C,1)$ summable to 1, but $s(x)$ is not $(C,1)$ summable to 1.

If the integral

$$\int_0^\infty f(t)\,dt = \ell \tag{4}$$

exists, then clearly $s(x)$ is $(C,1)$ summable to ℓ. The converse is not necessarily true. For example, the function s defined by $s(x) = \int_0^x \sin t\,dt$ is $(C,1)$ summable to 1, but it is not convergent in the ordinary sense. However, (3) may imply (4) by adding some suitable condition on $s(x)$. Such a condition is called a Tauberian condition and the resulting theorem is called a Tauberian theorem.

A real valued function $s(x)$ is slowly oscillating [3] if

$$\lim_{\lambda\to 1^+} \limsup_{x\to\infty} \max_{x\leq t\leq \lambda x} |s(t) - s(x)| = 0. \tag{5}$$

Note that $\sigma(x)$ is slowly oscillating for every slowly oscillating function $s(x)$.

On the other hand, $\omega_0(x) = O(1)$ implies that $s(x)$ is an slowly oscillating function. Indeed,

$$|s(t) - s(x)| = \left|\int_x^t s'(u)\,du\right| \leq \int_x^t |f(u)|\,du \leq \int_x^t \frac{du}{u} = \log\frac{t}{x}$$

By the definition of slowly oscillating function, we have

$$\lim_{\lambda\to 1^+} \limsup_{x\to\infty} \max_{x\leq t\leq \lambda x} |s(t) - s(x)| \leq 0.$$

The following classical Tauberian theorem is known as Littlewood type Tauberian theorem [4].

Theorem 1. *Let $s(x)$ be $(C,1)$ summable to ℓ. If*

$$\omega_0(x) = O(1)$$

then $\int_0^\infty f(t)\,dt$ converges to ℓ.

Different proofs of Theorem 1 are given by Laforgia [5], Çanak and Totur [6].

Çanak and Totur [1] have proved the following theorems for $(C,1)$ summability method.

Theorem 2. *Let $s(x)$ be $(C,1)$ summable to ℓ. If $s(x)$ is slowly oscillating, then $\int_0^\infty f(t)\,dt$ converges to ℓ.*

Theorem 3. *Let $s(x)$ be $(C,1)$ summable to ℓ. If $v(x)$ is slowly oscillating, then $\int_0^\infty f(t)\,dt$ converges to ℓ.*

The aim of this paper is to investigate some Tauberian conditions in terms of the general control modulo of the oscillatory behavior of integer order of continuous real functions on $[0,\infty)$ for $(C,1)$ summability of integrals. Also, we obtain a Tauberian theorem for a real bounded function $s(x)$.

2. A Lemma

For the proofs of the main theorems in the next section we need the following Lemma in [1] showing the differences between $s(x)$ and $\sigma(\lambda x)$ for $\lambda > 1$ and $0 < \lambda < 1$.

Lemma 1.

(i) *For $\lambda > 1$,*

$$s(x) - \sigma(\lambda x) = \frac{1}{\lambda - 1}(\sigma(\lambda x) - \sigma(x)) - \frac{1}{\lambda x - x}\int_x^{\lambda x}(s(t) - s(x))\,dt.$$

(ii) *For $0 < \lambda < 1$,*

$$s(x) - \sigma(\lambda x) = \frac{1}{1 - \lambda}(\sigma(x) - \sigma(\lambda x)) + \frac{1}{x - \lambda x}\int_{\lambda x}^{x}(s(x) - s(t))\,dt.$$

3. Main results

In this section, the main theorems of the paper will be presented.

Theorem 4. *Let $s(x)$ be $(C,1)$ summable to ℓ. If $\sigma(\omega_m(x))$ is slowly oscillating for some integer $m \geq 0$, then $\int_0^\infty f(t)\,dt$ converges to ℓ.*

Proof. Since $s(x)$ is $(C,1)$ summable to ℓ, then $\sigma(x)$ is also $(C,1)$ summable to ℓ. Thus, it follows from the identity (1) that $v(x)$ is $(C,1)$ summable to 0. By the definition of the general control modulo of oscillatory behavior of order m, we obtain that

$$\sigma(\omega_m(x)) \text{ is } (C,1) \text{ summable to } 0 \tag{6}$$

for some integer $m \geq 1$. On the other hand, since $\sigma(\omega_m(x))$ is slowly oscillating, we get, by Lemma 1 (i),

$$\begin{aligned} \sigma(\omega_m(x)) - \sigma_2(\omega_m(\lambda x)) &= \frac{1}{\lambda - 1}\left(\sigma_2(\omega_m(\lambda x)) - \sigma_2(\omega_m(x))\right) \\ &\quad - \frac{1}{\lambda x - x}\int_x^{\lambda x}\left(\sigma(\omega_m(t)) - \sigma(\omega_m(x))\right)dt. \end{aligned} \tag{7}$$

By (7) we have

$$\begin{aligned} |\sigma(\omega_m(x)) - \sigma_2(\omega_m(\lambda x))| &\leq \frac{1}{\lambda - 1}\left|\sigma_2(\omega_m(\lambda x)) - \sigma_2(\omega_m(x))\right| \\ &\quad + \max_{x \leq t \leq \lambda x}|\sigma(\omega_m(t)) - \sigma(\omega_m(x))|. \end{aligned} \tag{8}$$

Taking the lim sup of both sides of (8) as $x \to \infty$, we obtain

$$\begin{aligned} \limsup_{x\to\infty}|\sigma(\omega_m(x)) - \sigma_2(\omega_m(\lambda x))| &\leq \frac{1}{\lambda - 1}\limsup_{x\to\infty}\left|\sigma_2(\omega_m(\lambda x)) - \sigma_2(\omega_m(x))\right| \\ &\quad + \limsup_{x\to\infty}\max_{x \leq t \leq \lambda x}|\sigma(\omega_m(t)) - \sigma(\omega_m(x))|. \end{aligned} \tag{9}$$

Since $\sigma_2(\omega_m(x))$ converges by (6), the first term on the right-hand side of the inequality (9) vanishes. Therefore, the inequality (9) becomes

$$\limsup_{x\to\infty}|\sigma(\omega_m(x)) - \sigma_2(\omega_m(\lambda x))| \leq \limsup_{x\to\infty}\max_{x \leq t \leq \lambda x}|\sigma(\omega_m(t)) - \sigma(\omega_m(x))|. \tag{10}$$

Since $\sigma(\omega_m(x))$ is slowly oscillating, we have from (10) that $\lim_{x\to\infty}\sigma(\omega_m(x)) = 0$.

Analogously, since $s(x)$ is $(C, 1)$ summable to ℓ, we obtain that $\sigma_2(\omega_{m-1}(x))$ is $(C, 1)$ summable to 0. It follows from the identity

$$\sigma(\omega_m(x)) = \left(\frac{d}{dx}\right)_m v_m(x) = \frac{d}{dx}\left(\frac{d}{dx}\right)_{m-1} v_m(x) = \frac{d}{dx}\sigma_2(\omega_{m-1}(x))$$

that $\frac{d}{dx}\sigma_2(\omega_{m-1}(x)) = O(1)$. If we apply Theorem 1 to $\sigma_2(\omega_{m-1}(x))$, we see that $\sigma_2(\omega_{m-1}(x)) = o(1)$. From the identity

$$\sigma(\omega_{m-1}(x)) - \sigma_2(\omega_{m-1}(x)) = \sigma(\omega_m(x))$$

we have $\lim_{x\to\infty}\sigma(\omega_{m-1}(x)) = 0$. Continuing in this vein, we get $\sigma(\omega_0(x)) = v(x) = o(1)$ as $x \to \infty$. Since $s(x)$ is $(C, 1)$ summable to ℓ and $\lim_{x\to\infty} v(x) = 0$, $\lim_{x\to\infty} s(x) = \ell$. This completes the proof. □

Remark 5. Theorem 4 includes not only Theorem 3, but also Theorem 2. If $\sigma(\omega_m(x))$ is slowly oscillating for some integer $m \geq 0$, it follows from the identity $\sigma(\omega_{m+1}(x)) = v(\sigma(\omega_m(x)))$ that $\sigma(\omega_{m+1}(x))$ is slowly oscillating. However, the slow oscillation of $\sigma(\omega_m(x))$ does not imply the slow oscillation of $\sigma(\omega_{m-1}(x))$, since $\sigma(\omega_m(x)) = v(\sigma(\omega_{m-1}(x)))$. If we take $\sigma(\omega_{m-1}(x)) = \log(x+1) + \int_0^x \frac{\log(t+1)}{t+1}dt$, we have that $\sigma(\omega_{m-1}(x))$ is not slowly oscillating, but $\sigma(\omega_m(x)) = \log(x+1)$ is slowly oscillating. If we take $m = 0$ in Theorem 4, then we have Theorem 3.

An equivalent definition of slow oscillation of $s(x)$ given by Çanak and Totur [1] says that $s(x)$ is slowly oscillating if and only if $v(x)$ is slowly oscillating and bounded. If $v(x)$ is slowly oscillating, then $s(x)$ may not be slowly oscillating. For example, if we take $f(x)=\frac{1}{x+1}+\frac{\log(x+1)}{x+1}$, it is clear that $s(x)$ is not slowly oscillating. It follows from the identity (1) that $v(x)=\log(x+1)$ is slowly oscillating.

Corollary 6. *If $s(x)$ is $(C,1)$ summable to ℓ and $\frac{d}{dx}(\sigma_1(\omega_m(x)))=O\left(\frac{p'(x)}{p(x)}\right)$ as $x\to\infty$, where*

$$\lim_{\lambda\to1^+}\limsup_{x\to\infty}\frac{p(\lambda x)}{p(x)}=1, \tag{11}$$

then $\lim_{x\to\infty}s(x)=\ell$.

Proof. For any $x\le t\le\lambda x$, we have

$$|\sigma_1(\omega_m(t))-\sigma_1(\omega_m(x))|=\left|\int_x^t\frac{d}{dx}\sigma_1(\omega_m(u))du\right|\le C\int_x^t\frac{p'(u)}{p(u)}du=C\log\frac{p(t)}{p(x)},$$

whence we conclude that

$$\limsup_{x\to\infty}\max_{x\le t\le\lambda x}|\sigma_1(\omega_m(t))-\sigma_1(\omega_m(x))|\le C\log\limsup_{x\to\infty}\frac{p(\lambda x)}{p(x)}.$$

Taking the limit of both sides as $\lambda\to1^+$, we obtain

$$\lim_{\lambda\to1^+}\limsup_{x\to\infty}\max_{x\le t\le\lambda x}|\sigma_1(\omega_m(t))-\sigma_1(\omega_m(x))|=0,$$

i.e. $\sigma_1(\omega_m(x))$ is slowly oscillating. □

Corollary 7. *Let $s(x)$ be $(C,1)$ summable to ℓ. If $\omega_m(x)$ is slowly oscillating for some integer $m\ge0$, then $\int_0^\infty f(t)\,dt$ converges to ℓ.*

Proof. Since $\omega_m(x)$ is slowly oscillating, then $\sigma(\omega_m(x))$ is also slowly oscillating. Hence, the conditions in Theorem 4 are satisfied. □

In the following theorem, we obtain convergence of $s(x)$ out of the $(C,1)$ summability of $\sigma(x)$ instead of the $(C,1)$ summability of $s(x)$ with the same condition as in Theorem 4.

Theorem 8. *Let $\sigma(x)$ be $(C,1)$ summable to ℓ. If $\sigma(\omega_m(x))$ is slowly oscillating for some integer $m\ge0$, then $\int_0^\infty f(t)\,dt$ converges to ℓ.*

Proof. Since $\sigma(x)$ is $(C,1)$ summable to ℓ, from the identity (1), we have $v_1(x)$ is $(C,1)$ summable to 0. Thus, $\sigma_2(\omega_m(x))$ is $(C,1)$ summable to 0. Since $\sigma(\omega_m(x))$ is slowly oscillating for some nonnegative integer m, then we have $v(\sigma(\omega_m(x)))=\sigma(\omega_{m+1}(x))=O(1)$. It follows from the identity $\sigma(\omega_{m+1}(x))=x\frac{d}{dx}\sigma_2(\omega_m(x))$ that we have $x\frac{d}{dx}\sigma_2(\omega_m(x))=O(1)$. If we apply Theorem 1 to $\sigma_2(\omega_m(x))$, we see that

$$\sigma_2(\omega_m(x))=o(1). \tag{12}$$

Hence, $\sigma_1(\omega_m(x))$ is $(C,1)$ summable to 0 and $\sigma_1(\omega_m(x))$ is slowly oscillating for some nonnegative integer m. Analogously, continuing as in the proof of Theorem 4, we obtain $\sigma_1(\omega_m(x))=o(1)$ as $x\to\infty$. It follows from the identity $\sigma(\omega_m(x))=x\frac{d}{dx}\sigma_2(\omega_{m-1}(x))$ that we have $x\frac{d}{dx}\sigma_2(\omega_{m-1}(x))=O(1)$. If we apply Theorem 1 to $\sigma_2(\omega_{m-1}(x))$, we see that

$$\sigma_2(\omega_{m-1}(x))=o(1). \tag{13}$$

By the assumption and (13) we obtain from the identity

$$\sigma(\omega_{m-1}(x)) - \sigma_2(\omega_{m-1}(x)) = \sigma(\omega_m(x))$$

that $\sigma(\omega_{m-1}(x)) = o(1)$.

Continuing in this vein, we obtain $\sigma_2(\omega_1(x)) = o(1)$. Since $\sigma(x)$ is $(C,1)$ summable to ℓ, we have $v_2(x) = o(1)$. Therefore, from the identity

$$\sigma_2(\omega_1(x)) = \sigma_2(\omega_0(x)) - \sigma_3(\omega_0(x)) = v_1(x) - v_2(x)$$

we get $v_1(x) = o(1)$. Since $\sigma(x)$ is $(C,1)$ summable to ℓ, we have $\sigma_2(x) \to \ell$ as $x \to \infty$. From the identity $\sigma(x) - \sigma_2(x) = v_1(x)$, we obtain $\sigma(x) \to \ell$ as $x \to \infty$. Thus, $s(x)$ is $(C,1)$ summable to ℓ. Since the conditions in Theorem 4 are satisfied, the proof is completed. □

Theorem 9. *Let $s(x)$ be bounded. If $\frac{d}{dx}(\sigma(\omega_m(x)))$ is slowly oscillating for some integer $m \geq 0$, then $f(x)$ converges to 0 as $x \to \infty$.*

Proof. Since $s(x)$ is assumed to be bounded, $\sigma(\omega_m(x))$ is bounded for every nonnegative integer m. Thus, we have

$$\sigma\left(\frac{d}{dx}\sigma(\omega_m(x))\right) = \frac{1}{x}\int_0^x \frac{d}{dx}\left(\sigma(\omega_m(x))\right) = \frac{\sigma(\omega_m(x))}{x} = o(1), \quad x \to \infty.$$

Thus, we obtain that $\frac{d}{dx}(\sigma(\omega_m(x)))$ is $(C,1)$ summable to 0. By hypothesis, if we apply Theorem 1 to $\frac{d}{dx}(\sigma(\omega_m(x)))$, we see that $\frac{d}{dx}(\sigma(\omega_m(x)))$ converges to 0. If we write $\frac{d}{dx}(\sigma(\omega_{m-1}(x)))$ instead of $s(x)$ in the identity (1), we have

$$\frac{d}{dx}\left(\sigma(\omega_{m-1}(x))\right) - \frac{\sigma(\omega_m(x))}{x} = \frac{d}{dx}\left(\sigma(\omega_m(x))\right).$$

So, $\frac{d}{dx}(\sigma(\omega_{m-1}(x))) = o(1)$. Continuing in this manner, we obtain $\frac{d}{dx}(\sigma(\omega_0(x))) = v'(x) = o(1)$ as $x \to \infty$. On the other hand, boundedness of $s(x)$ implies boundedness of $v(x)$. Finally, applying the identity (1) to $\frac{d}{dx}s(x)$, we have

$$s'(x) = \frac{v(x)}{x} + v'(x) = o(1).$$

Hence $s'(x) = f(x) = o(1)$. This completes the proof. □

Acknowledgments. The authors are grateful to the referee for his/her valuable suggestions and constructive remarks in revising the paper.

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Received: February 17, 2012
Accepted: July 16, 2013