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TWO CLASSES OF SLANT SURFACES IN THE NEARLY KÄHLER SIX SPHERE

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Abstract. In this paper we find examples of slant surfaces in the nearly Kähler six sphere. First, we characterize two-dimensional small and great spheres which are slant. Their description is given in terms of the associative 3-form in $\operatorname{Im}\mathbb{O}$. Later on, we classify the slant surfaces of S^6 which are orbits of a maximal torus in G_2. Among them we find a one parameter family of minimal orbits with arbitrary slant angle.

1. Introduction

It is known that S^2 and S^6 are the only spheres that admit almost complex structures. The best known Hermitian almost complex structure J on S^6 is defined using octonionic multiplication. It is not integrable, but satisfies the condition $(\nabla_X J)X = 0$, for the Levi-Civita connection ∇ and every vector field X on S^6. The sphere S^6 with this structure J is usually referred to as nearly Kähler six sphere.

Submanifolds of the nearly Kähler sphere S^6 are the subject of intensive research. A. Gray [8] proved that almost complex submanifolds of nearly Kähler S^6 are necessarily two-dimensional and minimal. In [4] Bryant showed that any Riemannian surface can be embedded in the six sphere as an almost complex submanifold. Almost complex surfaces were further investigated in [2] and classified into four types.

Totally real submanifolds of S^6 can be of dimension two or three. Three-dimensional totally real submanifolds are investigated in [9] where N. Ejiri proved that they have to be minimal and orientable. In [3] the authors classify totally real, minimal surfaces of constant curvature. Totally real and minimal surfaces with Gauss curvature $K \in [0,1]$ (and compact) or those with K constant must have either $K = 0$ or $K = 1$ (see [7]).

Slant submanifolds are a generalization of totally real and almost complex submanifolds, that have slant angle $\frac{\pi}{2}$ and 0, respectively. Slant submanifolds with slant angle $\theta \in (0, \frac{\pi}{2})$ are called proper slant submanifolds. Note that the notion of

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a surface with constant Kähler angle coincides with the notion of a slant surface. The general theory regarding slant submanifolds and some classification theorems of slant surfaces in $\mathbb{C}^2$ can be found in [5]. Because of dimensional reasons (see [5]) a proper slant submanifold of the six sphere is two-dimensional. Very few examples of slant surfaces of S^6 are known and all of them are minimal. In [2] it is shown that rotation of an almost complex curve of type (III) results in a minimal slant surface that is linearly full in some $S^5 \subset S^6$. According to [13], a minimal slant surface of S^6 that has non-negative Gauss curvature $K \geq 0$ must have either $K \equiv 0$ or $K \equiv 1$. Classification of such surfaces is given in [14]. In the present paper we find two classes of slant surfaces of S^6 with $K \equiv 0$ and $K \equiv 1$. Some of them are minimal and therefore known, but most of them are not.

In Section 2 we recall some basic facts about octonions and define the notion of a slant submanifold (Definition 2.1). Lemma 2.2 is simple, but it is not in [5]. In Section 3 we consider slant two-dimensional spheres which are the intersection of an affine 3-plane and the six sphere. We give their characterization in terms of the associative 3-form in Theorem 3.1. Finally, in Section 4, we investigate two-dimensional orbits of a Cartan subgroup of the group G_2 on the sphere S^6. We found that these orbits are flat, two-dimensional tori which are always slant, with an arbitrary slant angle (see Theorem 4.1). In Theorem 4.2 we find a one-parameter family of minimal orbits. Note that a similar method was used in [11] to obtain three-dimensional orbits which are CR submanifolds of the six sphere S^6.

2. Preliminaries

Let $\mathbb{H}$ be the field of quaternions. The Cayley algebra, or algebra of octonions, is the vector space $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} \cong \mathbb{R}^8$ with multiplication defined in terms of quaternionic multiplication:

$$(q, r) \cdot (s, t) := (qs - \bar{t}r, tq + r\bar{s}), \quad q, s, r, t \in \mathbb{H}.$$

In the sequel we omit the multiplication sign. Conjugation of octonions is defined by

$$\overline{(q, r)} := (\overline{q}, -r), \quad q, r \in \mathbb{H},$$

and the inner product by

$$\langle x, y \rangle := \frac{1}{2}(x\overline{y} + y\overline{x}), \quad x, y \in \mathbb{O}. \tag{1}$$

If we denote by $1, i, j, k$ the standard orthonormal basis of $\mathbb{H}$, then $e_0 = (1, 0)$, $e_1 = (i, 0)$, $e_2 = (j, 0)$, $e_3 = (k, 0)$, $e_4 = (0, 1)$, $e_5 = (0, i)$, $e_6 = (0, j)$, $e_7 = (0, k)$ is an orthonormal basis of $\mathbb{O}$. It is easy to check that the multiplication in that basis is given by Table 1.

Octonions are not associative, so the *associator* is defined by

$$[x, y, z] := (xy)z - x(yz), \quad x, y, z \in \mathbb{O}.$$

Denote by

$$\operatorname{Im} \mathbb{O} := \{x \in \mathbb{O} \mid x + \overline{x} = 0\},$$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$-e_0$	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	$-e_3$	$-e_0$	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_2	$-e_1$	$-e_0$	e_7	$-e_6$	e_5	$-e_4$
e_4	$-e_5$	$-e_6$	$-e_7$	$-e_0$	e_1	e_2	e_3
e_5	e_4	$-e_7$	e_6	$-e_1$	$-e_0$	$-e_3$	e_2
e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	$-e_0$	$-e_1$
e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	$-e_0$

TABLE 1. Multiplication in the basis $e_0, \dots, e_7$ of $\mathbb{O}$

the subspace of imaginary octonions. Then we have the orthogonal decomposition

$$\mathbb{O} = \mathbb{R} \oplus \operatorname{Im} \mathbb{O} = \mathbb{R} \oplus \mathbb{R}^7. \tag{2}$$

On the subspace of imaginary octonions $\operatorname{Im} \mathbb{O}$ the vector product is defined by

$$x \times y := \frac{1}{2}(xy - yx),$$

that shares many properties with the vector product in $\mathbb{R}^3$.

We state some well known properties of octonions without a proof (see [10]).

Lemma 2.1.

1) *If* $x, y \in \operatorname{Im} \mathbb{O}$ *then*

$$xy = -\langle x, y \rangle + x \times y.$$

2) *For all* $x, y, z \in \mathbb{O}$ *we have*

$$\overline{x}(xy) = (\overline{x}x)y,$$

$$\langle xy, xz \rangle = \langle x, x \rangle \langle y, z \rangle = \langle yx, zx \rangle.$$

3) *If* $x, y, z \in \mathbb{O}$ *are mutually orthogonal unit vectors then*

$$x(yz) = y(zx) = z(xy).$$

The exceptional group G_2 is usually defined as the group of automorphisms of the octionions. Since it preserves the multiplication, it also preserves the inner product (1) and the decomposition (2) and therefore is a subgroup of the group $O(7)$. Actually, it is a subgroup of the group $SO(7)$.

For any point $p \in S^6 \subset \operatorname{Im} \mathbb{O}$ and a tangent vector $X \in T_pS^6$ we define the automorphism $J_p : T_pS^6 \to T_pS^6$ by

$$J_p(X) := p \cdot X = p \times X.$$

One can easily show that the six-dimensional sphere $(S^6, \langle\rangle, J)$ is an almost Hermitian manifold, i.e. J_p satisfies

$$J_p^2 = -\mathrm{Id}\,, \qquad \langle J_pX, J_pY \rangle = \langle X, Y \rangle,$$

for all $X, Y \in T_pS^6$.

The unit six-dimensional sphere $S^6 \subset \operatorname{Im}\mathbb{O} \cong \mathbb{R}^7$ posseses an almost complex structure J defined by

$$J_p(X) = pX = p \times X, \qquad p \in S^6, \quad X \in T_pS^6.$$

Obviously, the group G_2 preserves the structure J.

Definition 2.1. Let (M, g, J) be an almost Hermitian manifold and $N \subset M$ be a submanifold of M. For each $p \in N$ and $X \in T_pN$ we define the *Wirtinger angle* or *slant angle* by

$$\theta_p(X) := \angle(JX, T_pN).$$

We say that N is a *slant submanifold* if its slant angle θ is constant, i.e. it doesn't depend on the point $p \in N$ and the tangent vector $X \in T_pN$.

A slant submanifold with angle $\theta_p \equiv 0$ is usually called an *almost complex submanifold*, and a slant submanifold with the angle $\theta_p \equiv \frac{\pi}{2}$ is called a *totally real submanifold*. A slant submanifold that is neither almost complex nor totally real is called a *proper slant submanifold.*

It is known (see [5]) that a proper slant submanifold $N \subset M$ has even dimension, which must be less than half the dimension of M. The next lemma shows that in the case $\dim N = 2$, the slant angle of N is always independent of vector $X \in T_pM$.

Lemma 2.2. *Let (M, g, J) be an almost-Hermitian manifold and $N \subset M$ a surface. The slant angle $\theta_p(X)$ doesn't depend on the vector $X \in T_pN$ and*

$$\cos\theta_p(Z) = |g(X, JY)|$$

for all $Z \in T_pN$, where (X, Y) is any orthonormal basis of T_pN.

Proof. For each $Z \in T_pN$ there is an orthogonal decomposition

$$JZ = PZ + FZ,$$

where $PZ \in T_pN$ and $FZ \in T_p^{\perp}N$ are the tangent and normal components, respectively.

Therefore,

$$\cos\theta_p(Z) = \frac{g(JZ, PZ)}{\|JZ\|\|PZ\|} = \frac{\|PZ\|^2}{\|Z\|\|PZ\|} = \frac{\|PZ\|}{\|Z\|}.$$

Since J is an isometry we have

$$g(JX, Y) = -g(X, JY),$$

for all $X, Y \in T_pM$ and particularly for $X = Y$

$$g(JX, X) = 0. \tag{3}$$

Let (X, Y) be an orthonormal basis of T_pN. For any $Z = aX + bY \in T_pN$,

$$\begin{aligned}\|PZ\|^2 &= g(PZ, X)^2 + g(PZ, Y)^2 = g(JZ, X)^2 + g(JZ, Y)^2 \\ &= g(aJX + bJY, X)^2 + g(aJX + bJY, Y)^2 \\ &= (a^2 + b^2)g(X, JY)^2,\end{aligned}$$

so we have

$$\cos \theta_p(Z) = \frac{\sqrt{a^2+b^2}\,|g(X,JY)|}{\sqrt{a^2+b^2}} = |g(X,JY)|. \quad \square$$

3. Slant two-dimensional spheres in S^6

Definition 3.1. If $\pi \in G_{\mathbb{R}}(3, \operatorname{Im}\mathbb{O})$ is the imaginary part of a quaternionic subalgebra of octonions $\mathbb{O}$ we call π an associative 3-plane. Denote by $ASSOC \subset G_{\mathbb{R}}(3, \operatorname{Im}\mathbb{O})$ the set of all associative planes.

Since the quaternions are associative, the associator of any three vectors of an associative plane equals zero. Vice versa, if the associator of three vectors in $\operatorname{Im}\mathbb{O}$ vanishes, then these three elements span an associative plane.

On the vector space $\operatorname{Im}\mathbb{O} = \mathbb{R}^7$ we define the 3-form ϕ by the formula

$$\phi(x,y,z) := \langle x, yz\rangle.$$

In [10] this form is called an *associative 3-form* and it is shown that the form ϕ is calibration with the contact set $ASSOC$.

We use the following notation for the associative 3-form ϕ and the associator of a 3-dimensional plane $\pi \in \operatorname{Im}\mathbb{O}$:

$$\phi(\pi) := |\phi(f_1,f_2,f_3)|, \qquad \text{and} \qquad [\pi] := [f_1,f_2,f_3],$$

where f_1, f_2, f_3 is an orthonormal basis of π. One can show that these definitions do not depend on the choice of orthonormal basis f_1, f_2, f_3 of the plane π. Furthermore, both the form ϕ and the associator are G_2 invariant.

The associator and the associative 3-form ϕ are related by the formula

$$\phi^2(\pi) + \frac{1}{4}\|[\pi]\|^2 = 1, \tag{4}$$

which we prove in Lemma 3.1. It follows that $\phi(\pi) \in [0,1]$ for any 3-dimensional plane ϕ and that associative planes are characterized by the condition $\phi(\pi) = 1$.

In the remainder of this section, we characterize two-dimensional spheres that are intersections of a 3-dimensional affine plane and the sphere S^6. Intrinsically, they are the complete connected totally umbilic submanifolds of S^6 of dimension two (see for instance [1, Section 2.6]). For the proof of Theorem 3.1 we need the following lemma.

Lemma 3.1. *Let f_1, f_2, f_3 be an orthonormal basis of the plane π. The Gram matrix of the set of vectors*

$$f = (f_1, f_2, f_3, f_2f_3, f_3f_1, f_1f_2, [f_1,f_2,f_3])$$

is the matrix

$$\begin{pmatrix} I & \varphi I & 0 \\ \varphi I & I & 0 \\ 0 & 0 & 4(1-\varphi^2) \end{pmatrix},$$

where we abbreviate $\varphi = \phi(f_1,f_2,f_3)$ and I is the 3×3 identity matrix. Particularly, the set f spans $\operatorname{Im}\mathbb{O}$ if and only if $\phi(\pi) \neq 1$, i.e. the plane π is not associative.

Proof. Most of the inner products are simple to calculate using properties of octonions from Lemma 2.1. For example:

$$\langle f_2f_3, f_3f_1\rangle = -\langle f_2f_3, f_1f_3\rangle = \langle f_2, f_1\rangle|f_3|^2 = 0.$$

Now we prove the most complicated inner product $\langle[f_1, f_2, f_3], [f_1, f_2, f_3]\rangle$. First, note that all f_2f_3, f_3f_1, f_1f_2 are imaginary. Using simple transformations one can show

$$f_3(f_1f_2) = -2\varphi - (f_1f_2)f_3. \tag{5}$$

Using the previous formula we get

$$\begin{aligned}\langle[f_1, f_2, f_3], [f_1, f_2, f_3]\rangle &= 4(\langle(f_1f_2)f_3, (f_1f_2)f_3\rangle + 2\langle(f_1f_2)f_3, \varphi\rangle + \langle\varphi, \varphi\rangle)\\ &= 4(1 - 2\varphi^2 + \varphi^2) = 4(1 - \varphi^2).\end{aligned}$$

□

Theorem 3.1. *Let $S = \pi' \cap S^6$ be a two-dimensional sphere of radius $r \in (0, 1]$ and denote by π the 3-dimensional subspace parallel to the affine plane π'.*

a) *If the plane π is associative, then S is slant with slant angle $\theta = \arccos r$.*
b) *If π is not associative, then S is slant if and only if its center is a multiple of $[\pi]$, and in this case the slant angle is $\theta = \arccos(r\phi(\pi))$.*

In particular, if $r = 1$ (that is, $\pi = \pi'$ and S is a great sphere), then S is slant, and it is proper slant if π is not associative.

Proof. Let X, Y be an orthonormal basis of the tangent space T_pS, $p \in S$. According to the Lemma 2.2 we have

$$\cos\theta_p = |\langle X, JY\rangle| = |\langle X, pY\rangle| = |\phi(p, X, Y)|.$$

Let f_1, f_2, f_3 be an orthonormal basis of π. Since π' is parallel to π we can write $\pi' = \pi + \sqrt{1 - r^2}\,\xi$ for a unit vector $\xi \in \pi^\perp$. Therefore $p \in S$ is of the form

$$p = rp_0 + \sqrt{1 - r^2}\,\xi$$

for some point $p_0 = p_1f_1 + p_2f_2 + p_3f_3 \in S^6 \cap \pi$. As before, let X, Y be an orthonormal basis of the tangent space T_pS. Then the vectors p_0, X, Y form an orthonormal basis of π.

$$\begin{aligned}\cos\theta_p = |\phi(p, X, Y)| &= |\phi(r\,p_0 + \sqrt{1 - r^2}\,\xi, X, Y)|\\ &= |r\,\phi(p_0, X, Y) + \sqrt{1 - r^2}\,\phi(\xi, X, Y)|.\end{aligned}$$

Since $|\phi(p_0, X, Y)| = \phi(\pi)$, it remains to calculate $\phi(\xi, X, Y)$. Note that if the plane π is associative then $XY \in \pi$ and we have

$$\phi(\xi, X, Y) = \langle\xi, XY\rangle = 0.$$

Therefore, in the case of the associative plane π we have

$$\cos\theta_p = r\phi(\pi) = r,$$

which proves the statement a).

Let π be a non-associative plane. Since $X, Y \in \pi$ we can write them in the form

$$X = x_1 f_1 + x_2 f_2 + x_3 f_3,$$
$$Y = y_1 f_1 + y_2 f_2 + y_3 f_3.$$

The vectors X and Y are orthogonal to the point $p_0 \in \pi$, whose coordinates, with respect to $\{f_1, f_2, f_3\}$, are

$$(p_1, p_2, p_3) = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

Now, an easy calculation yields

$$\begin{aligned}\phi(\xi, X, Y) &= \langle \xi, XY\rangle = \langle \xi, (x_1f_1 + x_2f_2 + x_3f_3)(y_1f_1 + y_2f_2 + y_3f_3)\rangle \\ &= \langle \xi, f_2f_3p_1 - f_1f_3p_2 + f_1f_2p_3\rangle \\ &= \phi(\xi, f_2, f_3)p_1 - \phi(\xi, f_1, f_3)p_2 + \phi(\xi, f_1, f_2)p_3.\end{aligned}$$

The sphere S is slant if this expression doesn't depend on the point $p \in S$, that is, if and only if

$$\phi(\xi, f_2, f_3) = 0, \quad \phi(\xi, f_1, f_3) = 0, \quad \phi(\xi, f_1, f_2) = 0.$$

This condition means that the vector ξ is orthogonal to the vectors f_2f_3, f_3f_1, f_1f_2, and since $\xi \in \pi^{\perp}$, according to Lemma 3.1, the only possibility for the unit vector ξ is

$$\xi = \pm\frac{[\pi]}{|[\pi]|},$$

and therefore the statement b) holds. □

Lemma 3.2. *The two* 3*-dimensional planes* $\pi_1, \pi_2 \in G_{\mathbb{R}}(3, \operatorname{Im}\mathbb{O})$ *are* G_2*-equivalent if and only if* $\phi(\pi_1) = \phi(\pi_2)$.

Proof. If the planes π_1 and π_2 are equivalent by a G_2 transformation then we have $\phi(\pi_1) = \phi(\pi_2)$ because the form ϕ is G_2 invariant. Let us prove the converse.

If $\phi(\pi_1) = 1 = \phi(\pi_2)$, i.e. the planes are associative, they are G_2-equivalent. Namely, if the planes π_1 and π_2 are spanned by orthonormal bases f_1, f_2, f_1f_2 and g_1, g_2, g_1g_2 respectively, then any G_2 transformation that maps f_1, f_2 to g_1, g_2 also maps π_1 onto π_2.

Suppose that $\phi(\pi) = \varphi \neq 1$ and f_1, f_2, f_3 is an orthonormal basis of π. Denote by $F_1 = f_1$, $F_2 = f_2$, $F_3 = f_1f_2$. From Lemma 3.1, it follows that $F_4 = \frac{1}{2\sqrt{1-\varphi^2}}[f_1, f_2, f_3]$ is a unit vector orthogonal to F_1, F_2 and F_3. Following the Cayley-Dixon process, the set of vectors F_1, F_2, F_3, F_4, $F_5 = F_1F_4$, $F_6 = F_2F_4$, $F_7 = F_3F_4$ is G_2 basis of $\operatorname{Im}\mathbb{O}$, i.e. satisfies the same multiplication properties as the standard basis $e_1, \dots, e_7$ from Section 2. One can easily check the following relations:

$$F_5 = \frac{1}{\sqrt{1-\varphi^2}}(\varphi f_1 + f_2 f_3),$$
$$F_6 = \frac{1}{\sqrt{1-\varphi^2}}(\varphi f_2 + f_3 f_1),$$
$$F_7 = \frac{1}{\sqrt{1-\varphi^2}}(-f_3 + \varphi f_1 f_2). \tag{6}$$

There is a G_2 transformation that maps vectors F_1, F_2, F_4 to vectors e_1, e_2, e_4, respectively. According to the relation (6) the image of vector f_3 is $\varphi\, e_3 - \sqrt{1-\varphi^2}\, e_7$. Therefore, in the case $\phi(\pi) = \varphi \neq 1$, the plane π is G_2-equivalent to the plane

$$\pi_0^\varphi = \mathbb{R}\langle e_1, e_2, \varphi\, e_3 - \sqrt{1-\varphi^2}\, e_7\rangle.$$

We conclude that any two planes π_1, π_2 with $\phi(\pi_1) = \phi(\pi_2)$ are G_2-equivalent, as claimed. □

Corollary 3.1.1. *Let $\pi' \cap S^6$ and $\tau' \cap S^6$ be two-dimensional slant spheres from Theorem 3.1. They are G_2-equivalent if and only if they have the same radius and $\phi(\pi) = \phi(\tau)$.*

Remark 3.1.1. Up to G_2-equivalence there is only one almost complex real two-dimensional sphere. It is a great sphere belonging to the associative plane π, i.e. $\phi(\pi) = 1$.

Remark 3.1.2. Totally real two dimensional spheres are $S = \pi' \cap S^6$, where π' is an affine 3-plane parallel to the plane π with $\phi(\pi) = 0$. Up to a G_2-equivalence, there is a unique such sphere for each radius $r \in (0, 1]$. The sphere with radius $r = 1$ is minimal. It is found in the classification of totally real minimal surfaces of constant curvature (see [3, Theorem 6.5 (a)]).

Remark 3.1.3. Slant great spheres S from Theorem 3.1 are exactly those from the classification of minimal slant surfaces with $K \equiv 1$ in S^6 (see [14, Example 3.1]).

4. Two-dimensional slant orbits in S^6

In this section we consider orbits of a maximal torus, i.e. a Cartan subroup H of the group G_2 under the natural action on $S^6 = G_2/SU(3)$. Since such an action preserves both the metric on S^6 and its almost complex structure, all points on a fixed orbit have the same slant angle. Therefore, all such two-dimensional orbits are slant surfaces of S^6.

Since any two Cartan subgroups of G_2 are conjugate by some element of G_2, they have the same set of orbits. Therefore it is not a loss of generality if we pick any particular Cartan subgroup $H \subset G_2$ to work with.

Denote by $E_{[i,j]} = \frac{E_{ij} - E_{ji}}{2}$, $i, j = 1, 2, \ldots, 7,\ i < j$, the standard basis of the Lie algebra $\mathfrak{so}(7)$ of $SO(7)$. A basis of the Lie algebra $\mathfrak{g}_2$ of the Lie group G_2 can be found in [12]. Its Cartan subalgebra $\mathfrak{h}$ is spanned by

$$P_0 = E_{[3,2]} + E_{[6,7]}, \qquad Q_0 = E_{[4,5]} + E_{[6,7]}.$$

The elements of the corresponding group $H = S^1 \times S^1$ are of the form

$$g_{t,s} = \exp(tP_0 + sQ_0) = (\exp tP_0)(\exp sQ_0), \quad t, s \in \mathbb{R}.$$

It is easy to show that the action of the element $g_{t,s} \in H$ on a point $p = (x_1, x_2, x_3, y_0, y_1, y_2, y_3) \in S^6 \subset \mathbb{R}^7$ is given by

$$\begin{aligned} g_{t,s}p = (&x_1, x_2 \cos t - x_3 \sin t, x_3 \cos t + x_2 \sin t, \\ &y_0 \cos s + y_1 \sin s, y_1 \cos s - y_0 \sin s, \\ &y_2 \cos(s+t) + y_3 \sin(s+t), y_3 \cos(s+t) - y_2 \sin(s+t)). \end{aligned} \tag{7}$$

Note that the action of H preserves the x_1 coordinate, so the orbit $\mathcal{O}_p$ of any point p belongs to the hyperplane $x_1 = const$ and therefore to the sphere $S^5 = S^6 \cap \{x_1 = const\}$.

In the sequel we consider the points of the form $p = (x_1, 0, x_3, 0, y_1, y_2, y_3)$ that represent all the orbits. The tangent space of the orbit $\mathcal{O}_p$ in the point p is spanned by the vectors

$$\begin{aligned} \overline{X} &= \frac{d}{dt}(g_{t,s}p)|_{(t,s)=(0,0)} = (0, -x_3, 0, 0, 0, y_3, -y_2), \\ \overline{Y} &= \frac{d}{ds}(g_{t,s}p)|_{(t,s)=(0,0)} = (0, 0, 0, y_1, 0, y_3, -y_2). \end{aligned} \tag{8}$$

These two vectors are linearly independent, i.e. the orbit $\mathcal{O}_p$ of point $p = (x_1, 0, x_3, 0, y_1, y_2, y_3) \in S^6$ is two-dimensional if the following conditions are satisfied:

$$\begin{aligned} \alpha &= x_3^2 + y_1^2 \neq 0, \\ \beta &= x_3^2 + y_2^2 + y_3^2 \neq 0, \\ \gamma &= y_1^2 + y_2^2 + y_3^2 \neq 0. \end{aligned} \tag{9}$$

An orthonormal basis of tangent space $T_p\mathcal{O}_p$ reads

$$\begin{aligned} X &= \frac{1}{\sqrt{\beta}}(0, -x_3, 0, 0, 0, y_3, -y_2), \\ Y &= \frac{(0, x_3(y_2^2 + y_3^2), 0, \beta y_1, 0, y_3 x_3^2, -y_2 x_3^2)}{\sqrt{\beta}\sqrt{\alpha\beta - x_3^4}}. \end{aligned}$$

Using Lemma 2.2 we can get the slant angle at any point of the orbit $\mathcal{O}_p$, namely

$$\cos\theta_p = |\langle X, pY \rangle| = \frac{|x_3\, y_1\, y_2|}{\sqrt{\alpha\beta - x_3^4}}. \tag{10}$$

Careful analysis of the above expression shows that every slant angle $\theta \in [0, \frac{\pi}{2}]$ is achieved for some point p, so we have the following theorem.

Theorem 4.1. *Let $p = (x_1, 0, x_3, 0, y_1, y_2, y_3) \in S^6$ be a point satisfying relations (9) and $\mathcal{O}_p$ the orbit of p under the action (7) of Cartan subgroup $H \subset G_2$ on the sphere S^6. The orbit is a slant, flat torus with the slant angle given by the formula (10). The slant angles take all values in the interval $[0, \frac{\pi}{2}]$.*

Remark 4.1.1. The orbits $\mathcal{O}_p$ with the slant angle $\theta \equiv 0$, i.e. almost complex curves, are obtained for points $p = (0, 0, \pm\frac{1}{\sqrt{3}}, 0, \pm\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0)$. They are all G_2-equivalent. These orbits are necessarily minimal, as proven by Gray (see [8]), and confirmed in Theorem 4.2. This complex curve is linearly full in the totally geodesic sphere $S^5 \subset S^6$.

Remark 4.1.2. The orbits $\mathcal{O}_p$ with the slant angle $\theta \equiv \frac{\pi}{2}$, i.e. totally real surfaces, are obtained if $x_3 = 0$, $y_1 = 0$ or $y_2 = 0$. They are not necessarily minimal (for example if $x_1 \neq 0$). If $x_3 = 0$ or $y_1 = 0$ they belong to the sphere S^3, while if $y_2 = 0$ they are linearly full in some $S^5 \subset S^6$.

Now, we analyze the geometry of the orbit $\mathcal{O}_p$ as a submanifold of the sphere S^6. Starting from the basis (8) of $T_p\mathcal{O}_p$ one can calculate the induced connection and second fundamental form in the sphere S^6. Then, one can check that the Gauss curvature of the orbit $\mathcal{O}_p$ vanishes. This also follows trivially from the Gauss-Bonnet formula and the fact that the Gauss curvature is constant along the orbit that is topologically a torus.

One can check that the mean curvature vector of the orbit $\mathcal{O}_p$ in the point $p = (x_1, 0, x_3, 0, y_1, y_2, y_3)$ is given by the formula

$$H = (2x_1, 0, \frac{N(x_3, y_1)}{D}, 0, \frac{N(y_1, x_3)}{D}, y_2(2 - \frac{x_3^2 + y_1^2}{D}), y_3(2 - \frac{x_3^2 + y_1^2}{D})),$$

where

$$D = (y_1^2 + y_2^2 + y_3^2)x_3^2 + y_1^2(y_2^2 + y_3^2),$$
$$N(y_1, x_3) = y_1((2y_1^2 + 2y_2^2 + 2y_3^2 - 1)x_3^2 + (2y_1^2 - 1)(y_2^2 + y_3^2)).$$

There are many solutions to the minimality equation $H \equiv 0$, but most of them are found to be G_2-equivalent. From the formula (10) we also get the slant angle for the minimal orbits. The result is given by the following theorem.

Theorem 4.2. *Let $p = (x_1, 0, x_3, 0, y_1, y_2, y_3) \in S^6$ be a point satisfying relations (9), and $\mathcal{O}_p$ its orbit under the action (7). Up to a G_2-equivalence, the minimal orbits $\mathcal{O}_p$ correspond to points*

$$p_\alpha = (0, 0, \frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}, \frac{\cos\alpha}{\sqrt{3}}, \frac{\sin\alpha}{\sqrt{3}}), \quad \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}], \qquad \text{and}$$
$$\bar{p} = (0, 0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0).$$

The orbits of p_α are linearly full in the totally geodesic $S^5 \subset S^6$, have ellipse of curvature a circle, and slant angle $\theta \equiv |\alpha|$.

The orbit of $\bar{p}$ is a totally real Clifford torus in $S^3 \subset S^6$, and its ellipse of curvature is a segment.

Remark 4.2.1. The totally real minimal orbit corresponds to the point $p_{\frac{\pi}{2}}$. It is found in the classification of totally real minimal surfaces of constant curvature (see [3, Theorem 6.5 (c)]). Minimal orbits with arbitrary slant angle are those from the classification of minimal, flat, slant surfaces of S^6 (see [14, Example 3.2]).

Remark 4.2.2. Note that the orbits $\mathcal{O}_{p_\alpha}$ are obtained by certain $SO(7)$ rotation of the complex curve $\mathcal{O}_{p_0}$, as explained in [2].

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