REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 55, No. 1, 2014, Pages 19–24
Published online: April 10, 2014

POSINORMAL FACTORABLE MATRICES WITH A CONSTANT MAIN DIAGONAL

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Abstract. Sufficient conditions are found for a posinormal factorable matrix with a constant main diagonal to be hyponormal. Those conditions are satisfied by some Toeplitz matrices, and a non-Toeplitz example is also presented. Along the way, a more general result is also obtained.

1. Introduction

A lower triangular infinite matrix $M = [m_{ij}]$, acting through multiplication to give a bounded linear operator on ℓ^2, is *factorable* if its entries are

$$m_{ij} = \begin{cases} a_i c_j & \text{if } j \leq i, \\ 0 & \text{if } j > i, \end{cases}$$

where a_i depends only on i, and c_j depends only on j; the matrix M is *terraced* if $c_j = 1$ for all j. Note that $(C, 1)$, the Cesàro matrix of order one, is terraced with $a_i = 1/(i + 1)$ for all i.

The operator M is *posinormal* if $MM^* = M^*PM$ for some positive operator P, called the *interrupter*, and M is *hyponormal* if it satisfies $\langle (M^*M - MM^*)f, f\rangle \geq 0$ for all f in ℓ^2. Posinormal operators were introduced and studied in [13], where it was observed that the set of all posinormal operators on any Hilbert space is an enormous collection that includes every invertible operator and all the hyponormal operators. Some key facts about posinormal operators appear in the following results found in [13, Theorem 2.1 and Corollary 2.3].

Proposition 1.1. *For a bounded linear operator A on a Hilbert space H, the following statements are equivalent:*

1. *A is posinormal;*
2. $\operatorname{Ran} A \subseteq \operatorname{Ran} A^*$*;*
3. *$AA^* \leq \gamma^2 A^*A$ for some $\gamma \geq 0$; and*
4. *There exists a bounded operator T on H such that $A = A^*T$.*

Note that if A is hyponormal, then condition (3) is satisfied with $\gamma = 1$.

2010 *Mathematics Subject Classification.* Primary 47B99.

Key words and phrases. posinormal operator, hyponormal operator, factorable matrix, Toeplitz matrix.

Proposition 1.2. *If A is posinormal, then* $\operatorname{Ker} A \subseteq \operatorname{Ker} A^*$.

Progress in the study of posinormal operators has been surveyed in [8]. Other studies involving these operators can be found in [2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 17, 18, 19].

The fact that hyponormality implies posinormality is central to our work here. In an earlier paper [14], posinormality was used to determine sufficient conditions for a terraced matrix to be hyponormal. That approach was extended to a special subcollection of the factorable matrices in [15], and here we extend that approach to another special subcollection: those posinormal factorable matrices whose main diagonal is constant. Consequently, this study may be considered as a continuation of a larger program on posinormal and hyponormal operators.

2. Main result

We note that up to this point, the only case for which significant progress has been made extending the approach of [14] to factorable matrices $M = M(\{a_i\}, \{c_j\})$ is the case in which the interrupter P is diagonal (see [15]). In the case that we investigate now, P itself may not be diagonal, but the finite sections of $Q :\equiv I - P$ can be reduced to a diagonal matrix.

Assume that $a_i, c_j \neq 0$ for all i, j. To obtain P, we start with the matrix $B = [b_{ij}]$ defined by

$$b_{ij} = \begin{cases} c_i[1/c_j - a_{j+1}/(c_{j+1}a_j)] & \text{if } i \leq j, \\ -a_{j+1}/a_j & \text{if } i = j+1, \\ 0 & \text{if } i > j+1. \end{cases}$$

We know that if B is bounded on ℓ^2 and both $\{a_n\}$ and $\{a_n/c_n\}$ are positive decreasing sequences that converge to 0, then M is posinormal since $M^* = BM$ (see [15]). For M to be hyponormal, it must then be true that for all f in ℓ^2,

$$\langle (M^*M - MM^*)f, f\rangle = \langle (M^*M - (M^*B^*)(BM)f, f\rangle = \langle (I - B^*B)Mf, Mf\rangle \geq 0.$$

Consequently, we conclude that M will be hyponormal when $Q = I - P \geq 0$, where $P = B^*B$; we note that the range of M contains all the e_n's from the standard orthonormal basis for ℓ^2.

Using the entries of P as displayed in [15, Section 3], we find that the matrix $Q = [q_{ij}]$ has entries given by

$$q_{ij} = \begin{cases} \frac{c_i^2 c_{i+1}^2 (a_i^2 - a_{i+1}^2) - (\sum_{k=0}^{i} c_k^2)(c_{i+1}a_i - c_i a_{i+1})^2}{c_i^2 c_{i+1}^2 a_i^2} & \text{if } i = j, \\ \frac{(c_{i+1}a_i - c_i a_{i+1})[c_j(\sum_{k=0}^{j+1} c_k^2)a_{j+1} - c_{j+1}(\sum_{k=0}^{j} c_k^2)a_j]}{c_i c_{i+1} c_j c_{j+1} a_i a_j} & \text{if } i > j, \\ \frac{(c_{j+1}a_j - c_j a_{j+1})[c_i(\sum_{k=0}^{i+1} c_k^2)a_{i+1} - c_{i+1}(\sum_{k=0}^{i} c_k^2)a_i]}{c_i c_{i+1} c_j c_{j+1} a_i a_j} & \text{if } i < j. \end{cases}$$

In order to show that Q is positive, it suffices to show that Q_N, the N^{th} finite section of Q (involving rows $i = 0, 1, 2, \ldots, N$ and columns $j = 0, 1, 2, \ldots, N$), has positive determinant for each positive integer N. We assume that $c_{n+1}a_n \neq c_n a_{n+1}$ for all n, and we proceed in the following way. For $N \geq 1$ and $i = N-1, N-2, \dots, 2, 1, 0$ (in that order), multiply row i by

$$t(i) :\equiv c_i a_i (c_{i+2} a_{i+1} - c_{i+1} a_{i+2}) / [c_{i+2} a_{i+1} (c_{i+1} a_i - c_i a_{i+1})]$$

and subtract from row $i+1$. Call the new matrix Q'_N and reduce it in the following manner. For $j = N-1, N-2, \dots, 2, 1, 0$ (in that order), multiply column j by $t(j)$ and subtract from column $j+1$. The resulting matrix is tridiagonal with the following form:

$$Y_N :\equiv \begin{pmatrix} d_0 & s_0 & 0 & \dots & 0 & 0 \\ s_0 & d_1 & s_1 & \dots & 0 & 0 \\ 0 & s_1 & d_2 & \dots & . & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & . & \dots & d_{N-1} & s_{N-1} \\ 0 & 0 & 0 & \dots & s_{N-1} & d_N \end{pmatrix},$$

where $d_0 = \frac{c_0^2 c_1^2 (a_0^2 - a_1^2) - c_0^2 (c_1 a_0 - c_0 a_1)^2}{c_0^2 c_1^2 a_0^2}$, $d_n = 1 - \frac{a_{n+1}^2}{a_n^2} - \frac{(c_n^2 - c_{n-1}^2) a_{n-1}^2 (c_{n+1} a_n - c_n a_{n+1})^2}{c_{n+1}^2 a_n^2 (c_n a_{n-1} - c_{n-1} a_n)^2}$, and $s_{n-1} = \frac{(c_{n+1} a_n - c_n a_{n+1})(c_n a_n - c_{n-1} a_{n-1})}{c_{n+1} a_n (c_n a_{n-1} - c_{n-1} a_n)}$ for $1 \leq n \leq N$. Note that $\det Y_N = \det Q'_N = \det Q_N$. Our computations have proved the following result.

Theorem 2.1. *Suppose $M = M(\{a_i\}, \{c_j\})$ is a factorable matrix that acts as a bounded operator on ℓ^2 and that the following conditions are satisfied:*

(1) *both $\{a_n\}$ and $\{a_n/c_n\}$ are strictly decreasing sequences that converge to 0;*
(2) *the matrix B defined above is a bounded operator on ℓ^2; and*
(3) $\det Y_N \geq 0$ *for all N.*

Then M is hyponormal.

We observe that with the procedure used above, the entry in the northwest corner of Q_N has remained unchanged, whereas the procedure used in [14] left the entry in the southeast corner unchanged.

Before continuing on, we point out that results such as [1, Proposition 1] are sometimes useful in demonstrating that the tridiagonal matrix Y_N has a positive determinant; that is true of our first example.

Example 2.2. *Let M denote the factorable matrix associated with $a_i = 1/(i+2)^2$ and $c_j = j+1$ for all i, j. Theorem 2.1 can be used to show that M is hyponormal, since it can be verified that $d_n > 0$ for all n and that $d_{n-1} d_n \geq 4 s_{n-1}^2$ for all $n \geq 1$. The details are left to the interested reader.*

We note that the hyponormality of Example 2.2 cannot easily be seen directly from the definition.

Returning to our primary focus, in the case where $\{a_n c_n\}$ is a constant sequence, Y_N becomes a diagonal matrix, and that leads to the following corollary, which will be useful in our next two examples.

Corollary 2.3. *Suppose $M = M(\{a_i\}, \{c_j\})$ is a factorable matrix that acts as a bounded operator on ℓ^2 and that the following conditions are satisfied:*

(1) *both* $\{a_n\}$ *and* $\{a_n/c_n\}$ *are strictly decreasing sequences that converge to 0;*
(2) *the matrix* B *defined above is a bounded operator on* ℓ^2*;*
(3) $\{a_n c_n\}$ *is a constant sequence; and*
(4) $d_0 = [c_0^2 c_1^2(a_0^2 - a_1^2) - c_0^2(c_1 a_0 - c_0 a_1)^2]/[c_0^2 c_1^2 a_0^2] \geq 0$, *and* $d_n = 1 - a_{n+1}^2/a_n^2 - c_n a_{n-1}(c_{n+1}a_n - c_n a_{n+1})^2/[c_{n+1}^2 a_n^2(c_n a_{n-1} - c_{n-1}a_n)] \geq 0$ *for all* $n \geq 1$.

Then M *is hyponormal.*

Example 2.4 (Toeplitz matrix). *Consider the matrix* M *with* $a_i = r^i$ *and* $c_j = r^{-j}$ *for all* i, j *where* $0 < r < 1$. *It can be demonstrated that* M *and* B *are bounded, and it is clear that conditions (1) and (3) of the corollary are satisfied. It is straightforward to verify that* $d_0 = r^2(1 - r^2) > 0$ *and* $d_n = 0$ *for all* $n \geq 1$, *and hence* M *is hyponormal.*

We note that for the matrix M in Example 2.4, the scalar multiple $(1 - r^2)M$ can be shown to satisfy the inequality in [16, Theorem 1] ensuring hyponormality, although M itself does not satisfy that inequality. The use of our Corollary 2.3 has avoided that scaling problem while demonstrating hyponormality for this operator. We point out that the result in [16] was obtained without invoking posinormality.

The following is another example whose hyponormality cannot easily be seen directly from the definition.

Example 2.5. *If* M *is the factorable matrix associated with* $a_i = 1/[\sum_{k=0}^{i} 2^k]$ *and* $c_j = \sum_{k=0}^{j} 2^k$ *for all* i, j, *then* M *and* B *can be shown to be bounded (the details are left to the reader). Clearly* M *satisfies conditions (1) and (3) of Corollary 2.3. Since* $d_0 = 8/81$ *and*

$$d_n = 2^{n+2}[3(2^n) - 1][2^{2n+2} - 9(2^{n-1}) + 1]/\{[3(2^{n-1}) - 1](2^{n+2} - 1)^4\} > 0$$

for all $n \geq 1$, M *is hyponormal.*

In contrast with what was seen for Example 2.4, we observe that when M is the matrix from Example 2.5, there cannot exist a scalar $\alpha > 0$ such that αM satisfies the inequality in [16, Theorem 1]; for that would require $(1 - \alpha)(2^{k+2} - 1)^2 = (2^{k+1} - 1)^2$ for all k. So our work here using posinormality has demonstrated hyponormality for an example that does not satisfy that inequality from [16], even when one resorts to positive scalar multiples.

3. Remarks

Two classes of triangular infinite matrices that appear often in the literature are the Hausdorff matrices and the weighed mean matrices. The only Hausdorff matrices with constant main diagonal are those that are scalar multiples of the identity matrix. A weighted mean matrix is a lower triangular matrix with entries p_j/P_i, where $\{p_j\}$ is a nonnegative sequence with $p_0 > 0$, and $P_i = \sum_{j=0}^{i} p_j$. A weighted mean matrix is factorable, with $a_i = 1/P_i$ and $c_j = p_j$ for all i, j. It is not hard to show that there are no weighted mean matrices with constant main diagonal.

In closing, we note that the attention focused here on the role of the diagonal sequence $\{c_n a_n\}$ has led to an alternative version of [15, Theorem 8].

Theorem 3.1. *Suppose $M = M(\{a_i\}, \{c_j\})$ is a lower triangular factorable matrix that acts as a bounded operator on ℓ^2 and that the following conditions are satisfied:*

(1) *both $\{a_n\}$ and $\{\frac{a_n}{c_n}\}$ are positive decreasing sequences that converge to 0;*
(2) *the matrix $B = [b_{ij}]$ (defined in the previous section) is a bounded operator on ℓ^2;*
(3) *the sequence $\{(\sum_{k=0}^{n} c_k^2)\frac{a_n}{c_n}\}$ is constant; and*
(4) *the sequence $\{c_n a_n\}$ is nonincreasing.*

Then M is posinormal with a diagonal interrupter, and furthermore, M is hyponormal.

The change is in condition (4); using condition (3), it is not hard to show that the new condition (4) is equivalent to the original condition (4) as stated in [15].

Acknowledgment

The authors wish to express their gratitude to the referee, whose comments and suggestions helped improve the original version of this manuscript.

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Received: October 10, 2012
Accepted: September 24, 2013