REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 55, No. 1, 2014, Pages 25–53
Published online: April 25, 2014

VECTOR VALUED T(1) THEOREM AND LITTLEWOOD–PALEY THEORY ON SPACES OF HOMOGENEOUS TYPE

PABLO SEBASTIÁN VIOLA

Abstract. Singular integral operators associated to kernels valued on Hilbert spaces are studied in the setting of spaces of homogeneous type. By following the work of David and Journé (Ann. of Math. (2) **120** (1984), no. 2, 371–397), a $T1$-Theorem is obtained in this context. This result is applied to prove a Littlewood–Paley estimate.

1. Introduction and main result

In this paper, vector-valued singular integral operators T are studied, in the setting of a space of homogeneous type (X, d, μ) of order θ, where the measure μ is non-atomic.

The classical theory of Calderón-Zygmund's operators, contained in the celebrated work [CZ], was generalized in different directions and contexts in the last century. In 1984, David and Journé in [DJ] give necessary and sufficient conditions for the $L^2(\mathbb{R}^n)$ continuity of such operators. The central tool in that work is the Cotlar Lemma (see [C]), that allows them to develop the L^2 theory without using the Fourier transform. This fact later aids to David, Journé and Semmes ([DJS] and Aimar ([A]) to extend these results to the context of spaces of homogeneous type.

On the other hand, the theory of vector valued Calderón-Zygmund operators was started in the work of Benedeck, Calderón and Panzone in [BCP] and developed in [RFRT] and [RFT]. It has been shown that this theory has important applications in the study of the classical analysis and, in particular, in the theory of weights.

However, the generalization of the "$T1$-Theorem" in this context is obtained in 2004 by Figiel in [F] for the case of the Lebesgue spaces $L^p(\mathbb{R}^n, X)$, $1 < p < \infty$, where X is a Banach space satisfying the UMD condition.

Also, Hytönen and Weis (see [HW]) extend the $T1$-Theorem to the case of different Banach spaces, both satisfying the UMD condition. In the two works mentioned above there are no explicit examples of operators such that the vector

2010 *Mathematics Subject Classification.* 42B20, 42B25.

This work was partially supported by grants from Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Universidad Nacional del Litoral (UNL) and Universidad Nacional del Centro (UNICEN).

valued $T1$-Theorem can be applied. Recently in [VV] this approach is used to obtain L^p-boundedness of the oscillation of the Riesz transforms.

One purpose of this work is to analyze the behavior of the quadratic differences of operators T_ϵ, having pointwise $\lim_{\epsilon\to 0} T_\epsilon f(x) = Tf(x)$, in spaces with general measures, by using a vector valued $T1$-Theorem.

More precisely, given a family of operators $T = \{T_\varepsilon\}_{\varepsilon>0}$ such that the existence of the $\lim_{\varepsilon\to 0} T_\varepsilon f(x)$ for almost every x is known, we investigate the speed of convergence of the family $\{T_\varepsilon\}_{\varepsilon>0}$. To this end we consider square functions $\mathcal{G}(Tf)$ given by

$$\mathcal{G}(Tf)(x) = \Big(\sum_{i=1}^{\infty} \big|T_{\varepsilon_i} f(x) - T_{\varepsilon_{i+1}} f(x)\big|^2\Big)^{1/2}, \qquad \text{for } \varepsilon_i \downarrow 0. \tag{1}$$

The consideration of this problem goes back to the 1930's and it was mainly answered by Littlewood and Paley. During the last years, in order to measure the speed of convergence, other expressions such as the ρ-variation and the oscillation operators have been considered as well, see [JKRW, CJRW1, CJRW2, VV] and the references therein.

In the rest of the paper, $\mathbb{H}$ will denote a Hilbert space. When omitted, it can be assumed that $\mathbb{H}$ is some scalar, real or complex, field. The inner product in $\mathbb{H}$ will be denoted by $(\cdot,\cdot)_{\mathbb{H}}$.

In what follows we provide the definitions that we need to properly state our first Theorem.

Definition 1. *A function $K : X \times X \setminus \Delta \to \mathbb{H}$ is said a* **standard kernel** *if there is a number $0 < \delta \le \theta$ such that*

$$|K(x,y)|_{\mathbb{H}} \le C d(x,y)^{-1}, \tag{2}$$

for all $x, y \in X, x \neq y$, and

$$|K(x,y) - K(x',y)|_{\mathbb{H}} + |K(y,x) - K(y,x')|_{\mathbb{H}} \le C \frac{d(x,x')^\delta}{d(x,y)^{1+\delta}}, \tag{3}$$

for all $x, x', y \in X$, with $d(x,x') \le d(x,y)/2A$.

Here C is a positive constant that does not depend on x, x', y. We also say that K has **smoothness exponent** *δ.*

Definition 2 (Function spaces). *We say that a function $f : X \to \mathbb{H}$ belongs to $L^p_\mu(X,\mathbb{H})$, $1 \le p \le \infty$ if*

$$\|f\|_p := \Big(\int |f|^p_{\mathbb{H}}\, d\mu\Big)^{1/p} < \infty.$$

In a similar way the spaces L^∞ are defined.

Given a function $g : X \to \mathbb{H}$ and a number $0 < \beta \le \theta$, we define

$$|g|_{\beta,\mathbb{H}} := \sup_{x,y\in X, x\neq y} \frac{|g(x) - g(y)|_{\mathbb{H}}}{d(x,y)^\beta}.$$

We say that $g \in \Lambda^\beta(X,\mathbb{H})$ if the seminorm $|g|_{\beta,\mathbb{H}}$ is finite. In addition, we write $g \in \Lambda^\beta_0(X,\mathbb{H})$ if g is supported in a ball. In addition, we define the norm

$$\|g\|_{\beta,\mathbb{H}} := \|g\|_{L^\infty_\mu(X,\mathbb{H})} + |g|_{\beta,\mathbb{H}}.$$

The space of continuous linear functions on $\Lambda^\beta_0(X,\mathbb{H})$ is denoted by $(\Lambda^\beta_0(X,\mathbb{H}))'$.

Definition 3. *Let us consider a standard kernel K with smoothness exponent δ. We say that $T : \Lambda^\beta_0(X) \to (\Lambda^\beta_0(X,\mathbb{H}))'$ is a* **singular integral operator associated to the kernel** *K if T is linear continuous and for any $f \in \Lambda^\beta_0(X)$ and $g \in \Lambda^\beta_0(X,\mathbb{H})$ with disjoint supports, the following equality holds:*

$$\langle Tf, g\rangle = \iint (K(x,y), g(x))_{\mathbb{H}}\, f(y)\, d\mu(x) d\mu(y). \tag{4}$$

If T can be extended to a bounded operator $T : L^2_\mu(X) \to L^2_\mu(X,\mathbb{H})$, we say that T is a **Calderón-Zygmund operator***.*

Definition 4. *Given a singular integral operator T, its* **adjoint operator** *T^* is defined by*

$$\langle T^* g, f\rangle = \langle Tf, g\rangle ,$$

for all functions $f \in \Lambda^\beta_0(X)$ and $g \in \Lambda^\beta_0(X,\mathbb{H})$.

Definition 5. *We say that a singular integral operator T satisfies the* **Weak Boundedness Property** *with exponent β if, given a ball B, the following inequality holds*

$$|\langle Tf, g\rangle| \le C\, \mu(B)^{1+2\beta} \|f\|_{\Lambda^\beta(X)} \|g\|_{\Lambda^\beta(X,\mathbb{H})}, \tag{5}$$

for any $f \in \Lambda^\beta_0(X)$ and $g \in \Lambda^\beta_0(X,\mathbb{H})$ both supported in B.

We investigate conditions on an operator $T : \Lambda^\beta_0(X) \to \Lambda^\beta_0(X,\mathbb{H})'$ such that it can be extended to a bounded operator $T : L^2_\mu(X) \to L^2_\mu(X,\mathbb{H})$.

The main results of the paper are the following.

Theorem 6. *If T is a continuous linear operator, in the sense of Definition 3, associated to an $\mathbb{H}$-valued standard kernel K, satisfying a Weak Boundedness Property and such that*

$$T1 = 0_{\mathbb{H}}$$

and

$$T^*(1h) = 0,$$

for any vector $h \in \mathbb{H}$, then T is a Calderón-Zygmund operator.

Remark. The explicit expressions of $T1$ and $T^*(1h)$ are contained in Section 2, Proposition 8 below.

We will provide the proof of the main Theorem in Section 2. There, a key claim is used, whose proof is given along Section 3. Finally, in Section 4 we apply this result to prove a Littlewood–Paley estimate.

2. Proof of the main Theorem

Our assumptions on X, d, μ in Section 1 will be now precisely stated. First, X is a nonempty set provided with a quasi-distance d. There is a constant $A \geq 1$ such that

$$d(x,y) \leq A(d(x,z) + d(z,y)), \tag{6}$$

for any $x, y, z \in X$. The balls in X are denoted by $B_r(x)$.

The measure μ satisfies a doubling condition, that is, there is a positive constant C such that

$$\mu(B_{2r}(x)) \leq C\mu(B_r(x)). \tag{7}$$

We say that the measure μ is **non-atomic** if for every single point x, $\mu(\{x\}) = 0$. We also assume that $\mu(X) = \infty$. This is equivalent to the fact that X is non-bounded with respect to d.

The quasi-distance d has the **θ-regularity property** for some $0 < \theta \leq 1$. That is, there is a constant $C' > 0$ such that

$$|d(x,y) - d(x',y)| \leq C' r^{1-\theta} d(x,x')^{\theta}, \tag{8}$$

for all $x, x', y \in X$, $r > 0$, whenever $d(x,y) < r$, $d(x',y) < r$.

In addition, we will suppose that μ satisfies the following condition with respect to every d-ball $B_r(x)$: there is a constant $c > 0$ such that

$$\frac{1}{c} r \leq \mu(B_r(x)) \leq cr. \tag{9}$$

This condition, together with the regularity, can be assumed without loss of generality, since, as shown in [MS], an equivalent distance $\tilde{d}$ can be found such that the required properties are fulfilled.

Along the paper we often use, without mentioning it, the following result of linearity of integrals for vector-valued functions.

Theorem 7. (Hille) *Given a μ-integrable function ϕ and a continuous linear functional ℓ on $\mathbb{H}$, we have*

$$\ell\left(\int \phi(x)\, d\mu(x)\right) = \int \ell(\phi(x))\, d\mu(x). \tag{10}$$

See, for example, [DU].

As is usual in the theory of singular integrals, we define the action of T on the constant function 1. In fact, we will do that for any bounded Lipschitz function.

Proposition 8. *If $T : \Lambda_0^\beta(X) \to (\Lambda_0^\beta(X,\mathbb{H}))'$ is a singular integral operator associated to a standard kernel K, then T can be extended to a linear operator $T : \Lambda_b^\beta(X) \to (\Lambda_{00}^\beta(X,\mathbb{H}))'$. In addition, the adjoint operator T^* can be extended to a linear operator $T^* : \Lambda_b^\beta(X,\mathbb{H}) \to (\Lambda_{00}^\beta(X))'$.*

Here, $\Lambda_b^\beta(X,\mathbb{H})$ is the set of bounded functions $f \in \Lambda^\beta(X,\mathbb{H})$ and $\Lambda_{00}^\beta(X,\mathbb{H})$ consists of the functions $f \in \Lambda_0^\beta(X,\mathbb{H})$ with null μ-integral.

Proof. Let us consider a function $f \in \Lambda_b^\beta(X)$ and a function $g \in \Lambda_{00}^\beta(X, \mathbb{H})$ supported in a ball $B(x_0, R)$. Let us take $\xi \in \Lambda_0^\beta(X)$ such that $\xi \equiv 1$ on $B(x_0, 2AR)$, $\xi \equiv 0$ on $X \setminus B(x_0, 4AR^2)$, and $0 \leq \xi(y) \leq 1$ else. We say that ξ is a **cut function over the ball** $B(x_0, R)$. We define

$$\langle Tf, g\rangle = \langle T(f\xi), g\rangle + \langle T(f(1-\xi)), g\rangle\,, \tag{11}$$

where the second term means

$$\langle T(f(1-\xi)), g\rangle = \iint \langle (K(x,y) - K(x_0,y))(1-\xi(y))f(y), g(x)\rangle_{\mathbb{H}}\, d\mu(y)\, d\mu(x).$$

This term is finite, which can be seen after applying (3) and the regularity of μ to obtain

$$\begin{aligned}
&|\langle T(f(1-\xi)), g\rangle| \\
&\quad \leq C\|f\|_{L^\infty_\mu(X)}\, \|g\|_{L^\infty_\mu(X,\mathbb{H})} \int\limits_{X\setminus B(x_0,2AR)} d(x_0,y)^{-1-\delta}\, d\mu(y) \int\limits_{B(x_0,R)} d(x,x_0)^{\delta}\, d\mu(x) \\
&\quad \leq C\|f\|_{L^\infty_\mu(X)}\, \|g\|_{L^\infty_\mu(X,\mathbb{H})} \mu(B(x_0,R)).
\end{aligned}$$

Therefore, $\langle T(f(1-\xi)), g\rangle$ is well defined.

Finally, it is necessary to check that $\langle Tf, g\rangle$ does not depend on ξ. Indeed, for two cut functions ξ_1, ξ_2 over the balls $B_{R_1}(x_1)$ and $B_{R_2}(x_2)$, respectively, we have

$$\begin{aligned}
&(\langle T(f\xi_1), g\rangle + \langle T(f(1-\xi_1)), g\rangle) - (\langle T(f\xi_2), g\rangle + \langle T(f(1-\xi_2)), g\rangle) \\
&\quad = \int \Big(-\left[K(x_1,y)(1-\xi_1(y)) - K(x_2,y)(1-\xi_2(y))\right] f(y), \int g(x) d\mu(x)\Big)_{\mathbb{H}}\, d\mu(y) \\
&\quad = 0,
\end{aligned}$$

where we have used Theorem 7 and the fact that $\int g\, d\mu = 0_{\mathbb{H}}$.

For the adjoint operator T^* the proof is similar by taking again scalar valued cut functions. ☐

In what follows we use the approximation of the identity operators $S_j, \tilde{S}_j$, whose definition is given in (22), and the difference operators $\Delta_j = S_{j+1} - S_j, \tilde{\Delta}_j = \tilde{S}_{j+1} - \tilde{S}_j$. Now we prove the main Theorem. To this end, we assert the following claim, whose nontrivial proof will be accomplished along Section 3.

Claim 9. *For each j, let E_j be the operator defined by*

$$E_j = \tilde{S}_j T \Delta_j + \tilde{\Delta}_j T S_j - \tilde{\Delta}_j T \Delta_j. \tag{12}$$

Then, for every pair of integers j, k, the composite operators $E_j^* E_k$, $E_j E_k^*$ have operator norms

$$\|E_j^* E_k\|_{\mathcal{L}(L^2_\mu(X), L^2_\mu(X))} \leq C 2^{-|j-k|\delta},$$
$$\|E_j E_k^*\|_{\mathcal{L}(L^2_\mu(X,\mathbb{H}), L^2_\mu(X,\mathbb{H}))} \leq C 2^{-|j-k|\delta},$$

where C is a positive constant not depending on j, k.

Proof of Theorem 6. For each positive integer N we denote

$$R_N = \sum_{j=-N}^{N} (\tilde{S}_j T\Delta_j + \tilde{\Delta}_j T S_j - \tilde{\Delta}_j T\Delta_j).$$

After writing $R_N = \tilde{S}_N T S_N - \tilde{S}_{-(N+1)} T S_{-(N+1)}$ we obtain that

$$R_N f \to Tf, \quad \text{weakly, as } N \to \infty. \tag{13}$$

We will prove this shortly. Given $f \in \Lambda_0^\beta(X)$ and $g \in \Lambda_0^\beta(X, \mathbb{H})$, by Corollary 12 (Section 3) and the continuity of T we have that $\left\langle \tilde{S}_{-N} T S_{-N} f, g \right\rangle \to \langle Tf, g\rangle$, as $N \to \infty$. Also, by using (15) and (24) we get that $S_N f \to 0$ as $N \to \infty$ in the topology of $\Lambda_0^\beta(X)$. We have an analogous statement for $\tilde{S}_N g$, too. Then, the continuity of T implies $\left\langle \tilde{S}_N T S_N f, g \right\rangle \to 0$ as $N \to \infty$.

On the other hand, we can apply Claim 9 and Lemma 14 to conclude that there exists an operator $T^0 : L^2_\mu(X) \to L^2_\mu(X, \mathbb{H})$ such that $R_N f \to T^0 f$, strongly, as N goes to infinity. Therefore, from (13) we clearly have that $T = T^0$, and the Theorem is proved. □

3. Proof of Claim 9

3.1. **Preliminary definitions and results.** First, we give a list of preliminary results.

Theorem 10. *There is a function $\rho : X \times X \times (0, \infty) \to [0, \infty)$ and a positive constant c such that, for some $0 < \beta \le \theta$ and for any $t > 0$ and $x, y, x', y' \in X$, the following properties hold:*

$$\operatorname{supp} \rho(\cdot, \cdot, t) \subset \{(x, y) \in X \times X : d(x, y) < ct\}, \tag{14}$$

$$\sup\{\rho(x, y, t) : x, y \in X\} \le ct^{-1}, \tag{15}$$

$$\rho(x, y, t) = \rho(y, x, t) \tag{16}$$

$$|\rho(x, y, t) - \rho(x', y, t)| \le ct^{-\beta-1} d(x, x')^\beta, \tag{17}$$

$$|\rho(x, y, t) - \rho(x, y', t)| \le ct^{-\beta-1} d(y, y')^\beta, \tag{18}$$

$$\int \rho(x, y, t) d\mu(y) = 1, \tag{19}$$

$$\int \rho(x, y, t) d\mu(x) = 1, \tag{20}$$

$$|(\rho(x, y, t) - \rho(x', y, t)) - (\rho(x, y', t) - \rho(x', y', t))| \le ct^{-2\beta-1} d(x, x')^\beta d(y, y')^\beta. \tag{21}$$

The existence of a function ρ satisfying (14)-(19) is proved in [MST]. The approximation of the identity defined in [DJS] provides a function ρ satisfying all the properties (14)-(21). However the proof of (21) appears in [HS].

Given a Lipschitz function g, of any order, for each $t > 0$ we define

$$g_t(x) = \int \rho(x, y, t) g(y) d\mu(y).$$

Now we introduce the family of **approximation of the identity operators**. Given $f \in \Lambda_0^\beta(X)$ and $g \in \Lambda_0^\beta(X,\mathbb{H})$ we define

$$S_j f(x) = f_{2^j}(x), \qquad \tilde{S}_j g(x) = g_{2^j}(x). \tag{22}$$

Since these two operators are selfadjoint, their action can be extended to distributions. In particular, for $\phi_1, \phi_2 \in \Lambda_0^\beta(X)$ and $\psi_1, \psi_2 \in \Lambda_0^\beta(X,\mathbb{H})$, we have

$$\langle S_j T\phi_1, \phi_2\rangle = \langle T\phi_1, S_j\phi_2\rangle, \qquad \langle \tilde{S}_j T^*\psi_1, \psi_2\rangle = \langle T^*\psi_1, \tilde{S}_j\psi_2\rangle.$$

Also, we define the difference operators $\Delta_j = S_j - S_{j-1}$ and $\tilde{\Delta}_j = \tilde{S}_j - \tilde{S}_{j-1}$.

We have the following

Theorem 11. *If $g \in \Lambda_0^\beta(X,\mathbb{H})$, $0 < \beta \le \theta$, supported in a ball $B_r(x_0)$, then for any integer j the following properties hold.*

$$\operatorname{supp}(\tilde{S}_j g) \subset B_{\tilde{r}(j)}(x_0), \tag{23}$$

whenever $2^j < r$, where $\tilde{r}(j) = r + C'' r^{1-\theta} 2^{j\theta}$;

$$|\tilde{S}_j g(x) - \tilde{S}_j g(x')|_{\mathbb{H}} \le C'' 2^{-j(1+\theta)} \mu(B_r(x_0))^{1+\beta} d(x,x')^\theta \tag{24}$$

and

$$|(\tilde{S}_j g(x) - g(x)) - (\tilde{S}_j g(x') - g(x'))|_{\mathbb{H}} \le C(j) d(x,x')^\beta, \tag{25}$$

where $\lim_{j\to-\infty} C(j) = 0$.

Proof. The Theorem can be obtained by a straight generalization of Lemma 1.20 in [MST]. $\square$

Corollary 12. *Under the hypothesis of Theorem 11,*

$$\lim_{j\to-\infty} \|\tilde{S}_j g - g\|_{\beta,\mathbb{H}} = 0.$$

It is immediate that Theorem 11 and Corollary 12 hold for the operators S_j, since it is enough to consider $\mathbb{H}$ as a scalar field.

We need some results related to several L^2_μ spaces. Since they are Hilbert spaces, we formulate the Lemma below in an abstract form. If $\mathcal{G}$ and $\mathcal{H}$ are two Hilbert spaces, we denote by $\mathcal{L}(\mathcal{G},\mathcal{H})$ the set $\{T : \mathcal{G} \to \mathcal{H} \mid T \text{ linear and bounded}\}$ which has the operator norm:

$$\|T\|_{\mathcal{L}(\mathcal{G},\mathcal{H})} = \sup_{\|x\|_{\mathcal{G}}=1} \|Tx\|_{\mathcal{H}}.$$

Lemma 13. *Let $\mathcal{G},\mathcal{H}$ be Hilbert spaces. If $S : \mathcal{G} \to \mathcal{H}$ is a bounded linear operator then*

$$\|S\|^2_{\mathcal{L}(\mathcal{G},\mathcal{H})} \le \|(S^*S)\|_{\mathcal{L}(\mathcal{H})}, \qquad \|S^*\|^2_{\mathcal{L}(\mathcal{H},\mathcal{G})} \le \|(SS^*)\|_{\mathcal{L}(\mathcal{G})}. \tag{26}$$

The operators $SS^* : \mathcal{H} \to \mathcal{H}$ and $S^*S : \mathcal{G} \to \mathcal{G}$ are selfadoint. (27)

$$\|S\|_{\mathcal{L}(\mathcal{G},\mathcal{H})} \le \|(SS^*)^k\|^{1/2k}_{\mathcal{L}(\mathcal{H})}, \qquad \|S\|_{\mathcal{L}(\mathcal{G},\mathcal{H})} \le \|(S^*S)^k\|^{1/2k}_{\mathcal{L}(\mathcal{G})}. \tag{28}$$

Proof. To prove (26), we apply the Cauchy-Schwarz inequality to obtain

$$\|S\|^2_{\mathcal{L}(\mathcal{G},\mathcal{H})} = \|S^*\|^2_{\mathcal{L}(\mathcal{H},\mathcal{G})} = \sup_{\|x\|_{\mathcal{H}}=1} \left|(SS^*x, x)_{\mathcal{G}}\right| \leq \|SS^*\|_{\mathcal{L}(\mathcal{H})}.$$

To prove (28), we first observe that $\|(SS^*)^k\|_{\mathcal{L}(\mathcal{H})} = \|SS^*\|^k_{\mathcal{L}(\mathcal{H})}$ because SS^* is a self adjoint operator. Therefore, from (26) we have that

$$\|S\|_{\mathcal{L}(\mathcal{G},\mathcal{H})} \leq \|SS^*\|^{1/2}_{\mathcal{L}(\mathcal{H})} \leq \|(SS^*)^k\|^{1/2k}_{\mathcal{L}(\mathcal{H})}, \qquad \text{for all } k \geq 1. \qquad \square$$

Now we can prove a modified version of the Cotlar Lemma for two Hilbert spaces $\mathcal{G}, \mathcal{H}$.

Lemma 14. *Consider two Hilbert spaces $\mathcal{G}, \mathcal{H}$. If $\{T_j\}_{j\in Z}$ is a family of bounded linear operators $T_j : \mathcal{G} \to \mathcal{H}$ and if $\{a(j)\}_{j\in\mathbb{Z}}$ is a list of nonnegative numbers such that*

$$\|T_iT_j^*\|_{\mathcal{L}(\mathcal{H})} + \|T_i^*T_j\|_{\mathcal{L}(\mathcal{G})} \leq a(i-j),$$

then, for all pairs of integers m, n with $n \leq m$, we have

$$\|S_{n,m}\| := \Big\| \sum_{j=n}^{m} T_j \Big\|_{\mathcal{L}(\mathcal{G},\mathcal{H})} \leq \sum_{i=-\infty}^{\infty} a(i)^{1/2} =: \Gamma.$$

Moreover, the sum $S_{n,m}$ converges strongly to a bounded operator T^0 satisfying $\|T^0\|_{\mathcal{G},\mathcal{H}} \leq \Gamma$.

Proof. First, if we define the operator $S = \sum_{j=n}^m T_j$ then, given an integer $k > 0$, we can write

$$(SS^*)^k = \sum_{j_1,\cdots,j_{2k}=n}^{m} T_{j_1}T^*_{j_2}\cdots T_{j_{2k-1}}T^*_{j_{2k}}.$$

We now estimate

$$\begin{aligned}
&\|T_{j_1}T^*_{j_2}\cdots T_{j_{2k-1}}T^*_{j_{2k}}\|_{\mathcal{L}(\mathcal{H})}\\
&\qquad \leq \|T_{j_1}T^*_{j_2}\|_{\mathcal{L}(\mathcal{H})}\cdots \|T_{j_{2k-1}}T^*_{j_{2k}}\|_{\mathcal{L}(\mathcal{H})}\\
&\qquad \leq a(j_1-j_2)\cdots a(j_{2k-1}-j_{2k}).\\
&\|T_{j_1}T^*_{j_2}\cdots T_{j_{2k-1}}T^*_{j_{2k}}\|_{\mathcal{L}(\mathcal{H})}\\
&\qquad \leq \|T_{j_1}\|_{\mathcal{L}(\mathcal{G},\mathcal{H})}\|T^*_{j_2}T_{j_3}\|_{(\mathcal{G})}\cdots \|T^*_{j_{2k-2}}T^*_{j_{2k-1}}\|_{\mathcal{L}(\mathcal{G})}\|T^*_{j_{2k}}\|_{\mathcal{L}(\mathcal{H},\mathcal{G})}\\
&\qquad \leq a(0)^{1/2}a(j_2-j_3)\cdots a(j_{2k-2}-j_{2k-1})a(0)^{1/2}.
\end{aligned}$$

After that and by taking in account the inequality (28), the calculations can proceed in a standard way. $\square$

3.2. **The action of E_j over $\Lambda_0^\beta(X)\times\Lambda_0^\beta(X,\mathbb{H})$.** In what follows we left fixed an integer j and consider the operator

$$\mathcal{T}_j = \tilde{S}_jT\Delta_j = \tilde{S}_jTS_j - \tilde{S}_jTS_{j-1}.$$

Observe that $\mathcal{T}_j$ is just one of the terms of E_j. We show the proofs only for $\mathcal{T}_j$, since the other terms in E_j have similar behavior.

Lemma 15. *Given* $f \in \Lambda_0^\beta(X)$, $g \in \Lambda_0^\beta(X, \mathbb{H})$, *we have*

$$\left\langle \tilde{S}_j T S_j f, g \right\rangle = \iint \left\langle T\rho(\cdot, y, 2^j) f(y), \rho(\cdot, x, 2^j) g(x) \right\rangle \, d\mu(y) d\mu(x). \tag{29}$$

In addition, the integral is absolutely summable. Also,

$$\left\langle \tilde{S}_j T S_{j-1} f, g \right\rangle = \iint \left\langle T\rho(\cdot, y, 2^{j-1}) f(y), \rho(\cdot, x, 2^j) g(x) \right\rangle \, d\mu(y) d\mu(x).$$

Proof. Let us fix a point $x_0 \in X$. Suppose that $F \in \Lambda_0^\gamma(X)$, $G \in \Lambda_0^\gamma(X, \mathbb{H})$ and both functions are supported in a given ball $B_{R_1}(x_0)$.

Let us denote $\Phi(x, y) = \rho(x, y, 2^{-j})$. By (14), $\operatorname{supp}(\Phi) \subset \{(x, y) : d(x, y) < 2^{j+1}\}$. Let us denote $R = A(R_1 + c2^{j+1})$, $B = \overline{B_R(x_0)}$, then for each $y \in X$, $\Phi(x, y)F(y) = 0$ if $d(x, x_0) \geq R$.

The set $\mathcal{C}[B]$ of continuous functions supported in B is a Banach space with the supremum norm, which will be denoted by $\|\cdot\|_\infty$.

We consider the function $\mathcal{H} : X \to \mathcal{C}[B]$ given by

$$\mathcal{H}[y](\cdot) = \Phi(\cdot, y)F(y).$$

For each $y \in X$, we have $\mathcal{H}[y] \in \mathcal{C}[B]$. Also, if $y_k \to y$ as $k \to \infty$, we obtain that $\|\mathcal{H}[y_k] - \mathcal{H}[y]\|_\infty \to 0$. Hence, $\mathcal{H}[y]$ is a continuous function on the variable y.

Since $\mathcal{H}$ is continuous, it is Bochner μ-measurable. Now we write

$$\begin{aligned}
\int \|\mathcal{H}[y]\|_\infty \, d\mu(y) &= \int \|\mathcal{H}[y](\cdot)\|_{L^\infty_\mu(X)} \, d\mu(y) \\
&= \int \|\Phi(\cdot, y)\|_{L^\infty_\mu(X)} |F(y)| \, d\mu(y) \\
&\leq \|\Phi(\cdot, \cdot)\|_{L^\infty_\mu(X \times X, \mathbb{R})} \|F\|_{L^1_\mu(X)} < \infty.
\end{aligned}$$

Therefore $\mathcal{H}$ is $\mathcal{C}[B]$-summable in the sense of Bochner.

Then, given a linear functional $\ell : \mathcal{C}[B] \to \mathbb{C}$, Hille's Theorem implies that

$$\int \ell(\mathcal{H}[y]) \, d\mu(y) = \ell\left(\int \mathcal{H}[y] \, d\mu(y) \right).$$

Since $\Lambda_0^\beta(B) \subset \mathcal{C}[B]$, the functional $\ell(h) = (Th, G)$ is linear continuous for $h \in \Lambda_0^\beta(B, \mathbb{C})$. Since $\Lambda_0^\beta(B)$ is a dense subset of $\mathcal{C}[B]$, the functional ℓ can be continuously extended to the whole space $\mathcal{C}[B]$. Now, we can write

$$\begin{aligned}
\left(T\left(\int \Phi(\cdot, y)F(y) d\mu(y) \right), G \right) &= \left(T \int \mathcal{H}[y] d\mu(y), G \right) \\
&= \int \left(T\Phi(\cdot, y), G \right) \, F(y) \, d\mu(y).
\end{aligned}$$

By replacing Φ by its definition, the last equality says that

$$\begin{aligned}
(T(S_j F), G) &= \left(T\left(\int \rho(\cdot, y, 2^j) F(y) d\mu(y) \right), G \right) \\
&= \int \left(T\rho(\cdot, y, 2^j), G \right) \, F(y) \, d\mu(y).
\end{aligned} \tag{30}$$

Now, we work with vector-valued functions. With similar arguments as before, we can prove that

$$\left(T\Phi(\cdot,y), \int \Phi(\cdot,x)g(x)\,d\mu(x)\right) = \int \left(T\Phi(\cdot,y), \Phi(\cdot,x)g(x)\right)\,d\mu(x).$$

In order to attain this equality, we take the Banach space $\mathcal{C}[B,\mathbb{H}]$ consisting of continuous $\mathbb{H}$-valued functions supported in B with norm $\|g\|_{\infty,\mathbb{H}} = \sup\{|g(x)|_{\mathbb{H}} : x \in B\}$, and then by defining

$$\tilde{\mathcal{H}}[x] := \Phi(\cdot,x)g(x), \qquad \tilde{\ell}(h) := \left(T\Phi(\cdot,y), hg\right),$$

and taking $G = \tilde{S}_j g$ in (30), we obtain

$$\left(\tilde{S}_j T S_j f, g\right) = \left(T S_j f, \tilde{S}_j g\right) = \iint \left(T\rho(\cdot,y,2^j)f(y), \rho(\cdot,x,2^j)g(x)\right)\,d\mu(x)d\mu(y), \tag{31}$$

as desired. Now, the function

$$(x,y) \to \left(T\rho(\cdot,y,2^j)f(y), \rho(\cdot,x,2^j)g(x)\right)$$

has compact support in $X \times X$. Furthermore, for a given j the family of functions

$$\{\rho(\cdot,y,2^j)f(y)\}_{y\in X} \times \{\rho(\cdot,x,2^j)g(y)\}_{x\in X}$$

is entirely contained in a bounded subset of $\Lambda_0^\beta(X,\mathbb{C}) \times \Lambda_0^\beta(X,\mathbb{H})$. By applying the Weak Boundedness Property, the double integral in (29) becomes absolutely summable. □

This Lemma suggests the following definition.

Definition of kernels κ_j associated to the operators $\mathcal{T}_j$. Let x, y be points in X. We define a vector-valued kernel $\kappa_j(x,y)$ in the following way:

$$(\kappa_j(x,y), v)_{\mathbb{H}} = \left\langle T(\rho(\cdot,y,2^j) - \rho(\cdot,y,2^{j-1})), \rho(\cdot,x,2^j)v\right\rangle.$$

To verify that this is a good definition, observe that the right-hand side is a linear function $\ell(v)$ on $\mathbb{H}$. Also, the Weak Boundedness Property implies that $\ell(v)$ is bounded, since

$$|\ell(v)| \le |\left\langle T\rho(\cdot,y,2^j), \rho(\cdot,x,2^j)v\right\rangle| + |\left\langle T\rho(\cdot,y,2^{j-1}), \rho(\cdot,x,2^j)v\right\rangle| \le C_{x,y,j}|v|_{\mathbb{H}}.$$

By the Riesz Representation Theorem, there is an element $\xi \in \mathbb{H}$ such that $\ell(v) = (\xi, v)_{\mathbb{H}}$, for all $v \in \mathbb{H}$, and we write $\kappa_j(x,y) = \xi$.

Now, for functions $f \in \Lambda_0^\gamma(X)$, $g \in \Lambda_0^\gamma(X,H)$, Lemma 15 enables us to write

$$\langle \mathcal{T}_j f, g\rangle = \iint \left\langle T\psi(\cdot,y,2^j)\,f(y), \rho(\cdot,x,2^j)g(x)\right\rangle\,d\mu(y)d\mu(x), \tag{32}$$

where $\psi(x,y,2^j) = \rho(x,y,2^j) - \rho(x,y,2^{j-1})$.

We will say that $\kappa_j(x,y)$ is the **kernel** of the operator $\mathcal{T}_j$.

3.3. **Properties of the kernels κ_j.** From now on, we use the notation

$$p(r) = (1+r)^{-(1+\delta)}, \qquad p_j(r) = 2^{-j}p(2^{-j}r).$$

Lemma 16. *For each integer j, the function $p_j(x,y)$ is μ-integrable for each variable x, y.*

Proof. We consider integration on the variable y, for example.

$$\begin{aligned}
2^j \int p_j(d(x,y))d\mu(y) &= \int \frac{d\mu(y)}{\big(1+2^{-j}d(x,y)\big)^{1+\delta}} \\
&\leq \int_{d(x,y)<2^j} d\mu(y) + \sum_{n=0}^{\infty} \int_{2^{j+n}\leq d(x,y)<2^{j+n+1}} 2^{-(1+\delta)n} \\
&\leq \mu(B_{2^j(x)}) + \sum_{n=0}^{\infty} \mu(B_{2^{j+n+1}})2^{-(1+\delta)n} \\
&\leq C2^j.
\end{aligned}$$

In the last inequality we have used (9) □

An important consequence of Lemma 16 is that $\kappa_j(x,y)$ is integrable for each variable x, y, as we now show.

Proposition 17. *Given an integer j, if $x, y \in X$ then*

$$|\kappa_j(x,y)|_{\mathbb{H}} \leq Cp_j(d(x,y)).$$

Proof. First, we suppose $d(x,y) \leq 10cA2^j$, where c is the constant appearing in Theorem 10. Now, we apply the Weak Boundedness Property, with $\beta' = \delta$, (17) and (18) to estimate

$$\begin{aligned}
\big|(\kappa_j(x,y),v)_{\mathbb{H}}\big| &= \big|\big\langle T\psi(\cdot,y,2^j),\rho(x,\cdot,2^j)v\big\rangle\big| \\
&\leq 2^{j(1+2\delta)}c^2 2^{j(-1-\delta)}|v|_{\mathbb{H}} \\
&\leq Cp_j(d(x,y))|v|_{\mathbb{H}}.
\end{aligned}$$

The last inequality can be reached with the aid of the following estimation:

$$2^{-j} \leq (1+10cA)^{1+\delta}p_j(d(x,y)).$$

Now, we suppose that $d(x,y) > 10cA\,2^j$. In this case $\rho(x,\cdot,2^j)$ and $\psi(\cdot,y,2^j)$ have disjoint supports, as is easy to see. For any $v \in \mathbb{H}$ we can write

$$\begin{aligned}
(\kappa_j(x,y),v)_{\mathbb{H}} &= \big\langle T\psi(\cdot,y,2^j),\rho(x,\cdot,2^j)v\big\rangle \\
&= \iint \big(K(\xi,\zeta)\psi(\zeta,y,2^j),\rho(x,\xi,2^j)\,v\big)_{\mathbb{H}}\,d\mu(\zeta)d\mu(\xi) \\
&\quad - \iint \big(K(\xi,y),\rho(x,\xi,2^j)\,v\big)_{\mathbb{H}}\,\psi(\zeta,y,2^j)\,d\mu(\zeta)d\mu(\xi).
\end{aligned}$$

The last term is equal to 0 because (20) implies that $\int \psi(x,y,2^j)\,d\mu(x) = 0$.

In view of the support of $\rho(x,\cdot,2^j)$ we have $d(x,\xi) \leq c2^j < d(x,y)/10A$. Besides, $d(\xi,y) \leq A(d(x,\xi)+d(x,y)) \leq 3cA\,d(x,y)$.

Also, $d(\xi,y) > 7cd(\xi,x)$, since otherwise we would have $d(x,y) \leq A(d(x,\xi) + d(\xi,y)) \leq 10cAd(\xi,x)$, against our hypothesis.

Then we have $d(x,y) \leq A(d(x,\xi)+d(\xi,y)) \leq A(1+1/7c)d(\xi,y)$. Thus, we get $d(x,y) \approx d(\xi,y)$. This fact, together with (3), (15) and (9), implies

$$\begin{aligned}
&\left|(\kappa_j(x,y),v)_{\mathbb{H}}\right| \\
&\qquad \leq C \iint |K(\xi,\zeta)-K(\xi,y)|_{\mathbb{H}}\, \rho(x,\xi,2^j)\, |v|_{\mathbb{H}} |\psi(\zeta,y,2^j)|\, d\mu(\xi)d\mu(\zeta) \\
&\qquad \leq C \iint_{B(x,c2^j)\times B(y,c2^j)} \frac{d(\zeta,y)^{\delta}}{d(\xi,y)^{1+\delta}} 2^{-j}(2^{-j}+2^{-j+1})\, |v|_{\mathbb{H}}\, d\mu(\xi)d\mu(\zeta) \\
&\qquad \leq C2^{-2j} \iint_{B(x,c2^j)\times B(y,c2^j)} \frac{2^{j\delta}}{A(1/7c+1)d(x,y)^{1+\delta}}\, |v|_{\mathbb{H}}\, d\mu(\xi)d\mu(\zeta) \\
&\qquad \leq C\frac{2^{j\delta}}{d(x,y)^{1+\delta}}\, |v|_{\mathbb{H}} \leq Cp_j(d(x,y)).
\end{aligned}$$

□

Corollary 18. *For any integer j, the kernel $\kappa_j(x,y)$ is integrable for each variable x,y.*

Now we study the smoothness properties of the kernels $\kappa_j(x,y)$.

Proposition 19. *Let j be a fixed integer and x,w,y be points in X. Then*

$$|\kappa_j(x,y)-\kappa_j(w,y)|_{\mathbb{H}} \leq C\min(1,2^{-j\delta}d(x,w)^{\delta})(p_j(d(x,y))+p_j(d(w,y))).$$

Proof. If $d(w,x) \geq c2^j$, we can apply Proposition 17.

Thus, we can suppose that $d(w,x) \leq c2^j$.

For convenience, we define $\Delta_{j,x,w}(\xi) = \rho(x,\xi,2^j) - \rho(w,\xi,2^j)$.

We split the analysis in two cases. First, we consider $d(w,y) \leq A^2c2^{j+2}$. By applying the Weak Boundedness Property and the properties (17) and (21), we have for an arbitrary vector $v \in \mathbb{H}$,

$$\begin{aligned}
\left|\left\langle T\psi(\cdot,y,2^j), \Delta_{j,x,w}\, v\right\rangle\right| &\leq C(2^j)^{1+2\delta-2-3\delta} d(x,w)^{\delta}|v|_{\mathbb{H}} \\
&\leq Cp_j(d(w,y))2^{-j\delta}d(x,w)^{\delta}|v|_{\mathbb{H}}.
\end{aligned}$$

We have used that $2^{-j} \leq Cp_j(d(w,y))$.

Now, let us suppose $d(w,y) \geq A^2c2^{j+2}$. In this case, $\psi(\cdot,y,2^j)$ and $\Delta_{j,x,w}(\cdot)$ have disjoint supports. Indeed, if z were a point in the support of both functions, we would have $z \in (B_{c2^j}(x) \cup B_{c2^j}(w)) \cap B_{c2^j}(y)$. But, if $d(z,w) < c2^j$, $d(w,y) \leq A(d(w,z)+d(z,y)) \leq Ac2^{j+1}$; and if $d(z,x) < c2^j$ then

$$d(w,y) \leq A(A(d(z,x)+d(x,w))+d(z,y)) \leq A^2c2^{j+2},$$

in contradiction with the hypothesis.

Let us observe more carefully the earlier supports. By taking $\xi \in \operatorname{supp}(\Delta_{j,x,w})$ and $\zeta \in \operatorname{supp}(\psi(\cdot,y,2^j))$, we have $\xi \in B_{c2^j}(x) \cup B_{c2^j}(w)$ and then $d(w,\xi) \leq$

$\max\{A(d(w,x)+d(x,\xi)), c2^j\} \leq Ac2^{j+1}$. Also, $d(z,y) \leq c2^j$. Besides, both quantities $d(w,\xi)$ and $d(\zeta,y)$ are less than $d(\xi,y)/2A$, hence $d(\xi,y) \leq A(d(w,\xi) + d(w,y)) < (1/2+A)d(w,y)$.

Moreover, we have $d(\xi,y) \leq d(w,\xi)$, since otherwise $d(w,y) \leq A(d(\xi,w) + d(\xi,y)) < A^2c2^{j+2}$, against our hypothesis. So, we can write $d(w,y) \leq 2Ad(\xi,y)$. We conclude that

$$\frac{1}{\frac{1}{2}+A} d(\xi,y) \leq d(w,y) \leq 2Ad(\xi,y). \tag{33}$$

After this geometrical analysis, we continue with the main lines of the proof.

Since the term $(K(\xi,y), \Delta_{j,x,w}(\xi)\, v)_{\mathbb{H}}$ is constant with respect to the variable ζ, the following double integral is null:

$$\iint (K(\xi,y), \Delta_{j,x,w}(\xi)\, v)_{\mathbb{H}}\, \psi(\zeta,y,2^j)\, d\mu(\zeta) d\mu(\xi).$$

Thus, we can add this term and use (3), (15), (17) and (9) to obtain

$$\begin{aligned}
&\left|\left\langle T\psi(\cdot,y,2^j), \Delta_{j,x,w}\, v\right\rangle\right| \\
&\quad= \left|\iint \left((K(\xi,\zeta)-K(\xi,y))\, \psi(\zeta,y,2^j), \Delta_{j,x,y}(\xi)\, v\right)_{\mathbb{H}}\, d\mu(\zeta)d\mu(\xi)\right| \\
&\quad\leq C|v|_{\mathbb{H}} \iint_{\operatorname{supp}(\Delta_{j,x,w}\, \psi(\cdot,y,2^j))} \frac{d(\zeta,y)^\delta}{d(\xi,y)^{1+\delta}} c2^{-j(1+\delta)} d(x,w)^\delta 2c2^{-j+1}\, d\mu(\xi)d\mu(\zeta) \\
&\quad\leq C2^{-2j} d(x,w)^\delta\, 2^{-j\delta} \mu(\operatorname{supp}(\Delta_{j,x,w}))\mu(\operatorname{supp}(\psi(\cdot,y,2^j))) \frac{2^{j\delta}(\frac{1}{2}+A)}{d(w,y)^{1+\delta}}\, |v|_{\mathbb{H}} \\
&\quad\leq C2^{-j\delta} p_j(d(w,y)) d(x,w)^\delta.
\end{aligned}$$

By applying this inequality, we finally obtain

$$\begin{aligned}
\left|(\kappa_j(x,y)-\kappa(w,y),v)_{\mathbb{H}}\right| &= \left|\left\langle T\psi(\cdot,y,2^j), \Delta_{j,x,w}\, v\right\rangle\right| \\
&\leq C2^{-j\delta} p_j(d(w,y))\, d(x,w)^\delta\, |v|_{\mathbb{H}}.
\end{aligned}$$

□

In the following lemmas we give two estimates related to the functions p_j.

Lemma 20. *For every integer j and every point $y \in X$,*

$$\int_X p_j(d(z,y))\, d\mu(z) \leq C \min\{1, 2^{j\delta}\} \leq C < \infty,$$

where C is a constant not depending on j and y.

Proof. Given $y \in X$, for every integer N we denote $E_N(y) = B_{2^{N+1}}(y) \setminus B_{2^N}(y)$.

$$\begin{aligned}
\int p_j(d(z,y))\, d\mu(z) &= \int_X \frac{2^{-j}}{\left(1 + d(z,y)/2^j\right)^{1+\delta}}\, d\mu(z) \\
&= 2^{j\delta} \sum_{N\in\mathbb{Z}} \int_{E_N(y)} \left(2^j + d(z,y)\right)^{-1-\delta} d\mu(z) \\
&\leq 2^{j\delta} \sum_{N\in\mathbb{Z}} \frac{\mu(E_N(y))}{(2^j + 2^N)^{-1-\delta}} \\
&\leq C\, 2^{j\delta} \sum_{N\in\mathbb{Z}} 2^N (2^j + 2^N)^{-1-\delta}.
\end{aligned}$$

We can easily bound this summation by $C2^{-j\delta}$ if we split it as $\sum_{N<j} + \sum_{N\geq j}$. This gives the desired result. □

Lemma 21. *For all integers j,k, with $j \leq k$, and all $x \in X$, the following inequality holds:*

$$\int_X p_j(d(x,y)) \min(1, 2^{-k\delta} d(x,y)^\delta)\, d\mu(y) \leq C2^{-|j-k|\delta},$$

where C is a constant not depending on j,k or x.

Proof. For any $R > 0$, let us denote $A_R(x) = B_{2R}(x) \setminus B_R(x)$. We estimate

$$\begin{aligned}
&\int p_j(d(x,y)) \min(1, 2^{-k\delta} d(x,y)^\delta) d\mu(y) \\
&\qquad \leq \sum_{N\in\mathbb{Z}} \int_{A_{2^N}(x)} \frac{2^{-j}}{(1 + d(x,y)/2^j)^{1+\delta}} \min(1, 2^{-k\delta} d(x,y)^\delta) d\mu(y) \\
&\qquad \leq \sum_{N\in\mathbb{Z}} \mu(A_{2^N}(x)) \frac{2^{j\delta}}{(2^j + 2^N)^{1+\delta}} \min(1, 2^{(N+1-k)\delta}) \\
&\qquad \leq C \sum_{N\in\mathbb{Z}} 2^N \frac{2^{j\delta}}{(2^j + 2^N)^{1+\delta}} \min(1, 2^{(N+1-k)\delta}).
\end{aligned}$$

If $N \geq k$, $\min(1, 2^{(N+1-k)\delta}) = 1$. Besides,

$$\sum_{N\geq k} \frac{2^N 2^{j\delta}}{(2^j + 2^N)^{1+\delta}} \leq C2^{-|j-k|\delta}.$$

On the contrary, if $N < k$, then $\min(1, 2^{(N+1-k)\delta}) = 2^{(N+1-k)\delta}$, and then

$$\sum_{N\geq k} \frac{2^N 2^{j\delta}}{(2^j + 2^N)^{1+\delta}} \frac{2^{N\delta}}{2^{k\delta}} \leq C2^{-|j-k|(2\delta+1)} \leq C2^{-|j-k|\delta}.$$ □

In the following Proposition we show that the kernels κ_j have null integral.

Proposition 22. *For every integer j and for each $x \in X$*

$$\int \kappa_j(x,y)d\mu(y) = 0_{\mathbb{H}}, \tag{34}$$

and for each $y \in X$

$$\int \kappa_j(x,y)d\mu(x) = 0_{\mathbb{H}}. \tag{35}$$

Proof. Consider a vector $v \in \mathbb{H}$ with $|v|_{\mathbb{H}} = 1$. Let x be a given point in X. Since $\kappa_j(x,\cdot)$ is an integrable function, we can write

$$\begin{aligned}\left(\int \kappa_j(x,y)\,d\mu(y), v\right)_{\mathbb{H}} &= \int \langle \kappa_j(x,y), v\rangle_{\mathbb{H}}\,d\mu(y)\\ &= \int \left\langle T\psi(\cdot,y,2^j), \rho(x,\cdot,2^j)\,v\right\rangle\,d\mu(y)\\ &= \left\langle T\left(\int \psi(\cdot,y,2^j)\,d\mu(y)\right), \rho(x,\cdot,2^j)\,v\right\rangle\\ &= \left\langle T0, \rho(x,\cdot,2^j)\,v\right\rangle = 0.\end{aligned}$$

In the last equality we used that the linear functional $\ell(f) = \left\langle Tf, \rho(x,\cdot,2^j)\,v\right\rangle$ is continuous and then we can apply Hille's Theorem. Also, it is easy to check that $u \to \int \psi(u,y,2^j)\,d\mu(y)$ belongs to $\Lambda_0^{\beta}(X,\mathbb{R})$.

Since v is arbitrary, by the Riesz Representation Theorem we have (34).

Now we proceed with the second part of the Theorem. Let us take a fixed point $o \in X$ and a vector v with $|v|_{\mathbb{H}} = 1$.

Given $y \in X$ and $R > 0$ big enough, we can write

$$\begin{aligned}\left(\int_{B_R(o)} \kappa_j(x,y)\,d\mu(x), v\right)_{\mathbb{H}} &= \int_{B_R(o)} (\kappa_j(x,y), v)_{\mathbb{H}}\,d\mu(x)\\ &= \int_{B_R(o)} \left\langle T\psi(\cdot,y,2^j), \rho(x,\cdot,2^j)\,v\right\rangle\,d\mu(x)\\ &= \left\langle T\psi(\cdot,y,2^j), \int_{B_R(o)} \rho(x,\cdot,2^j)\,v\,d\mu(x)\right\rangle\\ &=: \left\langle T\psi(\cdot,y,2^j), h_R\,v\right\rangle,\end{aligned}$$

where we have defined

$$h_R(z) = \int_{B_R(o)} \rho(x,z,2^j)\,d\mu(z).$$

Since $\kappa_j(\cdot,y)$ is an integrable function, given $\epsilon > 0$ there is a radius $R_0 = R_0(\epsilon) >\ 0$ such that

$$\Big|\int_{d(o,x)>R} \kappa_j(x,y)\,d\mu(x)\Big|_{\mathbb{H}} \leq \int_{d(o,x)>R} |\kappa_j(x,y)|_{\mathbb{H}}\,d\mu(x) < \epsilon, \tag{36}$$

for all $R \geq R_0$.

Let us suppose that $d(o,z) < (R - cA2^j)/A$. If $x \in \operatorname{supp}(\rho(\cdot, z, 2^j))$ then $d(z,x) < c2^j$. Therefore

$$d(o,x) \leq A(d(o,z) + d(z,x)) < R.$$

Thus, $\operatorname{supp}(\rho(\cdot, z, 2^j)) \subset B_R(o)$, and then, by using (19), we have

$$h_R(z) - 1 = \int \rho(x, z, 2^j)\, d\mu(x) - 1 = 0.$$

In this way, the function $h_R - 1$ is supported in $X \setminus B(o, A(R - cA2^j))$.

Also, $\langle \psi(\cdot, y, 2^j), T^*(1v)\rangle = 0$ because we have supposed that $T^*(1v) = 0$ and $\int \psi(\cdot, y, 2^j) d\mu = 0$. This implies

$$\begin{aligned} \langle T\psi(\cdot, y, 2^j), h_R\, v\rangle &= \langle T\psi(\cdot, y, 2^j), h_R\, v\rangle - \langle \psi(\cdot, y, 2^j), T^*(1\, v)\rangle \\ &= \langle T\psi(\cdot, y, 2^j), (h_R - 1)\, v\rangle . \end{aligned}$$

If R is big enough, $h_R - 1$ and $\psi(\cdot, y, 2^j)$ have disjoint supports, thus the equalities (11) and (4) can be applied to obtain

$$\begin{aligned} &\left|\langle T\psi(\cdot, y, 2^j), (h_R - 1)\, v\rangle\right| \\ &= \left| \iint \big((K(u,z) - K(u,y))\psi(z, y, 2^j), (h_R(u) - 1)\, v\big)_{\mathbb{H}}\, d\mu(u) d\mu(z) \right| \\ &\leq C2^{-j} \int_{d(z,y) \leq c2^j} \int_{X \setminus B(o, (R - cA2^j)/A)} \frac{d(z,y)^\delta}{d(u,y)^{1+\delta}} d\mu(z) d\mu(u) \\ &\leq C2^{-j} 2^{j\delta} \mu(B_{c2^j}(y)) \int_{X \setminus B(o, R/2A)} d(u,y)^{-1-\delta}\, d\mu(u). \end{aligned}$$

We analyse the domain of the last integral. First, if $d(u,y) \geq d(o,u)$, obviously we have $d(o,u)^{-1} \geq d(u,y)^{-1}$.

Secondly, let us suppose $d(u,y) < d(o,u)$ and denote $\alpha_y = d(o,y)$. In this case we necessarily get $d(u,y) \geq R/2A^2 - \alpha_y$ since, otherwise, we would have

$$d(o,u) \leq A(d(o,y) + d(y,u)) < A(\alpha_y + R/2A^2 - \alpha_y) = R/2A,$$

against the condition $d(o,u) \geq R/2A$ on the domain of integration.

Then $u \in X \setminus B(y, R/2A^2 - \alpha_y) \subset X \setminus B_{CR}(y)$ for some constant C, while $R > 4A^2 \alpha_y$. Therefore we can estimate

$$\begin{aligned} &\int_{B \setminus B(o, R/2A)} d(u,y)^{-1-\delta}\, d\mu(u) \\ &\leq \int_{X \setminus B(o, R/2A)} d(o,u)^{-1-\delta}\, d\mu(u) + \int_{X \setminus B(y, CR)} d(u,y)^{-1-\delta}\, d\mu(u) \\ &\leq CR^{-\delta}. \end{aligned}$$

Finally,

$$\left|\langle T\psi(\cdot, y, 2^j), (h_R - 1)\, v\rangle\right| \leq C2^{j\delta} R^{-\delta}.$$

This expression tends to 0 as R goes to $+\infty$, then, given $\epsilon > 0$, there is $R_1 = R_1(\epsilon)$ such that

$$\left| \int_{d(x,o)<R} (\kappa_j(x,y), v)_{\mathbb{H}} \, d\mu(x) \right| < \epsilon, \tag{37}$$

for all $R \geq R_1$.

In consequence, by (36) and (37), for any $v \in \mathbb{H}$ we now have

$$\int (\kappa_j(x,y), v)_{\mathbb{H}} \, d\mu(x) = 0.$$

Since $\kappa_j(\cdot, y)$ is integrable, by Hille's Theorem we get (35). □

3.4. **Proof of Claim 9.** In the following Proposition we prove that $\mathcal{T}_j$ can be extended to a bounded operator in L^2 sense.

Proposition 23. *Given an integer j, the operator $\mathcal{T}_j = \tilde{S}_j T \Delta_j$ can be extended to a bounded operator from $L^2_\mu(X)$ into $L^2_\mu(X, \mathbb{H})$. In addition, $\mathcal{T}_j$ has the following integral formula:*

$$\mathcal{T}_j f = \int \kappa_j(\cdot, y) f(y) \, d\mu(y), \qquad f \in \Lambda^\beta_0(X). \tag{38}$$

Proof. Let f be a $\Lambda^\beta_0(X)$ function and g be a $\Lambda^\beta_0(X, \mathbb{H})$ function. We apply the Cauchy-Schwarz inequality, Proposition 17 and Lemma 20 to obtain

$$\begin{aligned}
|\langle \mathcal{T}_j f, g \rangle| &= \Big| \iint (\kappa_j(x,y) f(y), g(x))_{\mathbb{H}} \, d\mu(y) d\mu(x) \Big| \\
&\leq \int \Big(|f(y)|^2 d\mu(y) \Big)^{1/2} \Big(\int \Big[\int |(\kappa_j(x,y), g(x))_{\mathbb{H}}| \, d\mu(x) \Big]^2 d\mu(y) \Big)^{1/2} \\
&\leq C \|f\|_{L^2(X)} \Big(\int \Big[\int p_j(x,y) \|g(x)\|_{\mathbb{H}} \, d\mu(x) \Big]^2 d\mu(y) \Big)^{1/2} \\
&\leq C \|f\|_{L^2(X)} \Big(\int \Big[\int p_j(d(x,y)) \|g(x)\|^2_{\mathbb{H}} \, d\mu(x) \Big] \times \\
&\qquad \times \Big[\int p_j(d(x,y)) \, d\mu(x) \Big] \, d\mu(y) \Big)^{1/2} \\
&\leq C \|f\|_{L^2(X)} \Big(\int \|g(x)\|^2_{\mathbb{H}} \big[\int p_j(d(x,y)) \, d\mu(y) \big] \, d\mu(x) \Big)^{1/2} \\
&\leq C \|f\|_{L^2(X)} \|g\|_{L^2(X,\mathbb{H})}.
\end{aligned}$$

Since $\Lambda^\beta_0(X, \mathbb{H})$ is a dense subspace of $L^2_\mu(X, \mathbb{H})$, we can extend the bounded linear functional $\mathcal{T}_j f$ to any function $g \in L^2_\mu(X, \mathbb{H})$. Now, by the Riesz Representation Theorem, $\mathcal{T}_j f$ is identified with an $L^2_\mu(X, \mathbb{H})$ function and then $\mathcal{T}_j$ can be extended to a bounded operator $\mathcal{T}_j : L^2_\mu(X) \to L^2_\mu(X, \mathbb{H})$.

To verify (38), we take $f \in \Lambda_0^\beta(X)$, $g \in L^2_\mu(X,\mathbb{H})$, and we use that $\kappa_j(x,y)$ is integrable, for example in the variable y, and apply equality (32) to obtain

$$\begin{aligned}(\mathcal{T}_j f, g)_{L^2_\mu(X,\mathbb{H})} = \langle \mathcal{T}_j f, g\rangle &= \iint (\kappa_j(x,y)f(y), g(x))_{\mathbb{H}}\, d\mu(y)d\mu(x) \\ &= \int \left(\int \kappa_j(x,y)f(y)\,d\mu(y), g(x)\right)_{\mathbb{H}} d\mu(x) \\ &= \left(\int \kappa_j(\cdot,y)f(y)\,d\mu(y), g\right)_{L^2_\mu(X,\mathbb{H})}.\end{aligned}$$

This implies (38). □

We write an analogous result for the adjoint operators T_k^*.

Proposition 24. *Given an integer j, the adjoint operator $\mathcal{T}_j^*$ can be extended to an $(L^2_\mu(X,\mathbb{H}), L^2_\mu(X))$-operator. In addition, $\mathcal{T}_j^*$ has the following integral formula:*

$$\mathcal{T}_j^* g = \int (\kappa_j(x,\cdot), g(x))\; d\mu(x), \qquad g \in \Lambda_0^\beta(X,\mathbb{H}). \tag{39}$$

We omit the proof which is similar to the $\mathcal{T}_j$ case.

All the results already obtained are concerned with the family of operators $\mathcal{T}_j$. An analogous task can be repeated for the families $\{\tilde{\Delta}_j T S_j\}_{j\in\mathbb{Z}}$ and $\{\tilde{S}_{j-1} T \Delta_j\}_{j\in\mathbb{Z}}$. Besides, we can write each operator $\tilde{\Delta}_j T \Delta_j$ as $\tilde{S}_j T\Delta_j - \tilde{S}_{j-1}T\Delta_j$, which leads us to extend our earlier results to these operators.

In this way, it can be seen that each operator $E_j = \tilde{S}_j T \Delta_j + \Delta_j T S_j - \tilde{\Delta}_j T \Delta_j$ is bounded from $L^2_\mu(X)$ into $L^2_\mu(X,\mathbb{H})$, with integral representations

$$\begin{aligned} E_j f &= \int K_j(\cdot,y) f(y)\,d\mu(y), \qquad f \in \Lambda_0^\beta(X), \\ E_j g &= \int (K_j(x,\cdot), g(x))\; d\mu(x), \qquad g \in \Lambda_0^\beta(X,\mathbb{H}),\end{aligned}$$

where $K_j(x,y)$ are $\mathbb{H}$-valued kernels satisfying analogous properties as $\kappa_j(x,y)$ does, like the Propositions 17, 19 and 22.

As an easy consequence of these facts, we can write the following

Proposition 25. *Given two integers j and k, the composite operators $E_jE_k^*$, $E_k^*E_j$ are bounded on $L^2_\mu(X,\mathbb{H})$ and $L^2_\mu(X)$, respectively, and they can be written as*

$$(E_jE_k^* g)(x) = \iint K_j(x,y)\,(K_k(u,y), g(u))_{\mathbb{H}}\, d\mu(u)d\mu(y), \quad g \in \Lambda_0^\beta(X,\mathbb{H}), \tag{40}$$

$$(E_k^*E_j f)(y) = \iint (K_j(x,y), K_k(u,y)\, f(u))_{\mathbb{H}}\, d\mu(u)d\mu(x), \quad f \in \Lambda_0^\beta(X). \tag{41}$$

Now, we are in position to prove Claim 9.

Proof of Claim 9. First, we prove that $E_jE_k^*$ is bounded on $L^1_\mu(X,\mathbb{H})$. For $g \in L^1_\mu(X,\mathbb{H})$ we have

$$\begin{aligned}\int |E_jE_k^*g(x)|_{\mathbb{H}}\, d\mu(x) &\leq \iiint p_j(d(x,y))|g(u)|_{\mathbb{H}} \min\{1, 2^{-k\delta}d(x,y)^\delta\}\times \\ &\qquad\times (p_k(d(x,u)) + p_k(d(y,u)))\, d\mu(u)d\mu(y)d\mu(x) \\ &\leq C2^{-|j-k|\delta}\|g\|_{L^2_\mu(X,\mathbb{H})}.\end{aligned}$$

Now, let $g \in L^\infty_\mu(X,\mathbb{H})$. We call $\mathcal{E}_{k,j}g(x)$ to the right-hand side of (40). By applying Proposition 22 and Lemmas 20 and 21, it seems that $\mathcal{E}_{j,k}g$ is well defined and we can obtain the following estimations:

$$\begin{aligned}|\mathcal{E}_{j,k}g(x)|_{\mathbb{H}} &= \left|\iint K_j(x,y)\,(K_k(u,y), g(u))_{\mathbb{H}}\, d\mu(u)d\mu(y)\right| \\ &= \left|\iint K_j(x,y)\,(K_k(u,y) - K_k(u,x), g(u))_{\mathbb{H}}\, d\mu(u)d\mu(y)\right| \\ &\leq \iint |K_j(x,y)|_{\mathbb{H}}|K_k(u,y) - K_k(u,x)|_{\mathbb{H}}|g(u)|_{\mathbb{H}}\, d\mu(u)d\mu(y) \\ &\leq C\iint p_j(d(x,y)) \min\{1, d(x,y)^\delta 2^{-k\delta}\}\times \\ &\qquad\times (p_k(d(x,u)) + p_k(d(y,u)))\,|g(u)|_{\mathbb{H}}\, d\mu(u)d\mu(y) \\ &\leq C2^{-|j-k|\delta}\|g\|_{L^\infty_\mu(X,\mathbb{H})},\end{aligned}$$

where C is a constant not depending on j nor on k.

Since $E_jE_k^*g = \mathcal{E}_{j,k}g$ for $g \in \Lambda_0^\beta(X,\mathbb{H})$, we can invoke the interpolation theorem of Marcinkiewicz to obtain (12). □

4. Application to the Littlewood–Paley theory

In this Section we will apply Theorem 6 to obtain L^2-boundedness of the square function given by

$$\mathcal{S}f(x) = \Big(\sum_{j=-\infty}^{\infty} |\Delta_j f(x)|^2\Big)^{1/2}.$$

In order to do this, we consider the family $\{\rho(x,y,2^\ell)\}_{\ell\in\mathbb{Z}}$ given in Theorem 10 and denote $K = \{K_\ell\}_{\ell\in\mathbb{Z}}$, where

$$K_\ell(x,y) = \rho(x,y,2^\ell) - \rho(x,y,2^{\ell-1}).$$

We will prove that K is a standard kernel that takes values in the Hilbert space $\mathbb{H} = \ell^2(\mathbb{R})$.

Thus, given $0 < \beta \leq \theta$, $f \in \Lambda_0^\beta(X)$ and $g = \{g_\ell\}_{\ell\in\mathbb{Z}} \in \Lambda_0^\beta(X,\mathbb{H})$, we can define an operator T by means of

$$\langle Tf, g\rangle = \lim_{L\to\infty} \sum_{\ell=-L}^{L} \int \Big(\int K_\ell(x,y), f(y)\, d\mu(y)\Big)\, g_\ell(x)\, d\mu(x). \tag{42}$$

We will prove that T is a singular integral operator associated to the kernel K satisfying the conditions of Theorem 6, hence T will become a Calderón-Zygmund operator.

Proposition 26. *The function K is a standard kernel.*

Proof. First we check that K fulfills condition (2). Let $x, y \in X$ such that $x \neq y$. We have:

$$\begin{aligned}
|K(x,y)|_{\mathbb{H}} &= \Big(\sum_{\ell\in\mathbb{Z}} |\rho(x,y,2^{\ell}) - \rho(x,y,2^{\ell-1})|^2\Big)^{1/2} \\
&\leq C\Big(\sum_{\ell\in\mathbb{Z}} |\rho(x,y,2^{\ell})|^2\Big)^{1/2} \leq C\Big(\sum_{\ell\in\mathbb{Z}} \big(c2^{-\ell}\chi_{[0,c2^{\ell}]}d(x,y)\big)^2\Big)^{1/2} \\
&\leq C\Big(\sum_{\ell\in\mathbb{Z}, d(x,y)\leq c2^{\ell}} 2^{-2\ell}\Big)^{1/2} \leq Cd(x,y)^{-1}.
\end{aligned}$$

Second, we check condition (3). Let $x, x', y \in X$ such that $d(x,x') < d(x,y)/2A$. It is not hard to see that $d(x',y) \geq d(x,y)/4A$. Therefore,

$$\begin{aligned}
|K(x,y) - K(x',y)|_{\mathbb{H}} &\leq C\Big(\sum_{\ell\in\mathbb{Z}} |\rho(x,y,2^{\ell}) - \rho(x',y,2^{\ell})|^2\Big)^{1/2} \\
&\leq Cd(x,x')^{\beta}\Big(2^{-2(\beta+1)}\big(\chi_{[0,c2^{\ell}]}(d(x,y)) + \chi_{[0,c2^{\ell}]}(d(x',y))\big)\Big)^{1/2} \\
&\leq Cd(x,x')^{\beta}\big(d(x,y)^{-2\ell(1+\beta)} + d(x',y)^{-2\ell(1+\beta)}\big)^{1/2} \\
&\leq Cd(x,x')^{\beta}d(x,y)^{-1-\beta}.
\end{aligned}$$

Thus, K is a standard kernel with smoothness exponent β. □

Proposition 27. *The operator T given in* (42) *is well defined and is a singular integral operator associated to the standard kernel $K(x,y)$.*

Proof. Let $f \in \Lambda_0^{\beta}(X)$ and $g \in \Lambda_0^{\beta}(X,\mathbb{H})$ be Lipschitz functions supported in $B = B_R(x_0)$.

We split the series in (42) in two parts,

$$\Sigma_1 + \Sigma_2 = \sum_{-L\leq\ell<\log_2 R} + \sum_{\log_2 R<\ell\leq L}.$$

First, we estimate Σ_1. Since $\int K_\ell(x,y)d\mu(x) = 0$, by invoking (9), we get

$$
\begin{aligned}
\tilde{\Sigma}_1 &:= \int_B \sum_{\ell<\log_2 R} \Big| \int K_\ell(x,y)(f(y)-f(x))\,d\mu(y) \Big| |g_\ell(x)|\,d\mu(x) \\
&\leq C\|f\|_\beta \|g\|_{\infty,\mathbb{H}} \int_B \sum_{\ell<\log_2 R} \int 2^{-\ell}\chi_{[0,c2^\ell]}(d(x,y))d(x,y)^\beta\,d\mu(y)\,d\mu(x) \\
&\leq C\|f\|_\beta \|g\|_{\infty,\mathbb{H}} \sum_{\ell<\log_2 R} \int_B \int 2^{-\ell}\chi_{[0,c2^\ell]}(d(x,y))2^{\ell\beta}\,d\mu(y)d\mu(x) \\
&\leq C\|f\|_\beta \|g\|_{\infty,\mathbb{H}} \sum_{\ell<\log_2 R} \int_B 2^{-\ell}\mu(B_{c2^\ell}(x))2^{\ell\beta}\,d\mu(x) \\
&\leq C\|f\|_\beta \|g\|_{\infty,\mathbb{H}} R^\beta \mu(B) \\
&\leq C\|f\|_\beta \|g\|_{\beta,\mathbb{H}} \mu(B)^{1+2\beta}.
\end{aligned}
$$

Now we estimate Σ_2.

$$
\begin{aligned}
\tilde{\Sigma}_2 &:= \int \sum_{\ell\geq\log_2 R} \Big| \int K_\ell(x,y)f(y)\,d\mu(y) \Big| |g_\ell(x)|\,d\mu(x) \\
&\leq C\|f\|_\infty \|g\|_{\infty,\mathbb{H}} \int_B \sum_{\ell\geq\log_2 R} 2^{-\ell} \int_B d\mu(y)\,d\mu(x) \\
&\leq C\|f\|_\infty \|g\|_{\infty,\mathbb{H}} R^{-1} \int_B R\,d\mu(x) \\
&\leq C\|f\|_\beta \|g\|_{\beta,\mathbb{H}} \mu(B)^{1+2\beta}.
\end{aligned}
$$

Hence, by the theorem of the dominated convergence, the limit in (42) exists. Thus the operator T is well defined. Moreover,

$$
|\langle Tf, g\rangle| \leq C\|f\|_\beta \|g\|_{\beta,\mathbb{H}} \mu(B)^{1+2\beta}, \tag{43}
$$

where the constant C is independent of f, g.

The equality (43) clearly implies that T is linear continuous in the sense of Definition 3.

Finally, we will check the property (4). Consider functions $f \in \Lambda_0^\beta(X)$ and $g \in \Lambda_0^\beta(X,\mathbb{H})$ with disjoint supports.

We denote $F = \operatorname{supp}(f)$ and F_η will be the η-neighborhood of F. Observe that for each $\ell \in \mathbb{Z}$, $\operatorname{supp}(K_\ell(x,\cdot)) \subset B_{c2^{\ell+1}}(x)$. Then

$$
\operatorname{supp}\Big(\int |K_\ell(x,\cdot)|\,|f|\,d\mu \Big) \subset F_{c2^{\ell+1}}.
$$

There exists $\ell_0 \in \mathbb{Z}$ such that for any $\ell \leq \ell_0$, $F_{c2^{\ell+1}} \cap \operatorname{supp}(g) = \emptyset$. Thus, we have

$$\begin{aligned}
\lim_{L\to\infty} \int \sum_{-L}^{L} \Big(\int |K_\ell(x,\cdot)|\, |f|\, d\mu \Big) \, |g_\ell(x)| d\mu(x) & \\
= \int \sum_{\ell=\ell_0}^{\infty} \int |K_\ell(x,\cdot)||f| d\mu \, |g_\ell(x)| \, d\mu(x) & \qquad (44)\\
\leq \int \Big(\sum_{\ell_0}^{\infty} \Big(\int |K_\ell(x,\cdot)|\, |f|\, d\mu \Big)^2 \Big)^{1/2} |g(x)|_{\mathbb{H}} d\mu(x) & \\
\leq C \int \Big(\sum_{\ell_0}^{\infty} 2^{-2\ell} \|f\|_{L^1(X)} \Big)^{1/2} |g(x)|_{\mathbb{H}} d\mu(x) & \\
\leq C \|f\|_{L^1_\mu(X)} \|g\|_{L^1_\mu(X,\mathbb{H})}. & \qquad (45)
\end{aligned}$$

This shows that $\int \sum |K_\ell(x,\cdot)||f| d\mu \, g_\ell(x)$ is μ-integrable, which implies we can write the equality (4) for T. ☐

Proposition 28. *The operator T satisfies the conditions $T1 = 0_{\mathbb{H}}$ and $T^*(1h) = 0$, for any $h \in \mathbb{H}$.*

Proof. Consider a function $g \in \Lambda^\beta_{00}(X, \mathbb{H})$ and numbers $\eta > 0$, $L \in \mathbb{Z}$. Let h_η be some real valued Lipschitz function such that $0 \leq h_\eta \leq 1$, $h_\eta \equiv 1$ in the η-neighborhood of $\operatorname{supp}(g)$ and $h_\eta \equiv 0$ outside the 2η-neighborhood of $\operatorname{supp}(g)$. Let x_0 be a point in $\operatorname{supp}(g)$ and let $R > 0$ be such that $\operatorname{supp}(g) \subset B_R := B_R(x_0)$. We write

$$\begin{aligned}
\gamma_\ell &= \iint K_\ell(x,y) h_\eta(y) g_\ell(x) \, d\mu(x) d\mu(y), \\
\alpha_\ell &= \iint (K_\ell(x,y) - K_\ell(x_0,y)(1-h_\eta)(y) g_\ell(x) \, d\mu(x) d\mu(y), \\
\Omega &= \iint \langle (K(x,y) - K(x_0,y))(1-h_\eta(y)), g(x) \rangle \, d\mu(x) d\mu(y), \\
\langle T1, g \rangle &= \lim_{L\to\infty} \sum_{-L}^{L} \gamma_\ell + \Omega.
\end{aligned}$$

We write

$$\begin{aligned}
\Omega &= \iint \sum_{\ell=-\infty}^{\infty} (K_\ell(x,y) - K_\ell(x_0,y))(1-h_\eta(y)) g_\ell(x) \, d\mu(y) d\mu(x) \\
&= \iint \sum_{-\infty}^{-L} + \sum_{-L}^{L} + \sum_{L}^{\infty} = \sum_{\ell=-L}^{L} \alpha_\ell + I_\eta(L) + J_\eta(L),
\end{aligned}$$

where

$$I_\eta(L) = \iint \sum_{L}^{\infty} (K_\ell(x,y) - K_\ell(x_0,y)(1-h_\eta)(y) g_\ell(x)\, d\mu(x) d\mu(y) \tag{46}$$

$$J_\eta(L) = \iint \sum_{-\infty}^{L} (K_\ell(x,y) - K_\ell(x_0,y)(1-h_\eta)(y) g_\ell(x)\, d\mu(x) d\mu(y). \tag{47}$$

Therefore:

$$\begin{aligned}
\langle T1, g\rangle &= \lim_{L\to\infty} \left(\sum_{-L}^{L} (\gamma_\ell + \alpha_\ell) + I_\eta(L) + J_\eta(L) \right) \\
&= \lim_{L\to\infty} \Gamma(L) + \Lambda_\eta(L) + I_\eta(L) + J_\eta(L),
\end{aligned}$$

where

$$\Gamma(L) = \sum_{-L}^{L} \iint K_\ell(x,y)\, d\mu(y) g_\ell(x) d\mu(x) = 0,$$

because $\int K_\ell(x,y) d\mu(y) = 0$, and

$$\Lambda_\eta(L) = -\iint \sum_{-L}^{L} K_\ell(x_0,y)(1-h_\eta)(y) g_\ell(x)\, d\mu(x) d\mu(y) = 0,$$

because $\int g_\ell(x) d\mu(x) = 0$.

For $\eta \geq 2A^2R$, by using that $|K_\ell(x,y) - K_\ell(x_0,y)| \leq c2^{-\ell(\beta+1)} d(x,x_0)^\beta$, if L is big enough, we have

$$\begin{aligned}
&I_\eta(L) \\
&\leq \int \Big| \sum_{L}^{\infty} \int (K_\ell(x,y) - K_\ell(x_0,y))\, g_\ell(x)\, d\mu(x) \Big| (1-h_\eta(y))\, d\mu(y) \\
&\leq \int_{x\in \operatorname{supp}(g)} \sum_{L}^{\infty} \int_{d(x,y)>2AR} |K_\ell(x,y) - K_\ell(x_0,y)|\, |g_\ell(x)|\, (1-h_\eta(y))\, d\mu(y)\, d\mu(x) \\
&\leq CR^\beta \int_{x\in \operatorname{supp}(g)} \sum_{L}^{\infty} 2^{-\ell(\beta+1)} \int_{d(x,y)>AR,\, d(x,y)\leq c^2 2^\ell + cR} d\mu(y)\, |g_\ell(x)|\, d\mu(x),
\end{aligned}$$

where we have used that $\operatorname{supp}(K_\ell(x,\cdot) - K_\ell(x_0,\cdot)) \subset B_{A(c2^\ell+R)}(x)$. Now, since R is fixed for the given function g, and L is big enough, there is a constant C depending on R such that $d(x,y) \leq C_R 2^\ell$. Then, by applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
I_\eta(L) &\leq C\, R^\beta \int_{x\in \operatorname{supp}(g)} \sum_{L}^{\infty} 2^{-\ell\beta} |g_\ell(x)|\, d\mu(x) \\
&\leq CR^\beta 2^{-L\beta}\, \|g\|_{L^1_\mu(X,\mathbb{H})}.
\end{aligned}$$

This implies that
$$\lim_{L\to\infty} I_\eta(L) = 0. \tag{48}$$

Now, with $\eta \geq 2A^2R$, since $\operatorname{supp}(g_\ell) \cap \operatorname{supp}(1-h_\eta) = \emptyset$, and $\operatorname{supp}(K_\ell(x_0,\cdot) - K_\ell(x,\cdot)) \subset B_{A(c2^{-L}+R)}(x)$, thus $I_\eta(L) = 0$. Therefore, for $\eta \geq 2A^2R$ we get $\langle T1, g\rangle = 0$.

By repeating similar procedures for the adjoint operator T^*, if $f \in \Lambda^\beta_{00}(X)$ and $w = \{w_\ell\}_{\ell\in\mathbb{Z}}$ we get that $\langle T^*(1w), f\rangle = 0$. $\square$

Theorem 29. *The operator T defined in* (42) *is a Calderón-Zygmund operator.*

Proof. By Proposition 27, T is a singular integral operator. By Proposition 28, $T1 = 0_{\mathbb{H}}$ and for any $v \in \mathbb{H}$, $T^*(1v) = 0$. Besides, by inequality (43), T satisfies a Weak Boundedness Property. Thus, Theorem 6 holds for T. $\square$

Now we would want to prove the opposite inequality, that is:
$$\|f\|_{L^2_\mu(X)} \leq C\|\sum_j |\Delta_j f|^2\|_{L^2_\mu(X)}. \tag{49}$$

We need the Calderón reproducing formula contained in [HS, Theorem 3.9]. More precisely,

Theorem 30. *Suppose that $\{\tilde{D}_k\}_{k\in\mathbb{Z}}$ is a family satisfying Theorem 3.6 in* [HS]*. Then for all $f \in L^p_\mu(X)$, $1 < p < \infty$,*
$$\lim_{L\to\infty} \sum_{k=-L}^{L} \left\langle \tilde{D}_k \Delta_k f, g\right\rangle = \langle f, g\rangle, \tag{50}$$
where the series converges in the L^p_μ norm.

Here, the family of operators $\{\tilde{D}_k\}_{k\in\mathbb{Z}}$ is one obtained in [HS] whose respective kernels $\tilde{D}_k(x,y)$ satisfy the following properties, for $0 < \epsilon' < \beta \leq \theta$:
$$|\tilde{D}_k(x,y)| \leq C\frac{2^{-k\epsilon'}}{(2^{-k}+d(x,y))^{1+\epsilon'}}; \tag{51}$$
$$|\tilde{D}_k(x,y) - D_k(x',y)| \leq C\left(\frac{d(x,x')}{2^{-k}+d(x,y)}\right)^{\epsilon'} \frac{2^{-k\epsilon'}}{(2^{-k}+d(x,y))^{1+\epsilon'}}; \tag{52}$$
for $d(x,x') < (d(x,y)+2^{-k})/2A$;
$$\int \tilde{D}_k(x,y)d\mu(y) = \int \tilde{D}_k(x,y)d\mu(x) = 0. \tag{53}$$

We define the operator U in the following way:
$$\langle Uf, g\rangle := \lim_{L\to\infty} \sum_{k=-L}^{L} \iint \tilde{D}_k(x,y) f(y) g_k(x)\, d\mu(y)d\mu(x). \tag{54}$$

The kernel of this operator is

$$K(x,y) := \{\tilde{D}_k(x,y)\}_{k\in\mathbb{Z}}.$$

Theorem 31. *The operator U is a Calderón-Zygmund operator.*

Proof. We begin by studying the properties of size and smoothness of the kernel K. For the size, we have:

$$\begin{aligned}|K(x,y)|_{\ell^2}^2 &= \sum_{k\le \log_2 d(x,y)^{-1}} + \sum_{k\ge \log_2 d(x,y)^{-1}} |\tilde{D}_k(x,y)|^2\\ &\le C\sum_{k\le \log_2 d(x,y)^{-1}} 2^{2k} + C\sum_{k\ge \log_2 d(x,y)^{-1}} 2^{-2k\epsilon'} d(x,y)^{-2-2\epsilon'}\\ &\le Cd(x,y)^{-2}.\end{aligned}$$

For the smoothness, let $x, x', y \in X$ such that $d(x,x') \le d(x,y)/2A$; then, by using (52) and proceeding as above we have

$$\begin{aligned}|K(x,y)-K(x',y)|_{\ell^2}^2 = \sum_{k\in\mathbb{Z}} |\tilde{D}(x,y)-\tilde{D}(x',y)| &\le \sum_{k\le \log_2 d(x,y)^{-1}} + \sum_{k\ge \log_2 d(x,y)^{-1}}\\ &\le C\left[\frac{d(x,x')^{\epsilon'}}{d(x,y)^{1+\epsilon'}}\right]^2.\end{aligned}$$

Now we check that the operator U is continuous and well defined. We can proceed in a very similar way as in the case of the operator T in Subsection 4.1. To this end, we are going to prove that the limit corresponding to (54) exists and the following estimate holds:

$$|\langle Uf, g\rangle| \le C\|f\|_\beta \|g\|_{\beta,\mathbb{H}}\mu(B)^{1+2\beta'}, \tag{55}$$

for a suitable choice of the exponent β', where $f \in \Lambda_0^{\beta'}(X)$, $g \in \Lambda_0^{\beta'}(X,\mathbb{H})$, and $B = B_R(x_0)$ is the support of f and g.

We will take

$$0 < \beta' < \epsilon' < \beta \le \theta. \tag{56}$$

Let us take $f \in \Lambda_0^{\beta'}(X), g \in \Lambda_0^{\beta'}(X,\mathbb{H})$, both supported in $B = B(x_0,R)$ and

$$\begin{aligned}I_k &:= \iint \tilde{D}_k(x,y)f(y)g_k(x)\,d\mu(y)d\mu(x) &(57)\\ &= \int_{x\in \operatorname{supp}(g)} \int \tilde{D}_k(x,y)\,(f(y)-f(x))g_k(x)\,d\mu(y)d\mu(x), &(58)\end{aligned}$$

where we have used (53).

First, we give estimates for the case $2^{-k} \le 2AR$. In order to estimate I_k we split (58) in this way:

$$I_k = I_{k1} + I_{k2} := \int_B \int_{d(x,y)<2^{-k}} (\cdots) + \int\int_{d(x,y)\ge 2^{-k}} (\cdots).$$

Then

$$\begin{aligned} I_{k1} &\leq C\|f\|_{\beta'}\|g\|_{\infty,\mathbb{H}}\mu(B)\int_{d(x,y)<2^{-k}} \frac{2^{-k\epsilon'}2^{-k\beta'}}{(2^{-k})^{1+\epsilon'}}d\mu(y) \\ &\leq C\|f\|_{\beta'}\|g\|_{\beta',\mathbb{H}}2^{-k\beta'}\mu(B)^{1+\beta'}. \end{aligned}$$

$$\begin{aligned} I_{k2} &\leq C\|f\|_{\beta'}\|g\|_{\infty,\mathbb{H}}\mu(B)\int_{d(x,y)\geq 2^{-k}} 2^{-k\epsilon'}\frac{1}{d(x,y)^{1+\epsilon'-\beta'}}d\mu(y) \\ &\leq C\|f\|_{\beta'}\|g\|_{\beta',\mathbb{H}}2^{-k\beta'}\mu(B)^{1+\beta'}. \end{aligned}$$

Secondly, we study the case $2AR < 2^{-k}$. Let us observe that we only need to consider the case $d(x,y) \leq 2^{-k}$ in (57). Thus we write:

$$\begin{aligned} I_k &\leq C\|f\|_{\infty}\|g\|_{\infty,\mathbb{H}}\int_{B(x_0,R)}\int_{d(x,y)\leq 2AR}\frac{2^{-k\epsilon'}}{(2^{-k}+d(x,y))^{1+\epsilon'}}\,d\mu(y)d\mu(x). \\ &\leq C\|f\|_{\beta'}\|g\|_{\beta',\mathbb{H}}\mu(B)^{1+2\beta'}\,2AR\,2^k. \end{aligned}$$

Now we have:

$$\begin{aligned} |\,\langle Uf,g\rangle\,| &\leq C\|f\|_{\beta'}\|g\|_{\beta',\mathbb{H}}\mu(B)^{1+\beta'}\Big(\sum_{2^{-k}\leq 2AR} 2^{-k\beta'} + \mu(B)^{\beta'}\,2AR\sum_{2^{-k}>2AR}2^k\Big) \\ &\leq C\|f\|_{\beta'}\|g\|_{\beta',\mathbb{H}}\mu(B)^{1+2\beta'}. \end{aligned}$$

This proved the continuity and the Weak Boundedness Property of U, with exponent β'.

Now, let us suppose that f and g have disjoint supports. By considering the η-neighborhoods F_η of $F = \mathrm{supp}(f)$, we choose an integer ℓ_0 such that $F_{2^{\ell_0}} \cap \mathrm{supp}(g) = \emptyset$. By applying the Cauchy-Schwarz inequality, we can write

$$\begin{aligned} I &:= \int\sum_{k=-\infty}^{\infty}\int|\tilde{D}_k(x,y)|\,|f(y)|\,|g_k(x)|\,d\mu(y)\,d\mu(x) \\ &\leq \int\Big(\sum_{k=-\infty}^{\infty}\Big(\int|\tilde{D}_k(x,y)||f(y)|\,d\mu(y)\Big)^2\Big)^{1/2}|g(x)|_{\mathbb{H}}\,d\mu(x). \end{aligned}$$

We estimate

$$|\tilde{D}_k(x,y)| \leq \begin{cases} 2^k & \text{if } k \leq \ell_0, \\ 2^{-k\epsilon'}d(x,y)^{-1-\epsilon'} & \text{if } k > \ell_0. \end{cases}$$

We observe that the integral is null for $k > \ell_0$ and $d(x,y) \leq 2^{\ell_0}$. Hence,

$$I \leq C\|f\|_{L^\infty(X)}\|g\|_{L^\infty(X,\mathbb{H})}\mu(\mathrm{supp}(g))\Big(\sum_{k\leq\ell_0}(\mu(\mathrm{supp}(f))2^k)^2 + \sum_{k>\ell_0}2^{-2(\ell_0+k)\epsilon'}\Big)^{1/2}$$

is finite, so the operator U has associated the kernel K in the sense of (4).

Finally, let us check the condition $U1 = 0_{\mathbb{H}}$. The method is analogous to the case $T1 = 0_{\mathbb{H}}$, and we only shall do the estimates corresponding to (46) and (47).

We take $g \in \Lambda_{00}^{\beta'}(X, \mathbb{H})$ supported in $B = B_R(x_0)$. For $\eta > 2AR$, by picking L big enough we can write

$$\begin{aligned} I_\eta(L) &:= \sum_{k \geq L} \iint \left| \left(\tilde{D}_k(x,y) - \tilde{D}_k(x_0,y)\right)(1 - h_\eta(y))\, g_k(x) \right| d\mu(y) d\mu(x) \\ &\leq \int_{x \in \operatorname{supp}(g)} \int_{d(x,y) > 2AR} \frac{d(x,x_0)^{\epsilon'} 2^{-k\epsilon'}}{d(x,y)^{1+2\epsilon'}} |g_k(x)|\, d\mu(y) d\mu(x) \\ &\leq C R^{1-\epsilon'} \|g\|_{L^\infty_\mu(X,\mathbb{H})} \sum_{k \geq L} 2^{-k\epsilon'} \leq C_{R,g} 2^{-L\epsilon'}, \end{aligned}$$

$$\begin{aligned} J_\eta(L) &= \sum_{k \leq -L} \iint \left| \left(\tilde{D}_k(x,y) - \tilde{D}_k(x_0,y)\right)(1 - h_\eta(y))\, g_k(x) \right| d\mu(y) d\mu(x) \\ &\leq \sum_{k \leq -L} \int_{x \in \operatorname{supp}(g)} \int_{d(x,y) > 2AR} \frac{d(x,x_0)^{\epsilon'} 2^{-k\epsilon'}}{2^{-3k\epsilon'/2}\, d(x,y)^{1+\epsilon'/2}} |g_k(x)|\, d\mu(y) d\mu(x) \\ &\leq C R^{1+\epsilon'/2} \|g\|_{L^\infty(X,\mathbb{H})} \sum_{k \leq -L} 2^{k\epsilon'/2} \leq C_{R,g} 2^{-L\epsilon'/2}. \end{aligned}$$

□

By applying Theorems 29 and 31 and denoting

$$\tilde{\mathcal{G}} f(x) := \Big(\sum_{k=-\infty}^{\infty} |\tilde{D}_k f(x)|^2\Big)^{1/2},$$

we get

$$\begin{aligned} \|\mathcal{S} f\|_{L^2_\mu(X)} &= \|T f\|_{L^2_\mu(X,\mathbb{H})} \leq C \|f\|_{L^2_\mu(X)}, \\ \|\tilde{\mathcal{G}} f\|_{L^2_\mu(X)} &= \|U f\|_{L^2_\mu(X,\mathbb{H})} \leq C \|f\|_{L^2_\mu(X)}. \end{aligned}$$

Now, we are going to prove the inequality (49). From (50), since $\tilde{D}_j$ is a selfadjoint operator, and by applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|f\|_{L^2_\mu(X)} &= \Big\|\sum_j \tilde{D}_j \Delta_j f\Big\|_{L^2_\mu(X)} = \sup_{\|g\|_{L^2_\mu(X)=1}} \int \sum_j \tilde{D}_j \Delta_j f(x)\, g(x)\, d\mu(x) \\ &= \sup_{\|g\|_{L^2_\mu(X)=1}} \int \sum_j \Delta_j f(x) \tilde{D}_j\, g(x)\, d\mu(x) \\ &\leq \sup_{\|g\|_{L^2_\mu(X)=1}} \Big(\int [\sum_j |\Delta_j f(x)|]^2 d\mu(x) \Big)^{1/2} \Big(\int [\sum_j |\tilde{D}_j g(x)|]^2 d\mu(x) \Big)^{1/2} \\ &\leq C \|S(f)\|_{L^2_\mu(X)} \sup_{\|g\|_{L^2_\mu(g)=1}} \|\tilde{\mathcal{G}}(g)\|_{L^2_\mu(X)} \\ &\leq C \|S(f)\|_{L^2_\mu(X)}. \end{aligned}$$

From this, we can derive the Littlewood–Paley estimate

$$\|f\|_{L^2_\mu(X)} \approx \Big\| \Big(\sum_{\ell \in \mathbb{Z}} |\Delta_\ell f|^2 \Big) \Big\|_{L^2_\mu(X)}.$$

This result is obtained in [HS] and [T] without using any vector-valued $T1$ Theorem.

References

[A] H. Aimar, Singular integrals and approximate identities on spaces of homogeneous type, *Trans. Amer. Math. Soc.* **292** (1985), no. 1, 135–153. MR 0805957

[C] M. Cotlar, A unified theory of Hilbert transforms and ergodic theorems, *Rev. Mat. Cuyana* **1** (1955), 105–167 (1956). MR 0084632

[BCP] A. Benedek, A.-P. Calderón and R. Panzone, Convolution operators on Banach space valued functions, *Proc. Nat. Acad. Sci. U.S.A.* **48** (1962), 356–365. MR 0133653

[CJRW1] J. T. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl, Oscillation and variation for the Hilbert transform, *Duke Math. J.* **105** (2000), no. 1, 59–83. MR 1788042

[CJRW2] J. T. Campbell, R. L. Jones, K. Reinhold, and M. Wierdl, Oscillation and variation for singular integrals in higher dimensions, *Trans. Amer. Math. Soc.* **355** (2003), no. 5, 2115–2137. MR 1953540

[CZ] A. P. Calderon and A. Zygmund, On the existence of certain singular integrals, *Acta Math.* **88** (1952), 85–139. MR 0052553

[D] J. Duoandikoetxea, *Fourier analysis*, translated and revised from the 1995 Spanish original by David Cruz-Uribe, Graduate Studies in Mathematics, 29, *Amer. Math. Soc., Providence, RI*, 2001. MR 1800316

[DJ] G. David and J.-L. Journé, A boundedness criterion for generalized Calderón–Zygmund operators, Ann. of Math. (2) **120** (1984), no. 2, 371–397. MR 0763911

[DJS] G. David, J.-L. Journé and S. Semmes, Opérateurs de Calderón–Zygmund, fonctions para-accrétives et interpolation, *Rev. Mat. Iberoamericana* **1** (1985), no. 4, 1–56. MR 0850408

[DU] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Amer. Math. Soc., Providence, RI, 1977. MR 0453964

[F] T. Figiel, Singular integral operators: a martingale approach, in *Geometry of Banach spaces (Strobl, 1989)*, 95–110, *London Math. Soc. Lecture Note Ser.*, 158, *Cambridge Univ. Press, Cambridge*, 1990. MR 1110189

[HMMT] E. Harboure, R. A. Macías, M. T. Menárguez, and J. L. Torrea, Oscillation and variation for the Gaussian Riesz transforms and Poisson integral, *Proc. Roy. Soc. Edinburgh Sect. A* **135** (2005), no. 1, 85–104. MR 2119842

[HS] Y. S. Han and E. T. Sawyer, Littlewood–Paley theory on spaces of homogeneous type and the classical function spaces, *Mem. Amer. Math. Soc.* **110** (1994), no. 530, vi+126 pp. MR 1214968

[HTV] E. Harboure, J. L. Torrea and B. Viviani, On the search for weighted inequalities for operators related to the Ornstein-Uhlenbeck semigroup, *Math. Ann.* **318** (2000), no. 2, 341–353.. MR 1795566

[HW] T. Hytönen and L. Weis, A $T1$ theorem for integral transformations with operator-valued kernel, *J. Reine Angew. Math.* **599** (2006), 155–200. MR 2279101

[J] R. L. Jones, Ergodic theory and connections with analysis and probability, New York J. Math. **3A** (1997/98), *Proceedings of the New York Journal of Mathematics Conference*, June 9–13, 1997, 31–67. MR 1604581

[JKRW] R. L. Jones R. Kaufman, J. Rosenblatt, and M. Wierdl, Oscillation in ergodic theory, *Ergodic Theory Dynam. Systems* **18** (1998), no. 4, 889–935. MR 1645330

[JOR] R. L. Jones, I. V. Ostrovskii and J. M. Rosenblatt, Square functions in ergodic theory, *Ergodic Theory Dynam. Systems* **16** (1996), no. 2, 267–305. MR 1389625

[MS] R. A. Macías and C. Segovia, Lipschitz functions on spaces of homogeneous type, *Adv. in Math.* **33** (1979), no. 3, 257–270. MR 0546295

[MST] R. A. Macías, C. Segovia and J. L. Torrea, Singular integral operators with non-necessarily bounded kernels on spaces of homogeneous type, *Adv. Math.* **93** (1992), no. 1, 25–60. MR 1160842

[RFRT] J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea, Calderón–Zygmund theory for operator-valued kernels, *Adv. in Math.* **62** (1986), no. 1, 7–48. MR 0859252

[RFT] J. L. Rubio de Francia and J. L. Torrea, Vector extensions of operators in L^p spaces, *Pacific J. Math.* **105** (1983), no. 1, 227–235. MR 0688416

[S] E. M. Stein, *Topics in harmonic analysis related to the Littlewood–Paley theory*, Annals of Mathematics Studies, No. 63 *Princeton Univ. Press, Princeton, NJ*, 1970. MR 0252961

[T] X. Tolsa, Littlewood–Paley theory and the $T(1)$ theorem with non-doubling measures, *Adv. Math.* **164** (2001), no. 1, 57–116. MR 1870513

[VV] P. S. Viola and B. E. Viviani, An explicit expression for singular integral operators with non-necessarily doubling measures, *Collect. Math.* **63** (2012), no. 2, 217–242. MR 2909827

[W] R. Wittmann, Application of a theorem of M. G. Kreĭn to singular integrals, *Trans. Amer. Math. Soc.* **299** (1987), no. 2, 581–599. MR 0869223

Pablo Sebastián Viola
Facultad de Ciencias Exactas, UNICEN
Pinto 399, Tandil (7000), Buenos Aires, Argentina
pablosebviola@yahoo.com.ar

Received: October 11, 2012
Accepted: April 15, 2013