

A GEOMETRIC INEQUALITY FOR WARPED PRODUCT SEMI-SLANT SUBMANIFOLDS OF NEARLY COSYMPLECTIC MANIFOLDS

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ABSTRACT. Recently, we have shown that there do not exist warped product semi-slant submanifolds of cosymplectic manifolds [K.A. Khan, V.A. Khan and Siraj Uddin, *Balkan J. Geom. Appl.* 13 (2008), 55–65]. The nearly cosymplectic structure generalizes the cosymplectic one. Therefore the nearly Kaehler structure generalizes the Kaehler structure in almost Hermitian setting. It is interesting that the warped product semi-slant submanifolds exist in the nearly cosymplectic case while in the cosymplectic case they do not. In the beginning, we prove some preparatory results and finally we obtain an inequality such as $\|h\|^2 \geq 4q \csc^2 \theta \{1 + \frac{1}{9} \cos^2 \theta\} \|\nabla \ln f\|^2$ in terms of intrinsic and extrinsic invariants. The equality case is also considered.

1. INTRODUCTION

To study the manifolds with negative curvature Bishop and O'Neill introduced the idea of warped products [2]. Afterwards, this idea was used to model the standard space time, especially in the neighborhood of massive stars and black holes [11]. However, warped product spaces were developed in Riemannian manifolds enormously [8, 9]. The geometry of warped product submanifolds is intensified after B.Y. Chen's work on warped product CR-submanifolds of Kaehler manifolds [8]. Motivated by Chen's papers, many geometers studied warped product submanifolds in almost Hermitian as well as almost contact metric manifolds [1, 9, 13].

On the other hand, almost contact manifolds with Killing structure tensors were defined in [3] as nearly cosymplectic manifolds. Later on, Blair and Showers [5] studied the nearly cosymplectic structure (ϕ, ξ, η, g) on a Riemannian manifold \bar{M} with η closed from the topological viewpoint. An almost contact metric structure (ϕ, ξ, η, g) satisfying $(\bar{\nabla}_X \phi)X = 0$ is called a nearly cosymplectic structure. If we consider S^5 as a totally geodesic hypersurface of S^6 , then it is known that S^5 has

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a non cosymplectic nearly cosymplectic structure. It was shown that the normal nearly cosymplectic manifolds are cosymplectic (see [4]).

Next, the slant submanifolds of an almost contact metric manifold were defined and studied by J.L. Cabrerizo et al. [7]. The notion of semi-slant submanifolds of almost Hermitian manifolds was introduced by N. Papaghuic [12]. In fact, semi-slant submanifolds in almost Hermitian manifolds are defined on the line of CR-submanifolds. In the setting of almost contact metric manifolds, semi-slant submanifolds are defined and investigated by Cabrerizo et al. [6].

Recently, we studied warped product semi-slant submanifolds of cosymplectic manifolds [10]. We have seen that there do not exist warped product semi-slant submanifolds in cosymplectic manifolds. The nearly cosymplectic structure generalizes the cosymplectic ones. Therefore the nearly Kaehler generalizes the Kaehler structure. So, the geometric study of warped product semi-slant submanifolds in nearly cosymplectic is interesting. Such type of warped products exist in nearly cosymplectic case while in case of cosymplectic do not exist. In this paper, we prove that the warped product semi-slant submanifolds of the type $N_\theta \times_f N_T$ do not exist in a nearly cosymplectic manifold \bar{M} . However, we obtain some interesting results on the existence of the warped product submanifolds of the type $N_T \times_f N_\theta$ of a nearly cosymplectic manifold \bar{M} , where N_T and N_θ are the invariant and proper slant submanifolds of \bar{M} , respectively. We also establish a general sharp inequality for squared norm of the second fundamental form in terms of the warping function and the slant angle for the warped product semi-slant submanifolds in the form $N_T \times_f N_\theta$ in a nearly cosymplectic manifold \bar{M} .

2. PRELIMINARIES

A $(2n+1)$ -dimensional C^∞ manifold \bar{M} is said to have an *almost contact structure* if there exist on \bar{M} a tensor field ϕ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying [5]

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1. \quad (2.1)$$

There always exists a Riemannian metric g on an almost contact manifold \bar{M} satisfying the following compatibility condition

$$\eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.2)$$

where X and Y are vector fields on \bar{M} [5].

An almost contact structure (ϕ, ξ, η) is said to be *normal* if the almost complex structure J on the product manifold $\bar{M} \times \mathbb{R}$ given by

$$J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt}),$$

where f is a C^∞ -function on $\bar{M} \times \mathbb{R}$, has no torsion, i.e., J is integrable. The condition for normality in terms of ϕ , ξ and η is $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on \bar{M} , where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ . Finally the *fundamental 2-form* Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric structure (ϕ, ξ, η, g) is said to be *cosymplectic*, if it is normal and both Φ and η are closed [5]. The structure is said to be *nearly cosymplectic* if ϕ is Killing, i.e., if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = 0, \tag{2.3}$$

for any $X, Y \in T\bar{M}$, where $T\bar{M}$ is the tangent bundle of \bar{M} and $\bar{\nabla}$ denotes the Riemannian connection of the metric g . Equation (2.3) is equivalent to $(\bar{\nabla}_X \phi)X = 0$, for each $X \in T\bar{M}$. The structure is said to be *closely cosymplectic* if ϕ is Killing and η is closed. It is well known that an almost contact metric manifold is *cosymplectic* if and only if $\bar{\nabla}\phi$ vanishes identically, i.e., $(\bar{\nabla}_X \phi)Y = 0$ and $\bar{\nabla}_X \xi = 0$.

Proposition 2.1 ([5]). *On a nearly cosymplectic manifold the vector field ξ is Killing.*

From the above proposition, we have $g(\bar{\nabla}_X \xi, X) = 0$, for any vector field X tangent to \bar{M} , where \bar{M} is a nearly cosymplectic manifold.

Let M be a submanifold of an almost contact metric manifold \bar{M} with induced metric g and let ∇ and ∇^\perp be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ of M , respectively. Denote by $\mathcal{F}(M)$ the algebra of smooth functions on M and by $\Gamma(TM)$ the $\mathcal{F}(M)$ -module of smooth sections of TM over M . Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{2.5}$$

for each $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where h and A_N are the second fundamental form and the shape operator (corresponding to the normal vector field N) respectively for the immersion of M into \bar{M} . They are related as

$$g(h(X, Y), N) = g(A_N X, Y), \tag{2.6}$$

where g denotes the Riemannian metric on \bar{M} as well as induced on M .

For any $p \in M$, let $\{e_1, \dots, e_m, \dots, e_{2n+1}\}$ be an orthonormal frame for the tangent space $T_p \bar{M}$, such that e_1, \dots, e_m are tangent to M at p . We denote by H the mean curvature vector, that is

$$H(p) = \frac{1}{m} \sum_{i=1}^m h(e_i, e_i).$$

Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r), \quad i, j \in \{1, \dots, m\}, \quad r \in \{m+1, \dots, 2n+1\}$$

and

$$\|h\|^2 = \sum_{i,j=1}^m g(h(e_i, e_j), h(e_i, e_j)).$$

For any $X \in \Gamma(TM)$, we write

$$\phi X = PX + FX, \tag{2.7}$$

where PX is the tangential component and FX is the normal component of ϕX .

Similarly for any $N \in \Gamma(T^\perp M)$, we write

$$\phi N = BN + CN, \quad (2.8)$$

where BN is the tangential component and CN is the normal component of ϕN . The covariant derivative of the tensor field ϕ is defined as

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y, \quad (2.9)$$

for all $X, Y \in \Gamma(T\bar{M})$.

Now, denote by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ the tangential and normal parts of $(\bar{\nabla}_X \phi)Y$, i.e.,

$$(\bar{\nabla}_X \phi)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y \quad (2.10)$$

for all $X, Y \in \Gamma(TM)$. Making use of (2.7)-(2.10) and the Gauss and Weingarten formulae, the following equations may easily be obtained:

$$\mathcal{P}_X Y = (\bar{\nabla}_X P)Y - A_{FY}X - Bh(X, Y) \quad (2.11)$$

$$\mathcal{Q}_X Y = (\bar{\nabla}_X F)Y + h(X, PY) - Ch(X, Y), \quad (2.12)$$

where the covariant derivative of P and F are defined by

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y \quad (2.13)$$

$$(\bar{\nabla}_X F)Y = \nabla_X^\perp FY - F\nabla_X Y \quad (2.14)$$

for all $X, Y \in \Gamma(TM)$.

Similarly, for any $X \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, denoting the tangential and normal parts of $(\bar{\nabla}_X \phi)N$ by $\mathcal{P}_X N$ and $\mathcal{Q}_X N$ respectively, we obtain

$$\mathcal{P}_X N = (\bar{\nabla}_X B)N + PA_N X - A_{CN} X \quad (2.15)$$

$$\mathcal{Q}_X N = (\bar{\nabla}_X C)N + h(BN, X) + FA_N X, \quad (2.16)$$

where the covariant derivative of B and C are defined by

$$(\bar{\nabla}_X B)N = \nabla_X BN - B\nabla_X^\perp N \quad (2.17)$$

$$(\bar{\nabla}_X C)N = \nabla_X^\perp CN - C\nabla_X^\perp N. \quad (2.18)$$

It is straightforward to verify the following properties of \mathcal{P} and \mathcal{Q} , which we enlist here for later use:

$$(p_1) \quad (i) \mathcal{P}_{X+Y}W = \mathcal{P}_X W + \mathcal{P}_Y W, \quad (ii) \mathcal{Q}_{X+Y}W = \mathcal{Q}_X W + \mathcal{Q}_Y W,$$

$$(p_2) \quad (i) \mathcal{P}_X(Y+W) = \mathcal{P}_X Y + \mathcal{P}_X W, \quad (ii) \mathcal{Q}_X(Y+W) = \mathcal{Q}_X Y + \mathcal{Q}_X W,$$

$$(p_3) \quad (i) g(\mathcal{P}_X Y, W) = -g(Y, \mathcal{P}_X W), \quad (ii) g(\mathcal{Q}_X Y, N) = -g(Y, \mathcal{P}_X N),$$

for all $X, Y, W \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$.

On a submanifold M of a nearly cosymplectic manifold, by equations (2.3) and (2.10) we have

$$(a) \mathcal{P}_X Y + \mathcal{P}_Y X = 0, \quad (b) \mathcal{Q}_X Y + \mathcal{Q}_Y X = 0 \quad (2.19)$$

for any $X, Y \in \Gamma(TM)$.

A submanifold M of an almost contact metric manifold \bar{M} is said to be *invariant* if F is identically zero, that is, $\phi X \in \Gamma(TM)$, and *anti-invariant* if P is identically zero, that is, $\phi X \in \Gamma(T^\perp M)$, for any $X \in \Gamma(TM)$.

We shall always consider ξ to be tangent to the submanifold M . There is another class of submanifolds called slant. For each non zero vector X tangent to M at x , such that X is not proportional to ξ_x , we denote by $0 \leq \theta(X) \leq \pi/2$, the angle between ϕX and $T_x M$, called the slant angle. If the slant angle $\theta(X)$ is constant for all $X \in T_x M - \langle \xi_x \rangle$ and $x \in M$, then M is said to be a *slant submanifold* [7]. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither an invariant nor an anti-invariant submanifold.

We recall the following result for a slant submanifold.

Theorem 2.1 ([7]). *Let M be a submanifold of an almost contact metric manifold \bar{M} , such that ξ is tangent to M . Then M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$P^2 = \lambda(-I + \eta \otimes \xi). \tag{2.20}$$

Furthermore, if θ is the slant angle of M , then $\lambda = \cos^2 \theta$.

The following relations are straightforward consequences of relation (2.20):

$$g(PX, PY) = \cos^2 \theta (g(X, Y) - \eta(Y)\eta(X)) \tag{2.21}$$

$$g(FX, FY) = \sin^2 \theta (g(X, Y) - \eta(Y)\eta(X)) \tag{2.22}$$

for all $X, Y \in \Gamma(TM)$.

A submanifold M of an almost contact metric manifold \bar{M} is said to be *semi-slant* if there exist two orthogonal complementary distributions \mathcal{D}_1 and \mathcal{D}_2 satisfying:

- (i) $TM = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \langle \xi \rangle$;
- (ii) \mathcal{D}_1 is an invariant, i.e. $\phi \mathcal{D}_1 \subseteq \mathcal{D}_1$;
- (iii) \mathcal{D}_2 is a slant distribution with slant angle $\theta \neq \frac{\pi}{2}$.

A semi-slant submanifold M of an almost contact manifold \bar{M} is *mixed geodesic* if

$$h(X, Z) = 0 \tag{2.23}$$

for any $X \in \mathcal{D}_1$ and $Z \in \mathcal{D}_2$. Moreover, if μ is the ϕ -invariant subspace of the normal bundle $T^\perp M$, then in the case of a semi-slant submanifold, the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = F\mathcal{D}_2 \oplus \mu. \tag{2.24}$$

3. WARPED PRODUCT SEMI-SLANT SUBMANIFOLDS

Bishop and O'Neill [2] introduced the notion of warped product manifolds. These manifolds are natural generalizations of Riemannian product manifolds. They defined these manifolds as follows: Let (N_1, g_1) and (N_2, g_2) be two Riemannian manifolds, and $f > 0$ a differentiable function on N_1 . Consider the product manifold $N_1 \times N_2$ with its projections $\pi_1 : N_1 \times N_2 \rightarrow N_1$ and $\pi_2 : N_1 \times N_2 \rightarrow N_2$. Then the warped product of N_1 and N_2 denoted by $M = N_1 \times_f N_2$ is a Riemannian manifold $N_1 \times N_2$ equipped with the Riemannian structure such that

$$g(X, Y) = g_1(\pi_{1*}X, \pi_{1*}Y) + (f \circ \pi)^2 g_2(\pi_{2*}X, \pi_{2*}Y)$$

for each $X, Y \in \Gamma(TM)$, and \star is a symbol for the tangent map. Thus we have

$$g = g_1 + f^2 g_2. \quad (3.1)$$

The function f is called the *warping function* of the warped product [2, 11]. A warped product manifold $N_1 \times_f N_2$ is said to be *trivial* if the warping function f is constant. We recall the following general result for a warped product manifold for later use.

Lemma 3.1 ([2]). *Let $M = N_1 \times_f N_2$ be a warped product manifold. Then*

- (i) $\nabla_X Y \in TN_1$ is the lift of $\nabla_X Y$ on N_1 ,
- (ii) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$,
- (iii) $\nabla_Z W = \nabla_Z^{N_2} W - g(Z, W)\nabla \ln f$,

for each $X, Y \in \Gamma(TN_1)$ and $Z, W \in \Gamma(TN_2)$, where ∇ and ∇^{N_2} denote the Levi-Civita connections on M and N_2 , respectively, and $\nabla \ln f$ is the gradient of $\ln f$.

Let M be a Riemannian manifold of dimension k with the inner product g , and $\{e_1, \dots, e_k\}$ be an orthonormal frame on M . Then for a differentiable function f on M , the gradient ∇f of a function f on M is defined by

$$g(\nabla f, U) = Uf, \quad (3.2)$$

for any $U \in \Gamma(TM)$. As a consequence, we have

$$\|\nabla f\|^2 = \sum_{i=1}^k (e_i(f))^2, \quad (3.3)$$

where ∇f is the gradient of the function f on M .

Now, we consider warped product semi-slant submanifolds tangent to the structure vector field ξ which are either in the form $M = N_T \times_f N_\theta$ or $M = N_\theta \times_f N_T$ in a nearly cosymplectic manifold \bar{M} , where N_T and N_θ are invariant and proper slant submanifolds of a nearly cosymplectic manifold \bar{M} , respectively. On a warped product submanifold $M = N_1 \times_f N_2$ of a nearly cosymplectic manifold \bar{M} , we have the following result.

Theorem 3.1 ([13]). *A warped product submanifold $M = N_1 \times_f N_2$ of a nearly cosymplectic manifold \bar{M} is an usual Riemannian product if the structure vector field ξ is tangent to N_2 , where N_1 and N_2 are the Riemannian submanifolds of \bar{M} .*

From the above theorem for the existence of warped products we always consider the structure vector field ξ is tangent to the base. Now, we start with warped product semi-slant submanifolds of the type $M = N_\theta \times_f N_T$ of a nearly cosymplectic manifold \bar{M} .

Theorem 3.2. *There do not exist warped product semi-slant submanifolds $M = N_\theta \times_f N_T$ of a nearly cosymplectic manifold \bar{M} , where N_θ and N_T are proper slant and invariant submanifolds of \bar{M} , respectively.*

Proof. The proof is similar to the semi-invariant case, which we have proved in [13]. \square

We now discuss the other case. All the results before the geometric inequality are required since we shall use them in our main theorem.

Lemma 3.2. *Let $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold of a nearly cosymplectic manifold \bar{M} , where N_T and N_θ are invariant and proper slant submanifolds of \bar{M} , respectively. Then*

- (i) $\xi \ln f = 0$
- (ii) $g(h(X, Y), FZ) = 0$
- (iii) $g(h(PX, Z), FZ) = (X \ln f) \|Z\|^2$
- (iv) $g(h(X, Z), FPZ) = -g(h(X, PZ), FZ)$
- (v) $g(\mathcal{P}_X Z, PZ) = 2g(h(X, Z), FPZ)$

for any $X, Y \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$.

Proof. Let $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold of a nearly cosymplectic manifold \bar{M} . We assume that the structure vector field ξ is tangent to N_T ; then for any $Z \in \Gamma(TN_\perp)$, we have

$$\bar{\nabla}_Z \xi = \nabla_Z \xi + h(Z, \xi).$$

Taking the inner product with Z and using Proposition 2.1 and Lemma 3.1 (ii), we obtain

$$(\xi \ln f) \|Z\|^2 = 0.$$

This means that either M is invariant or $\xi \ln f = 0$, which proves (i).

Now, we consider $X, Y \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$; then we have

$$g(h(X, Y), FZ) = g(\bar{\nabla}_X Y, FZ) = -g(Y, \bar{\nabla}_X FZ).$$

Using (2.7) and then (2.9), we obtain

$$g(h(X, Y), FZ) = -g(Y, (\bar{\nabla}_X \phi)Z) - g(Y, \phi \bar{\nabla}_X Z) + g(Y, \bar{\nabla}_X PZ).$$

Then from (2.2), (2.4) and Lemma 3.1 (ii), the second and last terms of the right hand side vanish identically and hence by (2.10), we derive

$$g(h(X, Y), FZ) = -g(Y, \mathcal{P}_X Z).$$

Thus, on using the property p_3 (i), we get

$$g(h(X, Y), FZ) = g(\mathcal{P}_X Y, Z).$$

Hence, by skew-symmetry of $\mathcal{P}_X Y$ and symmetry of $h(X, Y)$, we get the second part of the lemma.

For the third part, consider for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$, we have

$$\begin{aligned} g(h(PX, Z), FZ) &= g(\bar{\nabla}_Z PX, \phi Z - PZ) \\ &= -g(PX, \bar{\nabla}_Z \phi Z) - g(\bar{\nabla}_Z PX, PZ). \end{aligned}$$

From (2.4), (2.9) and Lemma 3.1 (ii), the above equation is reduced to

$$g(h(PX, Z), FZ) = -g(PX, (\bar{\nabla}_Z \phi)Z) - g(PX, \phi \bar{\nabla}_Z Z) - (PX \ln f)g(Z, PZ).$$

On using the structure equation of nearly cosymplectic and the fact that Z and PZ are orthogonal vector fields, the first and last terms of the right hand side are identically zero. Then from (2.2) we derive

$$g(h(PX, Z), FZ) = g(\phi^2 X, \bar{\nabla}_Z Z).$$

Using (2.1), we get

$$g(h(PX, Z), FZ) = -g(X, \bar{\nabla}_Z Z) + \eta(X)g(\xi, \bar{\nabla}_Z Z).$$

By the property of Riemannian connection $\bar{\nabla}$, the above equation takes the form

$$g(h(PX, Z), FZ) = g(\bar{\nabla}_Z X, Z) - \eta(X)g(\bar{\nabla}_Z \xi, Z).$$

Then from (2.4), Proposition 2.1 and Lemma 3.1 (ii), we obtain

$$g(h(PX, Z), FZ) = (X \ln f)\|Z\|^2,$$

which is third part of the lemma.

For the other parts, consider

$$g(\nabla_{PZ} \phi X, Z) = g(\bar{\nabla}_{PZ} \phi X, Z)$$

for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$. Using the property of Riemannian connection $\bar{\nabla}$ and Lemma 3.1 (ii), we get

$$(\phi X \ln f)g(Z, PZ) = -g(\phi X, \bar{\nabla}_{PZ} Z).$$

Using the fact that Z and PZ are orthogonal vector fields, the above equation reduces to

$$0 = g(X, \phi \bar{\nabla}_{PZ} Z).$$

Then from (2.9), we derive

$$0 = g(X, \bar{\nabla}_{PZ} \phi Z) - g(X, (\bar{\nabla}_{PZ} \phi)Z).$$

By (2.7) and (2.10), we obtain

$$0 = g(X, \bar{\nabla}_{PZ} PZ) + g(X, \bar{\nabla}_{PZ} FZ) - g(X, \mathcal{P}_{PZ} Z).$$

Using (2.4), (2.5) and (2.19) (a), we get

$$0 = -g(\nabla_{PZ} X, PZ) - g(X, A_{FZ} PZ) + g(X, \mathcal{P}_Z PZ).$$

Then from the property p_3 (i) and Lemma 3.1 (ii), we obtain

$$0 = -(X \ln f)g(PZ, PZ) - g(h(X, PZ), FZ) - g(\mathcal{P}_Z X, PZ).$$

Again using (2.19) (a), (2.21) and the fact that ξ is tangent to N_T , we derive

$$g(\mathcal{P}_X Z, PZ) = (X \ln f) \cos^2 \theta \|Z\|^2 + g(h(X, PZ), FZ). \quad (3.4)$$

Interchanging Z by PZ and then using (2.20), (2.21) and the fact that ξ is tangent to N_T , we obtain

$$-\cos^2 \theta g(\mathcal{P}_X PZ, Z) = (X \ln f) \cos^4 \theta \|Z\|^2 - \cos^2 \theta g(h(X, Z), FPZ).$$

By the property p_3 (i), the above equation will be

$$g(\mathcal{P}_X Z, PZ) = (X \ln f) \cos^2 \theta \|Z\|^2 - g(h(X, Z), FPZ). \quad (3.5)$$

Thus, the fourth part of the lemma follows from (3.4) and (3.5).

Now, for part (v), we consider

$$g(h(X, PZ), FZ) = g(\bar{\nabla}_X PZ, FZ)$$

for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$. Using the property of Riemannian connection $\bar{\nabla}$ and then using (2.7), we have

$$g(h(X, PZ), FZ) = -g(PZ, \bar{\nabla}_X \phi Z) + g(PZ, \bar{\nabla}_X PZ).$$

Using (2.9), Lemma 3.1 (ii), (2.21) and the fact that ξ is tangent to N_T , we obtain

$$g(h(X, PZ), FZ) = -g(PZ, \phi \bar{\nabla}_X Z) - g(PZ, (\bar{\nabla}_X \phi)Z) + (X \ln f) \cos^2 \theta \|Z\|^2.$$

Then from (2.2) and (2.10), we get

$$g(h(X, PZ), FZ) = g(\phi PZ, \bar{\nabla}_X Z) - g(PZ, \mathcal{P}_X Z) + (X \ln f) \cos^2 \theta \|Z\|^2.$$

Using (2.4) and (2.7), we derive

$$\begin{aligned} g(h(X, PZ), FZ) &= g(P^2 Z, \nabla_X Z) + g(h(X, Z), FPZ) \\ &\quad - g(PZ, \mathcal{P}_X Z) + (X \ln f) \cos^2 \theta \|Z\|^2. \end{aligned}$$

Again, from the fact that ξ is tangent to N_T and using (2.20), the above equation reduces to

$$g(\mathcal{P}_X Z, PZ) = g(h(X, Z), FPZ) - g(h(X, PZ), FZ). \tag{3.6}$$

Thus, the fifth part of the lemma follows from (3.6) and the fourth part of this lemma. This proves the lemma completely. □

Theorem 3.3. *Let $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold in a nearly cosymplectic manifold \bar{M} , where N_T and N_θ are invariant and proper slant submanifolds of \bar{M} , respectively. Then,*

$$g(\mathcal{P}_X Z, PZ) = \frac{2}{3} (X \ln f) \cos^2 \theta \|Z\|^2$$

for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$.

Proof. From (3.4) and (3.5), we have

$$\begin{aligned} 2g(\mathcal{P}_X Z, PZ) &= 2(X \ln f) \cos^2 \theta \|Z\|^2 + g(h(X, PZ), FZ) \\ &\quad - g(h(X, Z), FPZ). \end{aligned} \tag{3.7}$$

Then, by Lemma 3.2 (iv), we obtain

$$2g(\mathcal{P}_X Z, PZ) = 2(X \ln f) \cos^2 \theta \|Z\|^2 - 2g(h(X, Z), FPZ). \tag{3.8}$$

Thus, from Lemma 3.2 (v) and (3.8), we obtain the desired result. □

From the above theorem we have the following consequence.

Corollary 3.1. *A warped product semi-slant submanifold $M = N_T \times_f N_\theta$ of a nearly cosymplectic manifold \bar{M} is simply a Riemannian product of N_T and N_θ if and only if $\mathcal{P}_X Z \in \Gamma(TN_T)$, for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$.*

Theorem 3.4. *Let $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold in a nearly cosymplectic manifold \bar{M} , where N_T and N_θ are invariant and proper slant submanifolds of \bar{M} , respectively. Then,*

$$g(h(X, Z), FPZ) = -g(h(X, PZ), FZ) = \frac{1}{3}(X \ln f) \cos^2 \theta \|Z\|^2,$$

for any $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$.

Proof. The first equality is nothing but Lemma 3.2 (iv), and the second equality follows directly from equation (3.8) and Lemma 3.2 (v). \square

From the above theorem we have the following corollaries.

Corollary 3.2. *A semi-slant warped product submanifold $M = N_T \times_f N_\theta$ of a nearly cosymplectic manifold is simply a Riemannian product of N_T and N_θ if and only if $h(X, Z) \in \Gamma(\mu)$, for all $X \in \Gamma(TN_T)$ and $Z \in \Gamma(TN_\theta)$, where μ is the invariant normal subbundle of $T^\perp M$.*

Corollary 3.3. *There does not exist a mixed geodesic warped product semi-slant submanifold of a nearly cosymplectic manifold.*

From Lemma 3.2 (i), (iv) and Theorem 3.4, we obtain

$$g(h(\xi, Z), FPZ) = g(h(\xi, PZ), FZ) = 0, \quad (3.9)$$

for any $Z \in \Gamma(TN_\theta)$.

Using the previous results, we derive the following geometric inequality for the length of the second fundamental form.

Theorem 3.5. *Let $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold of a nearly cosymplectic manifold \bar{M} , where N_T and N_θ are invariant and proper slant submanifolds of \bar{M} , respectively. Then*

- (i) *The squared norm of the second fundamental form h satisfies*

$$\|h\|^2 \geq 4q \csc^2 \theta \left\{ 1 + \frac{1}{9} \cos^2 \theta \right\} \|\nabla \ln f\|^2, \quad (3.10)$$

where $\nabla \ln f$ is the gradient of $\ln f$ and $2q$ is the dimension of N_θ .

- (ii) *If the equality sign of (3.10) holds, then N_T is totally geodesic in \bar{M} and N_θ is a totally umbilical submanifold of \bar{M} . Moreover, M is a minimal submanifold of \bar{M} .*

Proof. Let \bar{M} be a $(2n + 1)$ -dimensional nearly cosymplectic manifold and let $M = N_T \times_f N_\theta$ be a warped product semi-slant submanifold of \bar{M} . Let us denote by \mathcal{D} and \mathcal{D}_θ the tangent bundles on N_T and N_θ , respectively and let $\dim N_T = 2p + 1$ and $\dim N_\theta = 2q$, then $m = 2p + 2q + 1$. Let $\{X_1, \dots, X_p, X_{p+1} = \phi X_1, \dots, X_{2p} = \phi X_p, X_{2p+1} = \xi\}$ and $\{Z_1, \dots, Z_q, Z_{q+1} = \sec \theta PZ_1, \dots, Z_{2q} = \sec \theta PZ_q\}$ be the local orthonormal frames of \mathcal{D} and \mathcal{D}_θ , respectively. Then, the orthonormal frames of $F\mathcal{D}_\theta$ and μ are $\{Z_1^* = \csc \theta FZ_1, \dots, Z_q^* = \csc \theta FZ_q, Z_{q+1}^* = \csc \theta \sec \theta FPZ_1, \dots, Z_{2q}^* = \csc \theta \sec \theta FPZ_q\}$ and $\{N_{m+2q+1}^*, \dots, N_{2n+1}^*\}$, respectively, where N is the normal vector in the invariant normal subbundle of $T^\perp M$.

The dimensions of $F\mathcal{D}_\theta$ and μ will be $2q$ and $2n + 1 - m - 2q$, respectively. The length of the second fundamental form h is defined as

$$\|h\|^2 = \sum_{r=m+1}^{2n+1} \sum_{i,j=1}^m g(h(U_i, V_j), N_r)^2 \tag{3.11}$$

for any vector fields U_i, V_j tangent to M and N_r normal to M . Now, for the assumed frames of $F\mathcal{D}_\theta$ and μ , the above equation can be written as

$$\|h\|^2 = \sum_{r=m+1}^{m+2q} \sum_{i,j=1}^m g(h(U_i, V_j), Z_r^*)^2 + \sum_{r=m+2q+1}^{2n+1} \sum_{i,j=1}^m g(h(U_i, V_j), N_r^*)^2. \tag{3.12}$$

The first term in the right hand side of the above equation is the $F\mathcal{D}_\theta$ -component and the second term is the μ -component. Here, we equate only the $F\mathcal{D}_\theta$ -component term, thus we have

$$\|h\|^2 \geq \sum_{r=m+1}^{m+2q} \sum_{i,j=1}^m g(h(U_i, V_j), Z_r^*)^2. \tag{3.13}$$

Thus, for the given frames of \mathcal{D} and \mathcal{D}_θ , the above equation will be

$$\begin{aligned} \|h\|^2 \geq & \sum_{r=1}^{2q} \sum_{i,j=1}^{2p+1} g(h(X_i, X_j), Z_r^*)^2 + 2 \sum_{r=1}^{2q} \sum_{i=1}^{2p+1} \sum_{j=1}^{2q} g(h(X_i, Z_j), Z_r^*)^2 \\ & + \sum_{r=1}^{2q} \sum_{i,j=1}^{2q} g(h(Z_i, Z_j), Z_r^*)^2. \end{aligned} \tag{3.14}$$

By Lemma 3.2 (ii), the first term of the right hand side is identically zero and we shall compute the next term and leave the third term

$$\|h\|^2 \geq 2 \sum_{r=1}^{2q} \sum_{i=1}^{2p+1} \sum_{j=1}^{2q} g(h(X_i, Z_j), Z_r^*)^2.$$

As $j, r = 1, \dots, 2q$, then the above equation can be written for one summation as

$$\|h\|^2 \geq 2 \sum_{i=1}^{2p+1} \sum_{j=1}^{2q} g(h(X_i, Z_j), Z_j^*)^2.$$

Separating the $h(\xi, Z)$ -components, the above inequality will be

$$\|h\|^2 \geq 2 \sum_{i=1}^{2p} \sum_{j=1}^{2q} g(h(X_i, Z_j), Z_j^*)^2 + 2 \sum_{j=1}^{2q} g(h(\xi, Z_j), Z_j^*)^2. \tag{3.15}$$

Now, we solve the second term of the right hand side of (3.15) as follows:

$$\begin{aligned}
 \sum_{j=1}^{2q} g(h(\xi, Z_j), Z_j^*)^2 &= \csc^2 \theta \sum_{j=1}^q g(h(\xi, Z_j), FZ_j)^2 \\
 &\quad + \csc^2 \theta \sec^4 \theta \sum_{j=1}^q g(h(\xi, PZ_j), FPZ_j)^2 \\
 &\quad + \csc^2 \theta \sec^2 \theta \sum_{j=1}^q g(h(\xi, Z_j), FPZ_j)^2 \\
 &\quad + \csc^2 \theta \sec^2 \theta \sum_{j=1}^q g(h(\xi, PZ_j), FZ_j)^2.
 \end{aligned}
 \tag{3.16}$$

From (3.9), the last two terms of the right hand side of (3.16) are identically zero and we will compute the first two terms as follows. We know that

$$g(h(\xi, Z), FZ) = g(\bar{\nabla}_Z \xi, FZ) = -g(\xi, \bar{\nabla}_Z FZ).$$

Using (2.7) and then (2.9), we obtain

$$g(h(\xi, Z), FZ) = -g(\xi, (\bar{\nabla}_Z \phi)Z) - g(\xi, \phi \bar{\nabla}_Z Z) + g(\xi, \bar{\nabla}_Z PZ).$$

Using the nearly cosymplectic character the first term of the right hand side is zero; the second and last terms are also zero by using (2.1), property of Riemannian connection and either orthogonality of vectors Z and PZ or the fact that $\xi \ln f = 0$ or both. Hence

$$g(h(\xi, Z), FZ) = 0. \tag{3.17}$$

If we interchange Z by PZ , then $g(h(\xi, PZ), FPZ) = 0$. Put all these values in (3.16), we obtain

$$\sum_{j=1}^{2q} g(h(\xi, Z_j), FZ_j)^2 = 0. \tag{3.18}$$

Thus, from (3.15) and (3.18), we derive

$$\|h\|^2 \geq 2 \sum_{i=1}^{2p} \sum_{j=1}^{2q} g(h(X_i, Z_j), Z_j^*)^2.$$

Using the frame of $F\mathcal{D}_\theta$, the above inequality can be written as

$$\begin{aligned}
 \|h\|^2 &\geq 2 \csc^2 \theta \sum_{i=1}^p \sum_{j=1}^q g(h(X_i, Z_j), FZ_j)^2 \\
 &\quad + 2 \csc^2 \theta \sec^4 \theta \sum_{i=1}^p \sum_{j=1}^q g(h(X_i, PZ_j), FPZ_j)^2 \\
 &\quad + 2 \csc^2 \theta \sum_{i=1}^p \sum_{j=1}^q g(h(\phi X_i, Z_j), FZ_j)^2 \\
 &\quad + 2 \csc^2 \theta \sec^4 \theta \sum_{i=1}^p \sum_{j=1}^q g(h(\phi X_i, PZ_j), FPZ_j)^2 \\
 &\quad + 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^p \sum_{j=1}^q g(h(X_i, PZ_j), FZ_j)^2 \\
 &\quad + 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^p \sum_{j=1}^q g(h(X_i, Z_j), FPZ_j)^2 \\
 &\quad + 2 \csc^2 \theta \sec^2 \theta \sum_{i=1}^p \sum_{j=1}^q g(h(\phi X_i, PZ_j), FZ_j)^2 \\
 &\quad + 2 \csc^2 \theta \sec^4 \theta \sum_{i=1}^p \sum_{j=1}^q g(h(\phi X_i, Z_j), FPZ_j)^2.
 \end{aligned} \tag{3.19}$$

The first four terms of above inequality will be solved as follows. From Lemma 3.2 (iii), we have

$$g(h(\phi X, Z), FZ) = (X \ln f)g(Z, Z). \tag{3.20}$$

Interchanging X by ϕX and then using (2.1), we derive

$$-g(h(X, Z), FZ) + \eta(X)g(h(\xi, Z), FZ) = (\phi X \ln f)g(Z, Z).$$

But from (3.17), the second term of the left hand side of the above equation vanishes identically, thus we obtain

$$g(h(X, Z), FZ) = -(\phi X \ln f)g(Z, Z). \tag{3.21}$$

Interchanging Z by PZ in (3.20) and (3.21) and using (2.21) and the fact that ξ is tangent to N_T , we obtain the following equations, respectively

$$g(h(\phi X, PZ), FPZ) = (X \ln f) \cos^2 \theta g(Z, Z) \tag{3.22}$$

and

$$g(h(X, PZ), FPZ) = -(\phi X \ln f) \cos^2 \theta g(Z, Z). \tag{3.23}$$

The last four terms of (3.19) will be solved by Theorem 3.4 as follows:

$$g(h(X, Z), FPZ) = -g(h(X, PZ), FZ) = \frac{1}{3}(X \ln f) \cos^2 \theta g(Z, Z). \tag{3.24}$$

Interchanging X by ϕX in (3.24), we obtain

$$g(h(\phi X, Z), FPZ) = -g(h(\phi X, PZ), FZ) = \frac{1}{3}(\phi X \ln f) \cos^2 \theta g(Z, Z). \quad (3.25)$$

Put all these values in (3.19), we derive

$$\begin{aligned} \|h\|^2 &\geq 4 \csc^2 \theta \sum_{i=1}^p \sum_{j=1}^q \left\{1 + \frac{1}{9} \cos^2 \theta\right\} (X_i \ln f)^2 g(Z_j, Z_j)^2 \\ &\quad + 4 \csc^2 \theta \sum_{i=1}^p \sum_{j=1}^q \left\{1 + \frac{1}{9} \cos^2 \theta\right\} (\phi X_i \ln f)^2 g(Z_j, Z_j)^2. \end{aligned}$$

Hence, from (3.3) the above expression will be

$$\begin{aligned} \|h\|^2 &\geq 4 \csc^2 \theta \sum_{j=1}^q \left\{1 + \frac{1}{9} \cos^2 \theta\right\} \|\nabla \ln f\|^2 g(Z_j, Z_j)^2 \\ &= 4q \csc^2 \theta \left\{1 + \frac{1}{9} \cos^2 \theta\right\} \|\nabla \ln f\|^2, \end{aligned}$$

which is the inequality (3.10). If the equality holds in (3.10), then by (3.12) and (3.14), we obtain

$$h(\mathcal{D}, \mathcal{D}) = 0, \quad h(\mathcal{D}_\theta, \mathcal{D}_\theta) = 0, \quad h(\mathcal{D}, \mathcal{D}_\theta) \subset F\mathcal{D}_\theta. \quad (3.26)$$

Now, for any $Z, W \in \Gamma(\mathcal{D}_\theta)$ and $X \in \Gamma(\mathcal{D})$, we have

$$g(h^*(Z, W), X) = g(\nabla_Z W, X) = -g(W, \nabla_Z X) = -(X \ln f)g(Z, W),$$

where h^* is the second fundamental form of N_θ in M . Using (3.2), we obtain

$$g(h^*(Z, W), X) = -g(Z, W)g(\nabla \ln f, X),$$

where $\nabla \ln f$ is the gradient of $\ln f$. Thus from the last relation, we get

$$h^*(Z, W) = -\nabla \ln f g(Z, W). \quad (3.27)$$

Since N_T is a totally geodesic submanifold in M (by Lemma 3.1 (i)), using this fact with the first condition of (3.26), we get N_T is totally geodesic in \bar{M} . Also, the second condition of (3.26) with (3.27) implies that N_θ is totally umbilical in \bar{M} . Moreover all conditions of (3.26) imply that M is a minimal submanifold of \bar{M} . This completes the proof of the theorem. \square

If we consider $\theta = \frac{\pi}{2}$, then the inequality (3.10) generalizes the inequality which we have obtained for contact CR-warped products in [14].

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