REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 55, No. 1, 2014, Pages 71–81
Published online: June 2, 2014

LAPLACE TRANSFORM USING THE HENSTOCK-KURZWEIL INTEGRAL

SALVADOR SÁNCHEZ-PERALES AND JESÚS F. TENORIO

ABSTRACT. We consider the Laplace transform as a Henstock-Kurzweil integral. We give conditions for the existence, continuity and differentiability of the Laplace transform. A Riemann-Lebesgue Lemma is given, and it is proved that the Laplace transform of a convolution is the pointwise product of Laplace transforms.

1. Introduction

Given a function $f : [0, \infty) \to \mathbb{R}$, its Laplace transform at x is defined by $\mathscr{L}\{f\}(x) = \int_0^\infty f(t)e^{-tx}dt$. In this paper we consider this transform using the Henstock-Kurzweil integral. This integral generalizes the Riemann and Lebesgue integrals, as well as the Riemann and Lebesgue improper integrals. We prove some properties of the Laplace transform (existence, continuity and differentiability). A Riemann-Lebesgue Lemma is given, and various conditions are imposed on the functions so that the Laplace transform of a convolution is the pointwise product of Laplace transforms.

Throughout this paper we use the variable s to indicate that $\mathscr{L}\{f\}(s)$ is defined on a real number s, and we use z to indicate that $\mathscr{L}\{f\}(z)$ is defined on a complex number z.

2. Preliminaries

Let us begin by recalling the definition of the Henstock-Kurzweil integral. For finite intervals in $\mathbb{R}$ it is defined in the following way:

Definition 2.1. Let $f : [a, b] \to \mathbb{R}$ be a function. We say that f is Henstock-Kurzweil (shortly, HK-) integrable, if there exists $A \in \mathbb{R}$ such that, for each $\epsilon > 0$, there is a function $\gamma_\epsilon : [a, b] \to (0, \infty)$ (named a gauge) with the property that for any γ_ϵ-fine partition $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ of $[a, b]$ (i.e. $\{[x_{i-1}, x_i] : i = 1, \ldots, n\}$ is a non-overlapping partition of $[a, b]$ and for each i, $[x_{i-1}, x_i] \subset [t_i - \gamma_\epsilon(t_i), t_i + \gamma_\epsilon(t_i)\,]$), one has

$$|\Sigma_{i=1}^n f(t_i)(x_i - x_{i-1}) - A| < \epsilon. \tag{2.1}$$

2010 *Mathematics Subject Classification.* 44A10, 26A39.

Key words and phrases. Laplace transform, Henstock-Kurzweil integral, bounded variation functions.

The number A is the integral of f over $[a, b]$ and it is denoted as $A = \int_a^b f$.

In the unbounded case, we require the following definition.

Definition 2.2. Given a gauge function $\gamma : [a, \infty] \to (0, \infty)$ we say that a tagged partition $P = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n+1}$ of $[a, \infty]$ is γ-fine, if

(a) $a = x_0$, $x_{n+1} = t_{n+1} = \infty$,
(b) $[x_{i-1}, x_i] \subset [t_i - \gamma(t_i), t_i + \gamma(t_i)]$ for all $i = 1, 2, \ldots, n$,
(c) $[x_n, \infty] \subseteq [1/\gamma(t_{n+1}), \infty]$.

Definition 2.3. A function $f : [a, \infty] \to \mathbb{R}$ is Henstock-Kurzweil integrable on $[a, \infty]$, if there exists $A \in \mathbb{R}$ such that, for each $\epsilon > 0$, there is a gauge $\gamma_\epsilon : [a, \infty] \to (0, \infty)$ for which (2.1) is satisfied for every tagged partition P which is γ_ϵ-fine according to Definition 2.2.

Let f be a real function defined on an infinite interval $[a, \infty)$; we can suppose that f is defined on $[a, \infty]$ assuming that $f(\infty) = 0$. Thus f is Henstock-Kurzweil integrable on $[a, \infty)$ if f extended to $[a, \infty]$ is HK-integrable. For functions defined over intervals $(-\infty, a]$ and $(-\infty, \infty)$ we make similar considerations.

Let I be a finite or infinite interval. The space of all Henstock-Kurzweil integrable functions on I is denoted by $\mathcal{HK}(I)$. This space will be considered with the Alexiewicz semi-norm, which is defined as

$$\|f\|_I = \sup_{J \subseteq I} \left| \int_J f \right|, \quad f \in \mathcal{HK}(I),$$

where the supremum is being taken over all intervals J contained in I.

Definition 2.4. Let $\varphi : I \to \mathbb{C}$ be a function, where $I \subseteq \mathbb{R}$ is a finite interval. The variation of φ on the interval I is defined as

$$V_I \varphi = \sup \left\{ \sum_{i=1}^{n} |\varphi(x_i) - \varphi(x_{i-1})| \;\middle|\; P \text{ is partition of } I \right\}.$$

We say that the function φ is of bounded variation on I if $V_I \varphi < \infty$. Now, if φ is a function defined on an infinite interval I, then φ is of bounded variation on I, if φ is of bounded variation on each finite subinterval of I and there exists some $M > 0$ such that $V_{[a,b]} \varphi \leq M$ for all $[a, b] \subseteq I$. The variation of φ on I is $V_I \varphi = \sup\{V_{[a,b]} \varphi \mid [a, b] \subseteq I\}$. The space of all bounded variation functions on I is denoted by $\mathcal{BV}(I)$.

The following theorems are classical and will be used throughout this paper.

Theorem 2.5. [11, Lemma 24] *If g is a HK-integrable function on $[a, b] \subseteq \mathbb{R}$ and f is a function of bounded variation on $[a, b]$, then fg is HK-integrable on $[a, b]$ and*

$$\left| \int_a^b fg \right| \leq \inf_{t \in [a,b]} |f(t)| \left| \int_a^b g(t) dt \right| + \|g\|_{[a,b]} V_{[a,b]} f.$$

Theorem 2.6 (Chartier-Dirichlet's Test, [2])**.** *Let f and g be functions defined on $[a, \infty)$. Suppose that* *(i)* $g \in \mathcal{HK}([a,c])$ *for every* $c \geq a$, *and* G *defined by* $G(x) = \int_a^x g$ *is bounded on* $[a,\infty)$;

(ii) f *is of bounded variation on* $[a,\infty)$ *and* $\lim_{x\to\infty} f(x) = 0$.

Then $fg \in \mathcal{HK}([a,\infty))$.

Theorem 2.7 (Du Bois-Reymond's Test, [2]). *Let* f *and* φ *be functions defined on* $[a,\infty)$ *and suppose that:*

(1) $f \in \mathcal{HK}([a,c])$ *for all* $c \geq a$ *and* $F(x) = \int_a^x f$ *is bounded on* $[a,\infty)$;

(2) φ *is differentiable on* $[a,\infty)$ *and* φ' *is Lebesgue integrable on* $[a,\infty)$;

(3) $\lim_{x\to\infty} F(x)\varphi(x)$ *exists.*

Then $f\varphi \in \mathcal{HK}([a,\infty))$.

Definition 2.8 ([4]). Let $E \subseteq [a,b]$. A function $f : [a,b] \to \mathbb{R}$ is AC_δ on E, if for every $\epsilon > 0$, there exist $\eta_\epsilon > 0$ and a gauge δ_ϵ on E such that

$$\sum_{i=1}^{s} |f(v_i) - f(u_i)| < \epsilon,$$

whenever $P = \{([u_i, v_i], t_i)\}_{i=1}^s$ is a (δ_ϵ, E)-fine subpartition of $[a,b]$ (i.e., P is δ_ϵ-fine and the tags t_i belong to E) and $\sum_{i=1}^{s} |v_i - u_i| < \eta_\epsilon$.

We say that f is ACG_δ on $[a,b]$, if $[a,b]$ can be written as a countable union of sets on each of which the function f is AC_δ.

If h is a function in the variables (t,s), then we use the notation D_2h for the partial derivative of h with respect to the variable s.

Theorem 2.9. [10, Theorem 4] *Let* $a, b \in \mathbb{R}$. *If* $h : \mathbb{R} \times [a,b] \to \mathbb{C}$ *is such that*

(i) $h(t,\cdot)$ *is* ACG_δ *on* $[a,b]$ *for almost all* $t \in \mathbb{R}$,

(ii) $h(\cdot,s)$ *is HK-integrable on* $\mathbb{R}$ *for all* $s \in [a,b]$,

then $H := \int_{-\infty}^{\infty} h(t,\cdot)dt$ *is* ACG_δ *on* $[a,b]$ *and* $H'(s) = \int_{-\infty}^{\infty} D_2h(t,s)dt$ *for almost all* $s \in (a,b)$, *if and only if,*

$$\int_s^t \int_{-\infty}^{\infty} D_2h(t,s)dtds = \int_{-\infty}^{\infty} \int_s^t D_2h(t,s)dsdt$$

for all $[s,t] \subseteq [a,b]$. *In particular,*

$$H'(s_0) = \int_{-\infty}^{\infty} D_2h(t,s_0)dt,$$

when $H_2 := \int_{-\infty}^{\infty} D_2h(t,\cdot)dt$ *is continuous at* s_0.

3. Main results

Let I be a finite or infinite interval. We use the following notation:

- $L(I) = \{f \mid f$ is Lebesgue integrable on $I\}$,
- $L_{loc} = \{f \mid f$ is Lebesgue integrable on each $[a,b] \subseteq [0,\infty)\}$,
- $\mathcal{HK}(I) = \{f \mid f$ is HK-integrable on $I\}$,

- $\mathcal{HK}_{loc} = \{f \mid f \text{ is HK-integrable on each } [a,b] \subseteq [0,\infty)\}$,
- $\mathcal{B}(I) = \{f \mid f \text{ is bounded on } I\}$,
- $\mathcal{BV}(I) = \{f \mid f \text{ is of bounded variation on } I\}$,
- $\mathcal{BV}_0([0,\infty)) = \{f \in \mathcal{BV}([0,\infty)) \mid \lim_{t\to\infty} f(t) = 0\}$,
- $\mathcal{BV}_0[\infty] = \{f \mid f \in \mathcal{BV}([b,\infty)) \text{ for some } b > 0 \text{ and } \lim_{b\le t\to\infty} f(t) = 0\}$.

Talvila in [11] introduced the space $\mathcal{HK}_{loc} \cap \mathcal{BV}_0[\infty]$ to study the Fourier transform. Also Mendoza, Escamilla and Sánchez (see [5, 6, 7]) have studied this transform on the spaces $\mathcal{HK}([0,\infty)) \cap \mathcal{BV}([0,\infty))$ and $\mathcal{BV}_0([0,\infty))$. All these spaces satisfied the following inclusion relations:

(1) $\mathcal{HK}([0,\infty)) \cap \mathcal{BV}([0,\infty)) \subseteq \mathcal{BV}_0([0,\infty)) \subseteq \mathcal{HK}_{loc} \cap \mathcal{BV}_0[\infty]$;
(2) $\mathcal{HK}([0,\infty)) \cap \mathcal{BV}([0,\infty)) \nsubseteq L([0,\infty))$, (see [5, Example 2.1 (i)]);
(3) $\mathcal{HK}_{loc} \cap \mathcal{BV}_0[\infty] \nsubseteq \mathcal{HK}([0,\infty))$ and $\mathcal{HK}([0,\infty)) \nsubseteq \mathcal{HK}_{loc} \cap \mathcal{BV}_0[\infty]$ (see [5, Example 2.1 (ii)]).

In this paper, we analyse the Laplace transform on the spaces above.

It is well known that if $f \in L_{loc}$ and there exist constants $c \in \mathbb{R}$, $M > 0$ and $T > 0$ such that $|f(t)| \le Me^{ct}$ for all $t > T$, then $\mathscr{L}\{f\}(s)$ exists for all $s > c$. We give other existential conditions.

Observe that if $s \in \mathbb{R}^+$, then e^{-st}, as a function of a real variable t, is monotone and $\lim_{t\to\infty} e^{-st} = 0$. Therefore, from Chartier-Dirichlet's Test (Theorem 2.6), it follows that if $f \in \mathcal{HK}_{loc}$ and $F(x) = \int_0^x f(t)dt$, $0 \le x < \infty$, is bounded on $[0,\infty)$, then $\mathscr{L}\{f\}(s)$ exists for all $s > 0$. So the next theorem follows.

Theorem 3.1. *If $f \in \mathcal{HK}([0,\infty))$, then $\mathscr{L}\{f\}(s)$ exists for all $s \in [0,\infty)$.*

Proof. Its clear that if $f \in \mathcal{HK}([0,\infty))$, then $\mathscr{L}\{f\}(0)$ exists. Moreover, since $\left|\int_0^x f(t)dt\right| \le \|f\|_{[0,\infty)}$ for all $x \ge 0$, it follows that $\mathscr{L}\{f\}(s)$ exists for all $s \in [0,\infty)$. □

Example 3.2. For any $a > 0$, $\frac{\sin t}{t} \in \mathcal{HK}([a,\infty)) \setminus L([a,\infty))$. Thus the function $f_a : [0,\infty) \to \mathbb{R}$ defined by

$$f_a(t) = \begin{cases} \frac{\sin t}{t}, & \text{if } t \ge a, \\ 0, & \text{if } 0 \le t < a \end{cases}$$

belongs to $\mathcal{HK}([0,\infty))$. Therefore $\mathscr{L}\{f_a\}(s)$ exists for all $s \in [0,\infty)$.

The function $t \mapsto e^{-zt}$, now when $z \in \mathbb{C}$, is not of bounded variation on $[0,\infty)$, so in order to prove the existence of the Laplace transform we can't use Chartier-Dirichlet's Test. In [3] the next result is proved using Du Bois-Reymond's Test. For the sake of completeness, here we will give its proof.

Theorem 3.3. *If $f \in \mathcal{HK}_{loc}$ and $F(x) = \int_0^x f(t)dt$, $0 \le x < \infty$, is bounded on $[0,\infty)$, then $\mathscr{L}\{f\}(z)$ exists for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$.*

Proof. The condition $\operatorname{Re} z > 0$ implies that $\lim_{t\to\infty} e^{-zt} = 0$ and so $\frac{d}{dt}(e^{-zt})$ is Lebesgue integrable on $[0,\infty)$. Also $\lim_{x\to\infty} F(x)e^{-zx} = 0$. Therefore by Theorem 2.7, $\mathscr{L}\{f\}(z) = \int_0^\infty f(t)e^{-zt}dt$ exists. □

Corollary 3.4. *If $f \in \mathcal{HK}([0,\infty))$, then $\mathscr{L}\{f\}(z)$ exists for all $z \in \mathbb{C}$ such that* $\operatorname{Re} z > 0$.

Another condition for the existence of $\mathscr{L}\{f\}(z)$, that does not require $F(x) = \int_0^x f(t)dt$ to be bounded on $[0,\infty)$, is shown in the following theorem.

Theorem 3.5. *If $f \in \mathcal{HK}_{loc} \cap \mathcal{BV}_0[\infty]$ then $\mathscr{L}\{f\}(z)$ exists for all $z \in \mathbb{C}$ with* $\operatorname{Re} z > 0$.

Proof. There exists $b > 0$ such that $f \in \mathcal{BV}([b,\infty))$ and $\lim_{t\to\infty} f(t) = 0$. Since $\operatorname{Re} z > 0$, it follows that $\left|\int_b^x e^{-tz}dt\right| < \frac{2}{|z|}$ for all $x \in [b,\infty)$. This implies, from Theorem 2.6, that $\int_b^\infty f(t)e^{-tz}dt$ exists. Now since $f \in \mathcal{HK}_{loc}$ and e^{-tz} is of bounded variation on $[0,b]$, the integral $\int_0^b f(t)e^{-tz}dt$ also exists. □

Example 3.6. Let $f : [1,\infty) \to \mathbb{R}$ be a function defined by $f(t) = t^{-p}$, where $0 < p < 1$. Observe that $\lim_{x\to\infty} \int_0^x t^{-p}dt = \infty$ but $f \in \mathcal{HK}_{loc} \cap \mathcal{BV}_0[\infty]$. Thus, Corollary 3.4 does not apply; however from Theorem 3.5, $\mathscr{L}\{f\}(z)$ exists for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$.

Remark 3.7. If $f \in \mathcal{HK}([0,\infty)) \cap \mathcal{BV}([0,\infty))$, then $\mathscr{L}\{f\}(z)$ exists for all $z \in \mathbb{C}$ with $\operatorname{Re} z \geq 0$. It is clear that this condition is satisfied when $\operatorname{Re} z > 0$, because $\mathcal{HK}([0,\infty)) \cap \mathcal{BV}([0,\infty)) \subseteq \mathcal{HK}_{loc} \cap \mathcal{BV}_0[\infty]$. Now, if $\operatorname{Re} z = 0$, then $\mathscr{L}\{f\}(z) = \widehat{f\chi_{[0,\infty)}}(\operatorname{Im} z)$, where the latter expression is the Fourier transform of $f\chi_{[0,\infty)}$ evaluated at $\operatorname{Im} z$. From [6, Theorem 3.1], $\widehat{f\chi_{[0,\infty)}}(\operatorname{Im} z)$ exists, since $f \in \mathcal{HK}([0,\infty)) \cap \mathcal{BV}([0,\infty))$. Thus $\mathscr{L}\{f\}(z)$ also exists when $z \in \mathbb{C}$ with $\operatorname{Re} z = 0$.

Now, we prove that the Laplace transform is continuous on $[0,\infty)$ when $f \in \mathcal{HK}([0,\infty))$, and is continuous on $(0,\infty)$ when $f \in \mathcal{HK}_{loc} \cap \mathcal{BV}_0[\infty]$.

Proposition 3.8. *Let $f \in \mathcal{HK}([0,b])$, $b > 0$. If F_b is defined by $F_b(s) = \int_0^b f(t)e^{-ts}$, then F_b is continuous on $[0,\infty)$.*

Proof. Take $s_0 \in [0,\infty)$, and note that for $s \in [0,\infty)$ with $|s - s_0| < \delta$,

$$\begin{aligned} |F_b(s) - F_b(s_0)| &= \left| \int_0^b f(t)e^{-ts_0}[e^{-t(s-s_0)} - 1]dt \right| \\ &\leq \|f(\cdot)e^{-(\cdot)s_0}\|_{[0,b]} \left[\inf_{t\in[0,b]} |e^{-t(s-s_0)} - 1| + V_{[0,b]}[e^{-t(s-s_0)} - 1] \right] \\ &\leq \|f(\cdot)e^{-(\cdot)s_0}\|_{[0,b]} \left[V_{[0,b]}[e^{-t(s-s_0)} - 1] \right]. \end{aligned}$$

Since $|\frac{d}{dt}(e^{-t(s-s_0)} - 1)| = |s - s_0|e^{-t(s-s_0)} \leq |s - s_0|e^{\delta b}$, it follows that $V_{[0,b]}[e^{-t(s-s_0)} - 1] \leq 2|s - s_0|e^{\delta b}b$. Thus

$$|F_b(s) - F_b(s_0)| \leq 2b|s - s_0|e^{\delta b}\|f(\cdot)e^{-(\cdot)s_0}\|_{[0,b]},$$

and hence $\lim_{s\to s_0} F_b(s) = F_b(s_0)$. □

Proposition 3.9. *If $f \in \mathcal{HK}([a,b])$, then*

$$\left|\int_u^v f(t)e^{-ts}dt\right| \leq e^{-as}\|f\|_{[a,b]}, \quad \textit{for all } 0 \leq a \leq u \leq v \leq b \textit{ and } s \geq 0.$$

Proof. Take $a \leq u \leq v \leq b$ and $s \geq 0$. Since e^{-ts}, as a function of the variable t, is decreasing, it follows that $V_{[u,v]}e^{-ts} \leq e^{-us} - e^{-vs}$; thus by Theorem 2.5,

$$\begin{aligned}\left|\int_u^v f(t)e^{-ts}dt\right| &\leq \|f\|_{[u,v]}\left[\inf_{t\in[u,v]}|e^{-ts}| + V_{[u,v]}e^{-ts}\right]\\ &\leq \|f\|_{[u,v]}\left[e^{-vs} + (e^{-us} - e^{-vs})\right]\\ &\leq \|f\|_{[u,v]}e^{-us} \ \leq \ e^{-as}\|f\|_{[a,b]}.\end{aligned}$$

□

Theorem 3.10. *If either $f \in \mathcal{HK}([0,\infty))$ or $f \in \mathcal{HK}_{loc} \cap \mathcal{BV}_0[\infty]$, then $\mathscr{L}\{f\}$ is continuous on $(0,\infty)$.*

Proof. Take $s_0 \in (0,\infty)$ and let $\epsilon > 0$ be given. Consider $0 < \delta_1 < s_0/2$. First observe that for each $K > 0$ and $s \in (s_0 - \delta_1, s_0 + \delta_1)$,

$$\begin{aligned}|\mathscr{L}\{f\}(s) - \mathscr{L}\{f\}(s_0)| &\leq |F_K(s) - F_K(s_0)|\\ &\quad + \left|\int_K^\infty f(t)(e^{-ts} - e^{-ts_0})dt\right|.\end{aligned} \tag{3.1}$$

We claim that there exists $K_0 > 0$ such that $\left|\int_{K_0}^\infty f(t)(e^{-ts} - e^{-ts_0})dt\right| < \frac{\epsilon}{2}$ independently of $s \in (s_0 - \delta_1, s_0 + \delta_1)$.

Assumption 1: $f \in \mathcal{HK}([0,\infty))$. By Hake's Theorem, there exists $K_1 > 0$ such that

$$\|f\|_{[K_1,\infty)} < \frac{\epsilon}{2}. \tag{3.2}$$

Note that, for all $v \geq K_1$ and $s_0 - \delta_1 < s \leq s_0$,

$$\begin{aligned}\left|\int_{K_1}^v f(t)(e^{-ts} - e^{-ts_0})dt\right| &= \left|\int_{K_1}^v f(t)e^{-ts}[e^{-t(s_0-s)} - 1]dt\right|\\ &\leq \|f(\cdot)e^{-(\cdot)s}\|_{[K_1,v]}\left[\inf_{t\in[K_1,v]}|e^{-t(s_0-s)} - 1|\right.\\ &\quad \left. + V_{[K_1,v]}[e^{-t(s_0-s)} - 1]\right].\end{aligned} \tag{3.3}$$

From Proposition 3.9, $\|f(\cdot)e^{-(\cdot)s}\|_{[K_1,v]} \leq \|f\|_{[K_1,v]}$. Also, since $s \leq s_0$, it follows that the function $e^{-t(s_0-s)} - 1$ is decreasing and not positive. Thus, the right-hand side of the inequality (3.3) is bounded by

$$\|f\|_{[K_1,v]}\left[1 - e^{-K_1(s_0-s)} + (e^{-K_1(s_0-s)} - e^{-v(s_0-s)})\right]$$

and this equals $\|f\|_{[K_1,v]}\left[1 - e^{-v(s_0-s)}\right]$. So, taking the limit (as $v \to \infty$) produces

$$\left|\int_{K_1}^\infty f(t)(e^{-ts} - e^{-ts_0})dt\right| \leq \|f\|_{[K_1,\infty)},$$

for all $s_0-\delta_1 < s \le s_0$. Of course this inequality is also true when $s_0 \le s < s_0+\delta_1$.

Assumption 2: $f \in \mathcal{HK}_{loc} \cap \mathcal{BV}_0[\infty]$. First observe that for all $u, v \ge 0$ and $s \in (s_0-\delta_1, s_0+\delta_1)$,

$$\left|\int_u^v (e^{-ts}-e^{-ts_0})dt\right| \le \frac{6}{s_0}.$$

We set M to the right side of this inequality. Since $\lim_{t\to\infty} V_{[t,\infty)}f = 0$, it follows that there exists $K_2>0$ such that $V_{[K_2,\infty)}f < \frac{\epsilon}{2M}$. Then, from Theorem 2.5, it follows that for every $v \ge K_2$ and $s \in (s_0-\delta_1, s_0+\delta_1)$,

$$\begin{aligned}\left|\int_{K_2}^v f(t)(e^{-ts}-e^{-ts_0})dt\right| &\le M\left[\inf_{t\in[K_2,v]}|f(t)| + V_{[K_2,v]}f\right]\\ &\le M\left[|f(v)| + V_{[K_2,\infty)}f\right].\end{aligned}$$

This implies, since $\lim_{t\to\infty}|f(t)|=0$, that

$$\left|\int_{K_2}^\infty f(t)(e^{-ts}-e^{-ts_0})dt\right| \le M\cdot V_{[K_2,\infty)}f \;<\; M\frac{\epsilon}{2M} \;=\; \frac{\epsilon}{2}.$$

Therefore, with any of the two hypotheses, we see that our initial assertion is true. Finally, from Proposition 3.8, F_{K_0} is continuous, so there exists $\delta_2>0$ such that for every $s\in(0,\infty)$ with $|s-s_0|<\delta_2$, $|F_{K_0}(s)-F_{K_0}(s_0)|<\frac{\epsilon}{2}$. Let $\delta=\min\{\delta_1,\delta_2\}$; then by (3.1) we have that for all $s\in(s_0-\delta, s_0+\delta)$,

$$|\mathscr{L}\{f\}(s)-\mathscr{L}\{f\}(s_0)| < \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon. \qquad \square$$

It can be shown that if $f\in\mathcal{HK}_{loc}$ and $F(x)=\int_0^x f(t)dt$, $0\le x<\infty$, is bounded on $[0,\infty)$, then

$$\lim_{s\to s_0^+}\mathscr{L}\{f\}(s)=\mathscr{L}\{f\}(s_0)$$

for all $s_0\in[0,\infty)$.

We have already seen the continuity of the Laplace transform of a function, now we give another feature of this transform. This property indicates that not every arbitrary continuous function is a Laplace transform of some function.

Given $0\le a\le b$ we set

$$\Gamma([a,b])=\left\{f\in\mathcal{HK}([a,b]) \mid \lim_{s\to\infty}\int_a^b f(t)e^{-ts}dt=0\right\}.$$

It is clear that $L([a,b])\subseteq\Gamma([a,b])$; however $\Gamma([a,b])\nsubseteq L([a,b])$, see Example 3.12 below. Moreover, if $0<a\le b$, then, from Proposition 3.9, it follows that $\mathcal{HK}([a,b])=\Gamma([a,b])$. When $a=0$ the veracity of the equality $\mathcal{HK}([a,b])=\Gamma([a,b])$ is still an open question. A class of functions belonging to $\Gamma([0,b])$ is given in the next theorem:

Theorem 3.11. *Let f be a continuous function on $[0,b]$ such that f is differentiable except for an at most countable set A. If $g(t)=f(t)+tf'(t)$ for all $t\in[0,b]\setminus A$,* *then*

$$\lim_{s\to\infty}\int_0^b g(t)e^{-ts}dt = 0.$$

Proof. Let $G(t) = tf(t)$, then $G'(t) = g(t)$ for all $t \in [0,b]\setminus A$. From [2, Theorem 4.7], $g \in \mathcal{HK}([0,b])$ and by [2, Theorem 10.12], $g(t)e^{-ts} \in \mathcal{HK}([0,b])$. Integrating by parts we obtain

$$\int_0^b g(t)e^{-ts}dt = e^{-ts}G(t)\Big|_0^b + \int_0^b G(t)se^{-ts}dt$$
$$= e^{-bs}bf(b) + \int_0^b f(t)tse^{-ts}dt.$$

Let $h : [0,\infty) \to \mathbb{R}$ defined by $h(x) = xe^{-x}$, then h is a measurable function satisfying

$$\lim_{r\to\infty}\frac{1}{r}\int_0^r h(x)dx = \lim_{r\to\infty}\frac{1}{r}\left[1 - \frac{1}{e^r} - \frac{r}{e^r}\right] = 0;$$

so, since $f \in L[0,b]$, by the generalized Riemman-Lebesgue Lemma ([1, Theorem 4.4.1]),

$$\lim_{s\to\infty}\int_0^b h(ts)f(t)dt = 0.$$

Thus $e^{-bs}bf(b) + \int_0^b f(t)tse^{-ts}dt \to 0$ when $s \to \infty$. □

Example 3.12. Define $g : [0,1] \to \mathbb{R}$ by

$$g(t) = \begin{cases} \cos\left(\frac{\pi}{t}\right) + \frac{\pi}{t}\sin\left(\frac{\pi}{t}\right), & \text{if } t \in (0,1],\\ 0, & \text{if } t = 0.\end{cases}$$

Then $g \in \mathcal{HK}([0,1])$, but $g \notin L([0,1])$. Note that $g(t) = \cos\left(\frac{\pi}{t}\right) + t\frac{d}{dt}\left[\cos\left(\frac{\pi}{t}\right)\right]$. Thus by Theorem 3.11,

$$\lim_{s\to\infty}\int_0^1 g(t)e^{-ts}dt = 0,$$

and hence $g \in \Gamma([0,1])$.

Theorem 3.13 (Riemann-Lebesgue Lemma)**.** *Let $f : [0,\infty) \to \mathbb{R}$ be a function such that $f \in \Gamma([0,b])$ for some $b > 0$. If either*

(1) $f \in \mathcal{BV}_0([b,\infty))$ or
(2) $f \in \mathcal{HK}([b,\infty))$,

then $\lim_{s\to\infty}\mathscr{L}\{f\}(s) = 0$.

Proof. Let $s > 0$. If $f \in \mathcal{BV}_0([b,\infty))$, then $f(\cdot)e^{-(\cdot)s} \in \mathcal{HK}([b,\infty))$ and

$$\left|\int_b^\infty f(t)e^{-ts}dt\right| \le \frac{2}{s}[f(b) + V_{[b,\infty)}f].$$

Therefore the conclusion of the theorem follows since $f \in \Gamma([a,b])$. On the other hand, if $f \in \mathcal{HK}([b,\infty))$, then by Proposition 3.9,

$$\left| \int_b^\infty f(t)e^{-ts}dt \right| \leq \|f\|_{[b,\infty)} e^{-bs},$$

and again the conclusion of the theorem is satisfied. □

The following result is shown in [8, Theorem 3.3].

Lemma 3.14. *Let $a, b \in \mathbb{R}$. If $g : [0,\infty) \to \mathbb{R}$ and $h : [0,\infty) \times [a,b] \to \mathbb{C}$ are functions such that*

(i) $g \in \mathcal{BV}_0([0,\infty))$, h is measurable, bounded and
(ii) there exists $M > 0$ such that, for each $y \in [a,b]$,

$$\left| \int_0^v h(x,y)dx \right| \leq M,$$

for all $v \geq 0$,

then

$$\int_a^b \int_0^\infty g(x)h(x,y)\, dxdy = \int_0^\infty \int_a^b g(x)h(x,y)\, dydx.$$

Theorem 3.15. *If $f \in \mathcal{BV}_0([0,\infty))$ and $s_0 \in (0,\infty)$, then $\mathscr{L}\{f\}$ is differentiable at s_0, and*

$$(\mathscr{L}\{f\})'(s_0) = -\int_0^\infty tf(t)e^{-ts_0}dt. \tag{3.4}$$

Proof. There exist $a, b, M > 0$ with $a < s_0 < b$ such that, for each $s \in [a,b]$,

$$\left| \int_0^v e^{-ts}dt \right| < M \tag{3.5}$$

and

$$\left| \int_0^v te^{-ts}dt \right| < M \tag{3.6}$$

for all $v \geq 0$.

In order to show (3.4) we use Theorem 2.9. The function $f(t)e^{-t(\cdot)}$ is differentiable on $[a,b]$ for all $t \in [0,\infty)$, so $f(t)e^{-t(\cdot)}$ is ACG_δ on $[a,b]$ for all $t \in [0,\infty)$. By (3.5) and Theorem 2.6, $f(\cdot)e^{-(\cdot)s}$ is HK-integrable on $[0,\infty)$ for all $s \in [a,b]$. Then

$$(\mathscr{L}\{f\})'(s_0) = \int_0^\infty -tf(t)e^{-ts_0}dt,$$

if

$$\Gamma := \int_0^\infty -tf(t)e^{-t(\cdot)}dt$$

is continuous at s_0, and

$$\int_s^t \int_0^\infty -tf(t)e^{-ts}dtds = \int_0^\infty \int_s^t -tf(t)e^{-ts}dsdt$$

for all $[s,t] \subseteq [a,b]$. The first assertion follows using (3.6) and a similar argument as in the proof of Theorem 3.10 (Assumption 2). The second claim is true due to (3.6) and Lemma 3.14. $\square$

If f and g are functions defined on the interval $[0,\infty)$, then their convolution is the function $f * g$ defined by

$$f * g(y) = \int_0^y f(y-x)g(x)dx.$$

It is clear that if $f \in \mathcal{HK}([0,\infty))$ and $g \in \mathcal{BV}([0,\infty))$, then $f * g$ exists on $[0,\infty)$ by the Multiplier Theorem (see [2]), and $f * g(y) = g * f(y)$ for all $y \in [0,\infty)$. In [9], the equality $\mathscr{L}\{f * g\}(s) = \mathscr{L}\{f\}(s)\mathscr{L}\{g\}(s)$ is established, under certain conditions. However the authors use [11, Lemma 25(a)] and it has an omission unless f has compact support. We provide other conditions and a different proof.

Theorem 3.16. *If $f \in \mathcal{HK}([0,\infty)) \cap \mathcal{B}([0,\infty))$ and $g \in L([0,\infty)) \cap \mathcal{BV}([0,\infty))$, then*

$$\mathscr{L}\{f * g\}(z) = \mathscr{L}\{f\}(z)\mathscr{L}\{g\}(z)$$

for all $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$.

Proof. Take $z \in \mathbb{C}$ with $\operatorname{Re} z > 0$. From Corollary 3.4, it follows that $\mathscr{L}\{f\}(z)$ and $\mathscr{L}\{g\}(z)$ exist, and $\mathscr{L}\{f\}(z)\mathscr{L}\{g\}(z) = \int_0^\infty g(x) \int_x^\infty f(y-x)e^{-yz}dydx$.

Let $D = \{(x,y) \mid 0 \le x,\ x \le y\}$ and consider

$$h(x,y) = \begin{cases} f(y-x)e^{-yz}, & \text{if } (x,y) \in D, \\ 0, & \text{if } (x,y) \in ([0,\infty) \times [0,\infty)) \setminus D. \end{cases}$$

For each $a, x \ge 0$,

$$\left| g(x) \int_0^a h(x,y)dy \right| \le |g(x)| \| f(\cdot)e^{-(\cdot)z} \|.$$

Thus, since $g \in L([0,\infty))$, the Dominated Convergence Theorem and Hake's Theorem imply the second equality of the following:

$$\begin{aligned} \mathscr{L}\{f\}(z)\mathscr{L}\{g\}(z) &= \int_0^\infty \int_0^\infty g(x)h(x,y)\,dydx \\ &= \lim_{a\to\infty} \int_0^\infty \int_0^a g(x)h(x,y)\,dydx. \end{aligned} \tag{3.7}$$

On the other hand, since $\left|\int_0^v h(x,y)dx\right| \le \|f\|$ for all $v \ge 0$, it follows, from Lemma 3.14, that the right side of (3.7) is equal to

$$\lim_{a\to\infty} \int_0^a \int_0^\infty g(x)h(x,y)\,dxdy.$$

Therefore

$$\begin{aligned}\lim_{a\to\infty}\int_0^a g*f(y)e^{-yz}dy &= \lim_{a\to\infty}\int_0^a\int_0^y g(x)f(y-x)e^{-yz}dxdy\\ &= \lim_{a\to\infty}\int_0^a\int_0^\infty g(x)h(x,y)\,dxdy\\ &= \mathscr{L}\{f\}(z)\mathscr{L}\{g\}(z).\end{aligned}$$

Again using Hake's Theorem, we obtain that $g*f(\cdot)e^{-(\cdot)z}\in\mathcal{HK}([0,\infty))$ and $\mathscr{L}\{f*g\}(z)=\mathscr{L}\{f\}(z)\mathscr{L}\{g\}(z)$. □

References

[1] G. Bachman, L. Narici, E. Beckenstein, *Fourier and Wavelet Analysis*, Springer-Verlag, 1991. MR 1729490

[2] R. G. Bartle, *A Modern Theory of Integration*, Grad. Studies in Math., Vol. 32, Amer. Math. Soc., Providence, 2001. MR 1817647

[3] M.S. Chaudhary and Sanket A. Tikare, *On Gauge Laplace Transform*, Int. J. Math. Anal. 5, No. 35, 2011, 1733–1740. MR 2853594

[4] R. A. Gordon, *The Integrals of Lebesgue, Denjoy, Perron, and Henstock*, Grad. Studies in Math., Vol. 4, Amer. Math. Soc., Providence, 1994. MR 1288751

[5] F. J. Mendoza Torres, J. A. Escamilla Reyna and S. Sánchez Perales, *Inclusion Relations for the Spaces* $L(\mathbb{R})$, $\mathcal{HK}(\mathbb{R})\cap\mathcal{BV}(\mathbb{R})$ *and* $L^2(\mathbb{R})$, Russ. J. Math. Phys. 16, No. 2, 2009, 287–289. MR 2525413

[6] F. J. Mendoza Torres, J. A. Escamilla Reyna and S. Sánchez Perales, *Some results about the Henstock-Kurzweil Fourier transform*, Math. Bohem. 134 (2009), 379–386. MR 2597233

[7] F. J. Mendoza Torres, *On pointwise inversion of the Fourier transform of* BV_0 *functions*, Ann. Funct. Anal. 1, No. 2, 2010, 112–120. MR 2772044

[8] S. Sánchez-Perales, F. J. Mendoza Torres, J. A. Escamilla Reyna, *Henstock-Kurzweil Integral Transforms*, Int. J. Math. Math. Sci. vol. 2012, 11 pages. MR 2983789

[9] Sanket A. Tikare and M.S. Chaudhary, *On some results of HK-convolution*, International Electronic Journal of Pure and Applied Mathematics 2, No. 2, 2010, 93–102.

[10] E. Talvila, *Necessary and sufficient conditions for differentiating under the integral sign*, Amer. Math. Monthly, 108 (2001) 544–548. MR 1840661

[11] E. Talvila, *Henstock-Kurzweil Fourier transforms*, Illinois J. Math. 46 (2002), 1207–1226. MR 1988259

Salvador Sánchez-Perales
Instituto de Física y Matemáticas, Universidad Tecnológica de la Mixteca, Km. 2.5, Carretera a Acatlima, 69000 Huajuapan de León, Oaxaca, Mexico
es21254@yahoo.com.mx

Jesús F. Tenorio
Instituto de Física y Matemáticas, Universidad Tecnológica de la Mixteca, Km. 2.5, Carretera a Acatlima, 69000 Huajuapan de León, Oaxaca, Mexico
jtenorio@mixteco.utm.mx

Received: January 23, 2013
Accepted: August 19, 2013