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A CLASSIFICATION OF SOLVABLE QUADRATIC AND ODD QUADRATIC LIE SUPERALGEBRAS IN LOW DIMENSIONS

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ABSTRACT. We give an expansion of two notions of double extension and T^*-extension for quadratic and odd quadratic Lie superalgebras. Also, we provide a classification of quadratic and odd quadratic Lie superalgebras up to dimension 6. This classification is considered up to isometric isomorphism, mainly in the solvable case, and the obtained Lie superalgebras are indecomposable.

0. Introduction

Throughout the paper, all considered vector spaces are finite-dimensional complex vector spaces.

We recall a well-known example in Lie theory as follows. Let $\mathfrak{g}$ be a complex Lie algebra and $\mathfrak{g}^*$ its dual space. Denote by $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ the adjoint representation and by $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$ the coadjoint representation of $\mathfrak{g}$. The semidirect product $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^*$ of $\mathfrak{g}$ and $\mathfrak{g}^*$ by the representation ad^* is a Lie algebra with the bracket given by:

$$[X+f, Y+g] = [X,Y] + \mathrm{ad}^*(X)(g) - \mathrm{ad}^*(Y)(f), \quad \forall X, Y \in \mathfrak{g},\ f, g \in \mathfrak{g}^*.$$

Remark that $\overline{\mathfrak{g}}$ is also a quadratic Lie algebra with invariant symmetric bilinear form B defined by:

$$B(X+f, Y+g) = f(Y) + g(X), \quad \forall X, Y \in \mathfrak{g},\ f, g \in \mathfrak{g}^*.$$

In the way of how to generalize this example, A. Medina and P. Revoy gave the notion of double extension of quadratic Lie algebras to completely characterize all quadratic Lie algebras [MR85]. Another generalization is called T^*-extension, given by M. Bordemann, which is sufficient to describe solvable quadratic Lie algebras [Bor97]. In this paper, we shall just give an expansion of these two notions for Lie superalgebras. In particular, we present a way to obtain a quadratic Lie superalgebra from a Lie algebra and a symplectic vector space. It is regarded as a rather special case of the notion of generalized double extension of quadratic Lie

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superalgebras in [BBB]. In a slight change of the notion of T^*-extension, we give a manner of how to get an odd quadratic Lie superalgebra from a Lie algebra. For the application of double extensions and generalized double extensions in describing inductively quadratic and odd quadratic Lie superalgebras, we refer the reader to [BB99], [ABB09] or [ABB10]. For the definitions and basic facts of the Theory of Lie superalgebras the reader is referred to [Sch79].

We see that the classification problem of quadratic Lie algebras has appeared in many works. It is initiated for nilpotent quadratic Lie algebras of dimension ≤ 7 in [FS87] following the classification of nilpotent $\mathrm{O}(n)$-adjoint orbits of $\mathfrak{o}(n)$ and recently of dimension ≤ 10 in [Kat07] by applying a new proposal presented in [KO06]. Other approaches to the classification in the nilpotent case can be found in [NR97] for two-step nilpotent quadratic Lie algebras, and in [dBO12] for free nilpotent quadratic Lie algebras. For the solvable case, the classification of quadratic real Lie algebras is known in dimensions ≤ 6 in [BK03] where the authors used the method of double extension. Other classifications for special classes of solvable quadratic Lie algebras have been done, for example solvable admissible Lie algebras with small radical in [KO06], and solvable singular quadratic Lie algebras in [DPU12]. However, the classification of quadratic and odd quadratic Lie superalgebras in low dimensions does not seem to have been considered until now. In the present paper we deal with this problem up to dimension 6 mainly in the solvable case.

The paper is organized as follows. In Section 1 we shortly recall the definition of double extension of a quadratic Lie algebra and give some examples of 2-step double extensions. Although double extensions provide a useful description of quadratic Lie algebras, we know very little about them; for instance, for the class of 2-step double extensions a lot still remains to be discovered, and its complete classification is difficult to get. A quadratic Lie algebra is called a 2-step double extension if it is a one-dimensional double extension of a 1-step double extension but not a 1-step double extension. The notion of 1-step double extension is for quadratic Lie algebras obtained from one-dimensional double extensions of Abelian Lie algebras and such algebras have just been classified up to isomorphism and isometric isomorphism in [DPU12]. We also introduce the notion of T^*-extension and list all T^*-extensions of three-dimensional solvable Lie algebras. All of them are 1-step double extensions. Section 2 is devoted to quadratic Lie superalgebras where we give an expansion of the double extension and T^*-extension notions for the superalgebra case. We also give an exhaustive classification of solvable quadratic Lie superalgebras up to dimension 6. In the last section, we provide a way for constructing an odd quadratic Lie superalgebra and show an isometrically isomorphic classification of solvable odd quadratic Lie superalgebras up to dimension 6.

1. Quadratic Lie algebras

Definition 1.1. Let $\mathfrak{g}$ be a Lie algebra. A symmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is called:

(i) *non-degenerate* if $B(X,Y) = 0$ for all $Y \in \mathfrak{g}$ implies $X = 0$,

(ii) *invariant* if $B([X,Y],Z) = B(X,[Y,Z])$ for all $X,\ Y,\ Z \in \mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ is called *quadratic* if there exists a bilinear form B on $\mathfrak{g}$ such that B is symmetric, non-degenerate and invariant.

Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra. Since B is non-degenerate and invariant we have some simple properties of $\mathfrak{g}$ as follows:

Proposition 1.2.

(1) *If I is an ideal of $\mathfrak{g}$ then $I^\perp$ is also an ideal of $\mathfrak{g}$. Moreover, if I is non-degenerate then so is $I^\perp$ and $\mathfrak{g} = I \oplus I^\perp$. Conveniently, in this case we use the notation $\mathfrak{g} = I \overset{\perp}{\oplus} I^\perp$.*

(2) *$\mathscr{Z}(\mathfrak{g}) = [\mathfrak{g},\mathfrak{g}]^\perp$ where $\mathscr{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$. And then*

$$\dim(\mathscr{Z}(\mathfrak{g})) + \dim([\mathfrak{g},\mathfrak{g}]) = \dim(\mathfrak{g}).$$

A quadratic Lie algebra $\mathfrak{g}$ is called *indecomposable* if $\mathfrak{g} = \mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$, with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ ideals of $\mathfrak{g}$, implies $\mathfrak{g}_1$ or $\mathfrak{g}_2 = \{0\}$. Otherwise, we call $\mathfrak{g}$ *decomposable.*

We say that two quadratic Lie algebras $(\mathfrak{g}, B)$ and $(\mathfrak{g}', B')$ are isometrically isomorphic (or i-isomorphic, for short) if there exists a Lie algebra isomorphism A from $\mathfrak{g}$ onto $\mathfrak{g}'$ satisfying $B'(A(X), A(Y)) = B(X,Y)$ for all $X, Y \in \mathfrak{g}$. In this case, A is called an i-isomorphism.

1.1. **Double extensions.**

Definition 1.3. Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra and D a derivation of $\mathfrak{g}$. We say D a *skew-symmetric* derivation of $\mathfrak{g}$ if it satisfies $B(D(X), Y) = -B(X, D(Y))$ for all $X, Y \in \mathfrak{g}$.

Denote by $\mathrm{Der}_a(\mathfrak{g}, B)$ the vector space of skew-symmetric derivations of $(\mathfrak{g}, B)$; then $\mathrm{Der}_a(\mathfrak{g}, B)$ is a subalgebra of $\mathrm{Der}(\mathfrak{g})$, the Lie algebra of derivations of $\mathfrak{g}$. The notion of double extension is defined as follows (see[MR85]).

Definition 1.4. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{g}^*$ its dual space and $(\mathfrak{h}, B)$ a quadratic Lie algebra. Let $\psi : \mathfrak{g} \to \mathrm{Der}_a(\mathfrak{h}, B)$ be a Lie algebra endomorphism. Denote by $\phi : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}^*$ the linear mapping defined by $\phi(X,Y)Z = B(\psi(Z)(X), Y)$ for all $X, Y \in \mathfrak{h},\ Z \in \mathfrak{g}$. Consider the vector space $\overline{\mathfrak{h}} = \mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{g}^*$ and define a product on $\overline{\mathfrak{h}}$ by:

$$\begin{aligned}[X+F+f, Y+G+g]_{\overline{\mathfrak{h}}} = {}& [X,Y]_{\mathfrak{g}} + [F,G]_{\mathfrak{h}} + \mathrm{ad}^*(X)(g) - \mathrm{ad}^*(Y)(f) \\ &+ \psi(X)(G) - \psi(Y)(F) + \phi(F,G)\end{aligned}$$

for all $X, Y \in \mathfrak{g},\ f, g \in \mathfrak{g}^*$ and $F, G \in \mathfrak{h}$. Then $\overline{\mathfrak{h}}$ becomes a quadratic Lie algebra with the bilinear form $\overline{B}$ given by $\overline{B}(X+F+f, Y+G+g) = f(Y) + g(X) + B(F,G)$ for all $X, Y \in \mathfrak{g},\ f, g \in \mathfrak{g}^*$ and $F, G \in \mathfrak{h}$. The Lie algebra $(\overline{\mathfrak{h}}, \overline{B})$ is called the *double extension of* $(\mathfrak{h}, B)$ *by* $\mathfrak{g}$ *by means of* ψ.

Note that when $\mathfrak{h} = \{0\}$ then this definition is reduced to the notion of the semidirect product of $\mathfrak{g}$ and $\mathfrak{g}^*$ by the coadjoint representation.

Proposition 1.5. [MR85]

Let $(\mathfrak{g}, B)$ be an indecomposable quadratic Lie algebra such that it is not simple nor one-dimensional. Then $\mathfrak{g}$ is a double extension of a quadratic Lie algebra by a simple or one-dimensional algebra.

Sometimes, we use a particular case of the notion of double extension: a double extension by a skew-symmetric derivation. It is explicitly defined as follows.

Definition 1.6. Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra and $C \in \mathrm{Der}_a(\mathfrak{g})$. On the vector space $\bar{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}e \oplus \mathbb{C}f$ we define the product $[X, Y]_{\bar{\mathfrak{g}}} = [X, Y]_{\mathfrak{g}} + B(C(X), Y)f$, $[e, X] = C(X)$ and $[f, \bar{\mathfrak{g}}] = 0$ for all $X, Y \in \mathfrak{g}$. Then $\bar{\mathfrak{g}}$ is a quadratic Lie algebra with an invariant bilinear form $\bar{B}$ defined by:

$$\bar{B}(e, e) = \bar{B}(f, f) = \bar{B}(e, \mathfrak{g}) = \bar{B}(f, \mathfrak{g}) = 0,$$
$$\bar{B}(X, Y) = B(X, Y),$$
$$\bar{B}(e, f) = 1,$$

for all $X, Y \in \mathfrak{g}$. In this case, we call $\bar{\mathfrak{g}}$ the *double extension of* $\mathfrak{g}$ *by* C, or a *one-dimensional double extension of* $\mathfrak{g}$, for short.

A remarkable result is that one-dimensional double extensions are sufficient for studying solvable quadratic Lie algebras (see [Kac85] or [FS87]).

Example 1.7. Let $\mathfrak{g}_4$ be the diamond Lie algebra spanned by $\{X, P, Q, Z\}$ where the Lie bracket is defined by: $[X, P] = P$, $[X, Q] = -Q$, and $[P, Q] = Z$; then $\mathfrak{g}$ is quadratic with invariant bilinear form B given by $B(X, Z) = B(P, Q) = 1$, the others are vanish. Assume that D is a skew-symmetric derivation of $\mathfrak{g}_4$. By a straightforward computation, the matrix of D in the given basis is:

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ y & x & 0 & 0 \\ z & 0 & -x & 0 \\ 0 & -z & -y & 0 \end{pmatrix},$$

where $x, y, z \in \mathbb{C}$. Let $\bar{\mathfrak{g}}_4 = \mathfrak{g}_4 \oplus \mathbb{C}e \oplus \mathbb{C}f$ be the double extension of $\mathfrak{g}_4$ by D. Then the Lie bracket is defined on $\bar{\mathfrak{g}}_4$ as follows: $[e, X] = yP + zQ$, $[e, P] = xP - zZ$, $[e, Q] = -xQ - yZ$, $[X, P] = P + zf$, $[X, Q] = -Q + yf$, and $[P, Q] = Z + xf$.

In the above example, the quadratic Lie algebra $\bar{\mathfrak{g}}_4$ is decomposable since the elements $u = -e + xX - yP + zQ$ and f are central and $B(u, f) = -1$. Note that the skew-symmetric derivation D above is inner. Actually, in [FOS96], the authors proved a general result that any double extension by an inner derivation is decomposable. Here we give a short proof as follows.

Proposition 1.8. *Let $(\mathfrak{g}, B)$ be a quadratic Lie algebra and $C = \mathrm{ad}_{\mathfrak{g}}(X_0)$ be an inner derivation of $\mathfrak{g}$. Then the double extension $\bar{\mathfrak{g}}$ of $\mathfrak{g}$ by C is decomposable.*

Proof. If $\mathfrak{g} = \{0\}$ then the result is obvious. Assume $\mathfrak{g} \neq \{0\}$. It is straightforward to prove that the element $e - X_0$ is central. Moreover, $B(e - X_0, f) = 1$ and f is central so that $\bar{\mathfrak{g}}$ is decomposable by Proposition 1.2. □

Remark 1.9. As we know, the diamond Lie algebra is a one-dimensional double extension of an Abelian Lie algebra. We call this type of double extensions a *1-step double extension.* The above example shows that a double extension of a 1-step double extension may be a 1-step double extension. Therefore, we define a *2-step double extension* if it is a one-dimensional double extension of a 1-step double extension but not a 1-step double extension. Since a fact is that the 1-step double extensions are exhaustively classified in [DPU12], it remains k-step double extensions with $k > 1$. Note that every solvable quadratic Lie algebra up to dimension 6 is a 1-step double extension (see the classification in [BK03] or in Proposition 1.17).

Let $\mathfrak{g}_5$ be a non-Abelian nilpotent quadratic Lie algebra of dimension 5 spanned by $\{X_1, X_2, T, Z_1, Z_2\}$ with the Lie bracket defined by $[X_1, X_2] = T$, $[X_1, T] = -Z_2$, $[X_2, T] = Z_1$, and the bilinear form B given by $B(X_i, Z_i) = B(T, T) = 1$, $i = 1, 2$, zero otherwise. This algebra is a 1-step double extension. We can check that D is a skew-symmetric derivation of $\mathfrak{g}_5$ if and only if the matrix of D in the given basis is:

$$D = \begin{pmatrix} -x & -z & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 \\ -b & -c & 0 & 0 & 0 \\ 0 & -t & b & x & y \\ t & 0 & c & z & -x \end{pmatrix}$$

where $x, y, z, t, b, c \in \mathbb{C}$. Combined with Lemma 5.1 in [Med85]: two double extensions by skew-symmetric derivations D and D' respectively are i-isomorphic if D and D' are different by adding an inner derivation; we consider only skew-symmetric derivations having matrix:

$$D = \begin{pmatrix} -x & -z & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y \\ 0 & 0 & 0 & z & -x \end{pmatrix}.$$

Let $\bar{\mathfrak{g}}_5$ be the double extension of $\mathfrak{g}_5$ by D; then we have the following assertion.

Proposition 1.10. *If $x = y = 0$ or $x = z = 0$ then $\bar{\mathfrak{g}}_5$ is a 1-step double extension. Else, $\bar{\mathfrak{g}}_5$ is a 2-step double extension and indecomposable.*

Proof. If $x = y = z = 0$ then $\bar{\mathfrak{g}}_5$ is decomposable so it is obviously a 1-step double extension. We assume that $x = z = 0$ and $y \neq 0$. In this case, the Lie bracket on $\bar{\mathfrak{g}}_5$ is defined by:

$$[e, X_1] = -yX_2, \quad [e, Z_2] = yZ_1, \quad [X_1, X_2] = T, \quad [X_1, T] = -Z_2, \quad [X_2, T] = Z_1,$$

and $[X_1, Z_2] = -yf$. Then one has $\bar{\mathfrak{g}}_5 = \mathfrak{q} \oplus (\mathbb{C}X_1 \oplus \mathbb{C}Z_1)$ a 1-step double extension of $\mathfrak{q}$ spanned by $\{e, X_2, T, f, Z_2\}$ by the skew-symmetric $C : \mathfrak{q} \to \mathfrak{q}$, $C(e) = yX_2$, $C(X_2) = T$, $C(T) = -Z_2$ and $C(Z_2) = -yf$. Note that the case of $x = y = 0$ and $z \neq 0$ is completely similar to the case of $x = z = 0$ and $y \neq 0$.

If $y = z = 0$ and $x \neq 0$ then one has $[[\bar{\mathfrak{g}}_5, \bar{\mathfrak{g}}_5], [\bar{\mathfrak{g}}_5, \bar{\mathfrak{g}}_5]] = \mathrm{span}\{T, Z_1, Z_2, f\}$, so $\bar{\mathfrak{g}}_5$ is not a 1-step double extension. Remark that $\bar{\mathfrak{g}}_5$ is also indecomposable because otherwise $\bar{\mathfrak{g}}_5$ must be a 1-step double extension since all solvable quadratic Lie algebra up to dimension 6 are 1-step double extensions.

The remaining cases satisfy the condition $\dim([[\bar{\mathfrak{g}}_5, \bar{\mathfrak{g}}_5], [\bar{\mathfrak{g}}_5, \bar{\mathfrak{g}}_5]]) > 1$. Hence, they are all indecomposable 2-step double extensions and the result follows. □

As we have just seen in the above proof, we can obtain one quadratic Lie algebra from many different ways of double extension: maybe starting from non isomorphic Lie algebras or starting from a quadratic Lie algebra but with different skew-symmetric derivations, for instance, two double extensions of a quadratic Lie algebra $\mathfrak{g}$ by C and λC, λ non-zero (for non-trivial examples, see [FS87], [Med85] or [KO06]).

Let $(\mathfrak{g}_{2n+2}, B)$ be a quadratic Lie algebra of dimension $2n+2$, $n \geq 1$, spanned by $\{X_0, \ldots, X_n, Y_0, \ldots, Y_n\}$ where the Lie bracket is defined by $[Y_0, X_i] = X_i$, $[Y_0, Y_i] = -Y_i$ and $[X_i, Y_i] = X_0$, $1 \leq i \leq n$, the other are trivial, and the invariant bilinear form B is given by $B(X_i, Y_i) = 1$, $0 \leq i \leq n$, zero otherwise. Note that $\mathfrak{g}_{2n+2}$ is a 1-step double extension.

Now we assume $n \geq 2$ and let D be a derivation of $\mathfrak{g}_{2n+2}$ defined by $D(X_1) = X_1$, $D(Y_1) = -Y_1$ and the others are vanish. Then the double extension $\overline{\mathfrak{g}}$ of $\mathfrak{g}_{2n+2}$ by D has the Lie bracket: $[e, X_1] = X_1$, $[e, Y_1] = -Y_1$, $[Y_0, X_i] = X_i$, $[Y_0, Y_i] = -Y_i$, $[X_1, Y_1] = X_0 + f$ and $[X_i, Y_i] = X_0$, $2 \leq i \leq n$. It implies that $[[\overline{\mathfrak{g}}, \overline{\mathfrak{g}}], [\overline{\mathfrak{g}}, \overline{\mathfrak{g}}]] = \mathrm{span}\{X_0, X_0 + f\}$ so that $\overline{\mathfrak{g}}$ is a 2-step double extension.

Remark 1.11. It is obvious that $\mathfrak{g}_1 \overset{\perp}{\oplus} \mathfrak{g}_2$ with $\mathfrak{g}_1$ and $\mathfrak{g}_2$ non-Abelian 1-step double extensions is a 2-step double extension. However, this case is rather trivial since it is decomposable. For instance, the Lie algebra $\overline{\mathfrak{g}}$ defined as above is decomposable by $\overline{\mathfrak{g}} = \mathfrak{g}_4 \overset{\perp}{\oplus} \mathfrak{g}_{2n}$. More explicitly, one has

$$\overline{\mathfrak{g}} = \mathrm{span}\{e, X_1, Y_1, X_0 + f\} \ \overset{\perp}{\oplus} \ \mathrm{span}\{e - Y_0, X_i, Y_i, X_0\}, \quad 2 \leq i \leq n.$$

1.2. T^*-extensions.

Definition 1.12. [Bor97] Let $\mathfrak{g}$ be a Lie algebra and $\theta : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}^*$ a 2-cocycle of $\mathfrak{g}$, that is a skew-symmetric bilinear map satisfying:

$$\theta(X, Y) \circ \mathrm{ad}(Z) + \theta([X, Y], Z) + \mathrm{cycle}(X, Y, Z) = 0.$$

for all $X, Y, Z \in \mathfrak{g}$. Define on the vector space $T^*_\theta(\mathfrak{g}) := \mathfrak{g} \oplus \mathfrak{g}^*$ the product:

$$[X + f, Y + g] = [X, Y] + \mathrm{ad}^*(X)(g) - \mathrm{ad}^*(Y)(f) + \theta(X, Y)$$

for all $X, Y \in \mathfrak{g}$, $f,\ g \in \mathfrak{g}^*$; then $T^*_\theta(\mathfrak{g})$ becomes a Lie algebra and it is called the *T^*-extension of $\mathfrak{g}$ by means of θ*. In addition, if θ satisfies the *cyclic* condition, i.e. $\theta(X, Y)Z = \theta(Y, Z)X$ for all $X, Y, Z \in \mathfrak{g}$, then $T^*_\theta(\mathfrak{g})$ is quadratic with the bilinear form:

$$B(X + f, Y + g) = f(Y) + g(X), \quad \forall\ X, Y \in \mathfrak{g},\ f, g \in \mathfrak{g}^*.$$

Proposition 1.13. [Bor97]*Let* $(\mathfrak{g}, B)$ *be an even-dimensional quadratic Lie algebra over* $\mathbb{C}$. *If* $\mathfrak{g}$ *is solvable then* $\mathfrak{g}$ *is i-isomorphic to a* T^**-extension* $T^*_\theta(\mathfrak{h})$ *of* $\mathfrak{h}$*, where* $\mathfrak{h}$ *is the quotient algebra of* $\mathfrak{g}$ *by a totally isotropic ideal.*

As in Proposition 1.8, a double extension by an inner derivation is decomposable. We give here a similar situation for a T^*-extension decomposable as follows.

Proposition 1.14. *Assume that* $\mathfrak{g}$ *is a Lie algebra with* $\mathscr{Z}(\mathfrak{g}) \neq \{0\}$ *and* θ *is a cyclic 2-cocycle of* $\mathfrak{g}$. *If there are a nonzero* $X \in \mathscr{Z}(\mathfrak{g})$ *and a nonzero* $a \in \mathbb{C}$ *such that*

$$\theta(X, Y) = aX^* \circ \mathrm{ad}(Y), \quad \forall\, Y \in \mathfrak{g},$$

where X^* *denotes the dual form of* X*, then* $T^*_\theta(\mathfrak{g})$ *is decomposable.*

Proof. We can show that if there are such X and a then $X - aX^*$ is central. Moreover, $B(X - aX^*, X - aX^*) = -2a \neq 0$, then $T^*_\theta(\mathfrak{g})$ is decomposable. □

Example 1.15. Denote by $\mathfrak{h}_3$ the Lie algebra spanned by $\{X, Y, Z\}$ such that $[X, Y] = Z$. Assume $\{X^*, Y^*, Z^*\}$ the dual basis of $\{X, Y, Z\}$ and let θ be a non-trivial cyclic 2-cocycle of $\mathfrak{h}_3$. It is easy to compute that

$$\theta(X, Y) = \lambda Z^*, \quad \theta(Y, Z) = \lambda X^*$$

and $\theta(Z, X) = \lambda Y^*$, where λ is nonzero in $\mathbb{C}$. Then $T^*_\theta(\mathfrak{h}_3)$ is decomposable since $\theta(Z, T) = \lambda Z^* \circ \mathrm{ad}(T)$ for all $T \in \mathfrak{h}_3$.

Remark 1.16. Let $\mathfrak{g}$ be a Lie algebra, θ a cyclic 2-cocycle of $\mathfrak{g}$ and a nonzero $\lambda \in \mathbb{C}$. Then two T^*-extensions $T^*_\theta(\mathfrak{g})$ and $T^*_{\lambda\theta}(\mathfrak{g})$ are isomorphic by the isomorphism $A : T^*_\theta(\mathfrak{g}) \to T^*_{\lambda\theta}(\mathfrak{g})$, $A(X + f) = X + \lambda f$.

1.3. **T^*-extensions of three-dimensional solvable Lie algebras.** As we know, if $\mathfrak{g}$ is a solvable Lie algebra of dimension 3 then it is isomorphic to each of the following Lie algebras:

(1) $\mathfrak{g}_{3,0}$: Abelian,
(2) $\mathfrak{g}_{3,1}$: $[X, Y] = Z$,
(3) $\mathfrak{g}_{3,2}$: $[X, Y] = Y$ and $[X, Z] = Y + Z$,
(4) $\mathfrak{g}_{3,3}$: $[X, Y] = Y$ and $[X, Z] = \mu Z$ with $|\mu| \leq 1$.

Proposition 1.17. *Let* $\mathfrak{g}$ *be a solvable Lie algebra of dimension 3 and* θ *a cyclic 2-cocycle of* $\mathfrak{g}$. *If the* T^**-extension of* $\mathfrak{g}$ *by means of* θ *is indecomposable then it is i-isomorphic to each of the following quadratic Lie algebras:*

(1) $\mathfrak{g}_{6,1}$*:* $[X, Y] = Z$, $[X, Z^*] = -Y^*$ *and* $[Y, Z^*] = X^*$,
(2) $\mathfrak{g}_{6,2}$*:* $[X, Y] = Y$, $[X, Z] = Y + Z$, $[X, Y^*] = -Y^* - Z^*$, $[X, Z^*] = -Z^*$ *and* $[Y, Y^*] = [Z, Y^*] = [Z, Z^*] = X^*$,
(3) $\mathfrak{g}_{6,3}$*:* $[X, Y] = Y$, $[X, Z] = \mu Z$, $[X, Y^*] = -Y^*$, $[X, Z^*] = -\mu Z^*$, $[Y, Y^*] = X^*$ *and* $[Z, Z^*] = \mu X^*$ *where* $|\mu| \leq 1$.

Proof.

(1) If $\mathfrak{g}$ is Abelian then the non-Abelian T^*-extensions of $\mathfrak{g}$ are 2-nilpotent. In the case of dimension 6, the classification of 2-nilpotent quadratic Lie algebras is shown in [NR97], where there is only one Lie algebra i-isomorphic to $\mathfrak{g}_{6,1}$.

Next we shall show that if $\mathfrak{g}$ is a non-Abelian solvable Lie algebra of dimension 3 and θ is a non-vanishing cyclic 2-cocycle of $\mathfrak{g}$, then θ is defined by:

$$\theta(X,Y) = \lambda Z^*, \quad \theta(Y,Z) = \lambda X^*, \quad \theta(Z,X) = \lambda Y^*$$

with nonzero $\lambda \in \mathbb{C}$. This is a straightforward computation so we omit it.

(2) Consider the Lie algebra $\mathfrak{g}_{3,1} : [X,Y] = Z$ and let θ be a non-vanishing cyclic 2-cocycle of $\mathfrak{g}_{3,1}$. As in Example 1.15, T^*-extension of $\mathfrak{g}$ by means of θ is decomposable.

For the Lie algebra $\mathfrak{g}_{3,2} : [X,Y] = Y$ and $[X,Z] = Y + Z$, we can check that the Lie bracket on $T^*_\theta(\mathfrak{g}_{3,2})$ is defined by:

$$[X,Y] = Y + \lambda Z^*, \quad [X,Z] = Y + Z - \lambda Y^*, \quad [X,Y^*] = -Y^* - Z^*,$$
$$[X,Z^*] = -Z^*, \quad [Y,Z] = \lambda X^*, \quad \text{and} \quad [Y,Y^*] = [Z,Y^*] = [Z,Z^*] = X^*.$$

In this case, replacing $Y + \frac{\lambda}{2} Z^*$ by Y, $Z - \frac{\lambda}{2} Y^*$ by Z, and keeping the other basis vectors (this is an isometry) to get the Lie algebra $\mathfrak{g}_{6,2}$.

(3) For the Lie algebra $\mathfrak{g}_{3,3}$, by a straightforward computation, we obtain the T^*-extension of $\mathfrak{g}_{3,3}$ having the Lie bracket:

$$[X,Y] = Y + \lambda Z^*, \quad [X,Z] = \mu Z - \lambda Y^*, \quad [X,Y^*] = -Y^*,$$
$$[X,Z^*] = -\mu Z^*, \quad [Y,Z] = \lambda X^*, \quad [Y,Y^*] = X^*, \quad \text{and} \quad [Z,Z^*] = \mu X^*.$$

We consider two cases: $\mu \neq -1$ and $\mu = -1$. If $\mu \neq -1$, replacing $Y + \frac{\lambda}{\mu+1} Z^*$ by Y, $Z - \frac{\lambda}{\mu+1} Y^*$ by Z and keeping the other (this is an isometry) to get (3). If $\mu = -1$, replacing $\frac{1}{\lambda} Z$ by Z, λZ^* by Z^* and keeping the other (this is an isometry) to turn to (2).

Finally, it is easy to check that if $\theta = 0$ and $T^*_0(\mathfrak{g})$ is indecomposable then $T^*_0(\mathfrak{g})$ coincides with each of the given Lie algebras. □

Remark 1.18.

(1) The Lie algebras $\mathfrak{g}_{6,i}$, $1 \leq i \leq 3$, given in Proposition 1.17 are all solvable quadratic Lie algebras in the classification by the method of double extension [BK03]. In this case, since double extensions of the diamond Lie algebra is decomposable, it remains to consider double extensions of the 4-dimensional Abelian Lie algebra, and then the classification result follows the classification of linear maps $A \in \mathfrak{o}(4)$ with totally isotropic kernel up to conjugation and up to non-zero multiples.

(2) To get a complete classification of quadratic Lie algebras of dimension ≤ 5, we refer the reader to [BK03] or [DPU12] where there are only three indecomposable non-Abelian ones: the Lie algebra $\mathfrak{o}(3)$, the diamond Lie algebra $\mathfrak{g}_4$, and the 5-dimensional nilpotent Lie algebra $\mathfrak{g}_5$.

(3) Similar to double extensions, the proof of Proposition 1.17 shows that by the T^*-extension process it is possible to get i-isomorphic quadratic Lie algebras from non isomorphic Lie algebras. However, it seems more difficult to classify quadratic Lie algebras by T^*-extensions than by double extensions.

2. Quadratic Lie superalgebras

Definition 2.1. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. We assume that there is a non-degenerate invariant supersymmetric bilinear form B on $\mathfrak{g}$, i.e. B satisfies:

(1) $B(Y,X) = (-1)^{xy}B(X,Y)$ (supersymmetric) for all $X \in \mathfrak{g}_x,\ Y \in \mathfrak{g}_y$,
(2) $B([X,Y],Z) = B(X,[Y,Z])$ (invariant) for all $X,\ Y,\ Z \in \mathfrak{g}$.

If B is even, that is $B(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}) = 0$, then we call $\mathfrak{g}$ a *quadratic Lie superalgebra*. In this case, $B_0 = B|_{\mathfrak{g}_{\bar{0}}\times\mathfrak{g}_{\bar{0}}}$ and $B_1 = B|_{\mathfrak{g}_{\bar{1}}\times\mathfrak{g}_{\bar{1}}}$ are non-degenerate, consequently $(\mathfrak{g}_{\bar{0}}, B_0)$ is a quadratic Lie algebra and $(\mathfrak{g}_{\bar{1}}, B_1)$ is a symplectic vector space. The invariancy of B implies that $B_1(\mathrm{ad}_X(Y), Z) = -B_1(Y, \mathrm{ad}_X(Z))$ for all $X \in \mathfrak{g}_{\bar{0}},\ Y,\ Z \in \mathfrak{g}_{\bar{1}}$. Therefore, the adjoint representation is a homomorphism from the Lie algebra $\mathfrak{g}_{\bar{0}}$ onto the Lie subalgebra $\mathfrak{sp}(\mathfrak{g}_{\bar{1}}, B_1)$ of $\mathrm{End}(\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}})$.

If B is odd, i.e. $B(\mathfrak{g}_\alpha, \mathfrak{g}_\alpha) = 0$ for all $\alpha \in \mathbb{Z}_2$, then we call $\mathfrak{g}$ an *odd quadratic Lie superalgebra*. In this case, $\mathfrak{g}_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}}$ are totally isotropic subspaces of $\mathfrak{g}$ and $\dim(\mathfrak{g}_{\bar{0}}) = \dim(\mathfrak{g}_{\bar{1}})$.

Proposition 1.2 is still right here so that we have similarly sequential properties for quadratic Lie superalgebras and odd quadratic Lie superalgebras.

For the notion of 1-step double extension in the quadratic Lie superalgebras case, we refer the reader to [DU14] (the definition of double extension in the general case is given in [BB99]).

Clearly, the definition of the semidirect product of a Lie algebra $\mathfrak{g}$ and its dual space $\mathfrak{g}^*$ can be comprehended that the Lie algebra $\mathfrak{g}$ is "glued" by $\mathfrak{g}^*$. The notion of double extension is more general by such "gluing" but combined with a quadratic Lie algebra $(\mathfrak{h}, B)$. From this point of view, what happens when we glue the dual space $\mathfrak{g}^*$ to $\mathfrak{g}$ combined with a *symplectic* vector space $(\mathfrak{h}, B)$ to obtain a quadratic Lie superalgebra? Obviously, in this case, the even part is $\mathfrak{g} \oplus \mathfrak{g}^*$ and the odd part is $\mathfrak{h}$.

The following lemma is straightforward.

Lemma 2.2. *Let $\mathfrak{g}$ be a Lie algebra and $(\mathfrak{h}, B_{\mathfrak{h}})$ a symplectic vector space with symplectic form $B_{\mathfrak{h}}$. Let $\psi : \mathfrak{g} \to \mathrm{End}(\mathfrak{h})$ be a Lie algebra endomorphism satisfying:*

$$B_{\mathfrak{h}}(\psi(X)(Y), Z) = -B_{\mathfrak{h}}(Y, \psi(X)(Z)), \quad \forall\ X \in \mathfrak{g},\ Y, Z \in \mathfrak{h}.$$

Denote by $\phi : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}^*$ the bilinear map defined by $\phi(X,Y)Z = B_{\mathfrak{h}}(\psi(Z)(X), Y)$ for all $X, Y \in \mathfrak{h}$, $Z \in \mathfrak{g}$; then ϕ is symmetric.

Proposition 2.3. *Keep notions as in the above lemma and define on the vector space $\overline{\mathfrak{g}} := \mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathfrak{h}$ the bracket:*

$$\begin{aligned}[X+f+F, Y+g+G]_{\overline{\mathfrak{g}}} = [X,Y]_{\mathfrak{g}} &+ \mathrm{ad}^*(X)(g) - \mathrm{ad}^*(Y)(f) \\ &+ \psi(X)(G) - \psi(Y)(F) + \phi(F,G),\end{aligned}$$

for all $X, Y \in \mathfrak{g}$, $f, g \in \mathfrak{g}^*$ *and* $F, G \in \mathfrak{h}$. *Then* $\overline{\mathfrak{g}}$ *becomes a quadratic Lie superalgebra with* $\overline{\mathfrak{g}}_{\bar{0}} = \mathfrak{g} \oplus \mathfrak{g}^*$, $\overline{\mathfrak{g}}_{\bar{1}} = \mathfrak{h}$ *and the bilinear form* $\overline{B}$ *defined by:*

$$\overline{B}(X + f + F, Y + g + G) = f(Y) + g(X) + B_{\mathfrak{h}}(F, G)$$

for all $X, Y \in \mathfrak{g}$, $f, g \in \mathfrak{g}^*$ *and* $F, G \in \mathfrak{h}$.

Proof. The proof is a straightforward but lengthy computation so we omit it. $\square$

Now we combine the above proposition with Definition 1.12 and Proposition 1.8 to get a more general result as follows.

Proposition 2.4. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\theta : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}^*$ *a 2-cocycle of* $\mathfrak{g}$. *Assume* $(\mathfrak{h}, B_{\mathfrak{h}})$ *is a symplectic vector space with symplectic form* $B_{\mathfrak{h}}$. *Let* $\psi : \mathfrak{g} \to \mathrm{End}(\mathfrak{h})$ *be a Lie algebra endomorphism satisfying* $B_{\mathfrak{h}}(\psi(X)(Y), Z) = -B_{\mathfrak{h}}(Y, \psi(X)(Z))$ *for all* $X \in \mathfrak{g}$, $Y, Z \in \mathfrak{h}$. *Denote by* $\phi : \mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}^*$ *the bilinear map defined by* $\phi(X, Y)Z = B_{\mathfrak{h}}(\psi(Z)(X), Y)$ *for all* $X, Y \in \mathfrak{h}$, $Z \in \mathfrak{g}$, *and define on the vector space* $\overline{\mathfrak{g}} := \mathfrak{g} \oplus \mathfrak{g}^* \oplus \mathfrak{h}$ *the bracket:*

$$\begin{aligned}[X + f + F, Y + g + G]_{\overline{\mathfrak{g}}} = [X, Y]_{\mathfrak{g}} &+ \mathrm{ad}^*(X)(g) - \mathrm{ad}^*(Y)(f) + \theta(X, Y) \\ &+ \psi(X)(G) - \psi(Y)(F) + \phi(F, G),\end{aligned}$$

for all $X, Y \in \mathfrak{g}$, $f, g \in \mathfrak{g}^*$ *and* $F, G \in \mathfrak{h}$. *Then* $\overline{\mathfrak{g}}$ *becomes a Lie superalgebra with* $\overline{\mathfrak{g}}_{\bar{0}} = \mathfrak{g}\oplus\mathfrak{g}^*$ *and* $\overline{\mathfrak{g}}_{\bar{1}} = \mathfrak{h}$. *Moreover, if* θ *is cyclic then* $\overline{\mathfrak{g}}$ *is a quadratic Lie superalgebra with the bilinear form* $\overline{B}(X + f + F, Y + g + G) = f(Y) + g(X) + B_{\mathfrak{h}}(F, G)$ *for all* $X, Y \in \mathfrak{g}$, $f, g \in \mathfrak{g}^*$ *and* $F, G \in \mathfrak{h}$.

Note that the construction in the previous proposition is not well-characterized for every quadratic Lie superalgebra, for example quadratic Lie superalgebras have $[[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}], [\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]] \neq \{0\}$. In order to describe all quadratic Lie superalgebras we need to use the notion of *generalized double extensions* of quadratic Lie superalgebras, where $\mathfrak{g}$ is replaced by an arbitrary Lie superalgebra with a supersymmetric invariant (not necessarily non-degenerate) bilinear form, $\mathfrak{h}$ is replaced by a quadratic Lie superalgebra, $\psi : \mathfrak{g} \to \mathrm{Der}_a(\mathfrak{h})$ is an even linear map satisfying the following condition:

$$[\psi(X), \psi(Y)] - \psi([X, Y]_{\mathfrak{g}}) = \mathrm{ad}_{\mathfrak{h}}(\phi(X, Y))$$

for all homogeneous $X, Y \in \mathfrak{g}$, where $\phi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{h}$ is an even bilinear superantisymmetric map, and the 2-cocycle and cyclicity conditions of θ are generalized to the *twisted cocycle* and *supercyclicity* conditions. For more details of the definition and for the proof of the following proposition, the reader is referred to [BBB].

Proposition 2.5. *Let* $(\mathfrak{g}, B)$ *be a quadratic Lie superalgebra,* $\mathfrak{J}$ *be a totally isotropic graded ideal of* $\mathfrak{g}$, *and* $\mathfrak{J}^\perp$ *its orthogonal space. Then* $(\mathfrak{g}, B)$ *is i-isomorphic to the generalized double extension of the quadratic Lie superalgebra* $(\mathfrak{g}_1, B_1)$ *by* $\mathfrak{g}_2$, *where* $\mathfrak{g}_1 := \mathfrak{J}^\perp/\mathfrak{J}$, B_1 *is the quadratic form induced by the restriction of* B *to* $\mathfrak{J}^\perp \times \mathfrak{J}^\perp$ *and* $\mathfrak{g}_2 := \mathfrak{g}/\mathfrak{J}^\perp$.

2.1. **Quadratic Lie superalgebras of dimension 4.** Let $\mathfrak{g}$ be a quadratic Lie superalgebra. If $\dim(\mathfrak{g}_{\bar{0}}) \leq 1$ then $\mathfrak{g}$ is Abelian [DU14]. Therefore, in the non-Abelian four-dimensional case, we only consider $\dim(\mathfrak{g}_{\bar{1}}) = 0$ or $\dim(\mathfrak{g}_{\bar{0}}) = \dim(\mathfrak{g}_{\bar{1}}) = 2$.

If $\dim(\mathfrak{g}_{\bar{1}}) = 0$ then $\mathfrak{g}$ is isomorphic to $\mathfrak{o}(3) \oplus \mathbb{C}$ or the diamond Lie algebra $\mathfrak{g}_4$.

If $\dim(\mathfrak{g}_{\bar{0}}) = \dim(\mathfrak{g}_{\bar{1}}) = 2$ then there is a classification up to i-isomorphism given in [DU14] as follows:

(1) $\mathfrak{g}^s_{4,1} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = \mathrm{span}\{X_{\bar{0}}, Y_{\bar{0}}\}$ and $\mathfrak{g}_{\bar{1}} = \mathrm{span}\{X_{\bar{1}}, Y_{\bar{1}}\}$ such that the non-zero bilinear form $B(X_{\bar{0}}, Y_{\bar{0}}) = B(X_{\bar{1}}, Y_{\bar{1}}) = 1$ and the non-trivial Lie super-bracket $[Y_{\bar{1}}, Y_{\bar{1}}] = -2X_{\bar{0}}$, $[Y_{\bar{0}}, Y_{\bar{1}}] = -2X_{\bar{1}}$.

(2) $\mathfrak{g}^s_{4,2} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = \mathrm{span}\{X_{\bar{0}}, Y_{\bar{0}}\}$ and $\mathfrak{g}_{\bar{1}} = \mathrm{span}\{X_{\bar{1}}, Y_{\bar{1}}\}$ such that the non-zero bilinear form $B(X_{\bar{0}}, Y_{\bar{0}}) = B(X_{\bar{1}}, Y_{\bar{1}}) = 1$ and the non-trivial Lie super-bracket $[X_{\bar{1}}, Y_{\bar{1}}] = X_{\bar{0}}$, $[Y_{\bar{0}}, X_{\bar{1}}] = X_{\bar{1}}$, $[Y_{\bar{0}}, Y_{\bar{1}}] = -Y_{\bar{1}}$.

Remark 2.6. We can obtain the above result by the method of double extension: $\mathfrak{g}$ can be seen as a double extension of the two-dimensional symplectic vector space $\mathfrak{q}$ by a map $C \in \mathfrak{sp}(\mathfrak{q})$. Since there are only two cases —C nilpotent or C semisimple— the result follows.

2.2. **Quadratic Lie superalgebras of dimension 5.** Let $\mathfrak{g}$ be a quadratic Lie superalgebra of dimension 5. For the non-Abelian case, we only consider two situations: $\dim(\mathfrak{g}_{\bar{1}}) = 0$ and $\dim(\mathfrak{g}_{\bar{1}}) = 2$. The case $\dim(\mathfrak{g}_{\bar{1}}) = 0$ is shown in [DPU12], Appendix B, where there are three non-isomorphic Lie algebras. There is only one indecomposable that is the five-dimensional nilpotent quadratic Lie algebra $\mathfrak{g}_5$.

Assume $\dim(\mathfrak{g}_{\bar{1}}) = 2$ and $\mathfrak{g}$ is indecomposable. First we recall a result in [BB99] that is useful for our classification as follows.

Proposition 2.7. *Let* $\mathfrak{g}$ *be an indecomposable quadratic Lie superalgebra such that* $\dim(\mathfrak{g}_{\bar{1}}) = 2$ *and* $\mathscr{Z}(\mathfrak{g}_{\bar{0}}) \neq \{0\}$, *where* $\mathscr{Z}(\mathfrak{g}_{\bar{0}}) = \{X \in \mathfrak{g}_{\bar{0}} \mid [X,Y] = 0$ *for all* $Y \in \mathfrak{g}_{\bar{0}}\}$. *Then* $\mathscr{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} \neq \{0\}$. *Moreover, if* $\dim([\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]) \geq 2$ *then* $\mathscr{Z}(\mathfrak{g}_{\bar{0}}) \subset \mathscr{Z}(\mathfrak{g})$.

If $\mathfrak{g}_{\bar{0}}$ is solvable then $\mathfrak{g}_{\bar{0}}$ must be Abelian [PU07]. Using Proposition 2.7, we prove the following lemma.

Lemma 2.8. *Let* $\mathfrak{g}$ *be an indecomposable quadratic Lie superalgebra such that* $\mathfrak{g}_{\bar{0}} \neq \{0\}$. *Assume that* $\mathfrak{g}_{\bar{0}}$ *is Abelian and* $\dim(\mathfrak{g}_{\bar{1}}) = 2$. *Then* $\dim(\mathfrak{g}_{\bar{0}}) = 2$.

By the above lemma and by $\mathfrak{g}$ indecomposable, $\mathfrak{g}_{\bar{0}}$ must be not solvable and then $\mathfrak{g}_{\bar{0}} \simeq \mathfrak{o}(3)$. Choose a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}_{\bar{0}}$ such that $B(X_i, X_j) = \delta_{ij}$, $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$ and $[X_3, X_1] = X_2$. Recall that the adjoint representation ad is a homomorphism from $\mathfrak{g}_{\bar{0}}$ onto $\mathfrak{sp}(\mathfrak{q}_{\bar{1}})$. If it is not an isomorphism then we can assume that $\mathrm{ad}(X_1) = x\,\mathrm{ad}(X_2) + y\,\mathrm{ad}(X_3)$; then

$$\mathrm{ad}(X_3) = [\mathrm{ad}(X_1), \mathrm{ad}(X_2)] = -y\,\mathrm{ad}(X_1)$$

and

$$\mathrm{ad}(X_2) = [\mathrm{ad}(X_3), \mathrm{ad}(X_1)] = -x\,\mathrm{ad}(X_1).$$

That implies $\mathrm{ad}(X_2) = [\mathrm{ad}(X_1), \mathrm{ad}(X_3)] = 0$. Similarly, $\mathrm{ad}(X_3) = 0$ and so $\mathrm{ad}(X_1) = 0$. This implies that $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \{0\}$, which is a contradiction since $\mathfrak{g}$

is indecomposable. Therefore, the map ad must be an isomorphism from $\mathfrak{g}_{\overline{0}}$ onto $\mathfrak{sp}(\mathfrak{q}_{\overline{1}})$ and $\mathfrak{g} \simeq \mathfrak{osp}(1,2)$.

2.3. **Solvable quadratic Lie superalgebras of dimension 6.** If $\dim(\mathfrak{g}_{\overline{1}}) = 0$ then the classification in the solvable case is given in Proposition 1.17. We only consider two non trivial cases: $\dim(\mathfrak{g}_{\overline{1}}) = 2$ and $\dim(\mathfrak{g}_{\overline{1}}) = 4$.

2.3.1. $\dim(\mathfrak{g}_{\overline{1}}) = 2$. We assume that $\mathfrak{g}$ is indecomposable. By Lemma 2.8, $\mathfrak{g}_{\overline{0}}$ is non-Abelian. If $\mathfrak{g}_{\overline{0}}$ is not solvable then $\mathfrak{g}_{\overline{0}} = \mathfrak{s} \oplus \mathfrak{r}$ with $\mathfrak{s}$ semi-simple, $\mathfrak{r}$ the radical of $\mathfrak{g}$ and $\mathfrak{s} \simeq \mathfrak{sl}(2)$. Since $B([\mathfrak{s}, \mathfrak{s}^\perp], \mathfrak{s}) = B([\mathfrak{s}, \mathfrak{s}], \mathfrak{s}^\perp) = B(\mathfrak{s}, \mathfrak{s}^\perp) = 0$, then $\mathrm{ad} : \mathfrak{s} \to \mathrm{End}(\mathfrak{s}^\perp, \mathfrak{s}^\perp)$ is a one-dimensional representation of $\mathfrak{s}$. Hence, $\mathrm{ad}(\mathfrak{s})|_{\mathfrak{s}^\perp} = 0$ and $[\mathfrak{s}^\perp, \mathfrak{s}] = \{0\}$. This implies that $\mathfrak{s}$ and $\mathfrak{s}^\perp$ are non-degenerate ideals of $\mathfrak{g}_{\overline{0}}$. Note that $\mathfrak{s}^\perp = \mathscr{Z}(\mathfrak{g}_{\overline{0}})$. By Proposition 2.7, $\mathscr{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\overline{0}} \neq \{0\}$ so $\mathfrak{s}^\perp \subset \mathscr{Z}(\mathfrak{g})$ and then $\mathfrak{g}$ is decomposable. This is a contradiction. Therefore, $\mathfrak{g}_{\overline{0}}$ is solvable and i-isomorphic to the diamond Lie algebra $\mathfrak{g}_4$ given in Example 1.7. Particularly, $\mathfrak{g}_{\overline{0}} = \mathrm{span}\{X, P, Q, Z\}$ such that $B(X, Z) = B(P, Q) = 1$, $B(X, Q) = B(P, Z) = 0$, the subspaces spanned by $\{X, P\}$ and $\{Q, Z\}$ are totally isotropic. The Lie bracket on $\mathfrak{g}_{\overline{0}}$ is defined by $[X, P] = P$, $[X, Q] = -Q$ and $[P, Q] = Z$. It is obvious that $(\mathscr{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\overline{0}}) \subset \mathscr{Z}(\mathfrak{g}_{\overline{0}})$. By Proposition 2.7, one has $(\mathscr{Z}(\mathfrak{g}) \cap \mathfrak{g}_{\overline{0}}) = \mathbb{C}Z$.

If $\mathrm{ad}(X)|_{\mathfrak{g}_{\overline{1}}} = 0$ then $\mathrm{ad}(P)|_{\mathfrak{g}_{\overline{1}}} = \mathrm{ad}([X, P])|_{\mathfrak{g}_{\overline{1}}} = [\mathrm{ad}(X), \mathrm{ad}(P)]|_{\mathfrak{g}_{\overline{1}}} = 0$. Similarly, $\mathrm{ad}(Q)|_{\mathfrak{g}_{\overline{1}}} = 0$ and then $[\mathfrak{g}_{\overline{0}}, \mathfrak{g}_{\overline{1}}] = 0$. This implies $[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] = 0$ by the invariance of B. This is a contradiction since $\mathfrak{g}$ is indecomposable. Hence, $\mathrm{ad}(X)|_{\mathfrak{g}_{\overline{1}}} \neq 0$.

Note that $\mathrm{ad}(X)|_{\mathfrak{g}_{\overline{1}}}$, $\mathrm{ad}(P)|_{\mathfrak{g}_{\overline{1}}}$ and $\mathrm{ad}(Q)|_{\mathfrak{g}_{\overline{1}}} \in \mathfrak{sp}(\mathfrak{g}_{\overline{1}}, B|_{\mathfrak{g}_{\overline{1}} \times \mathfrak{g}_{\overline{1}}}) = \mathfrak{sp}(2)$, where

$$\mathfrak{sp}(2) = \mathrm{span}\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}.$$

Set the subspace $V = \mathrm{span}\{\mathrm{ad}(X)|_{\mathfrak{g}_{\overline{1}}}, \mathrm{ad}(P)|_{\mathfrak{g}_{\overline{1}}}, \mathrm{ad}(Q)|_{\mathfrak{g}_{\overline{1}}}\}$ of $\mathfrak{sp}(2)$. Assume that

$$\mathrm{ad}(P)|_{\mathfrak{g}_{\overline{1}}} = a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\text{and} \quad \mathrm{ad}(Q)|_{\mathfrak{g}_{\overline{1}}} = a' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b' \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c' \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Since $[\mathrm{ad}(P)|_{\mathfrak{g}_{\overline{1}}}, \mathrm{ad}(Q)|_{\mathfrak{g}_{\overline{1}}}] = \mathrm{ad}([P, Q])|_{\mathfrak{g}_{\overline{1}}} = \mathrm{ad}(Z)|_{\mathfrak{g}_{\overline{1}}} = 0$, one has

$$\begin{cases} ab' - a'b = 0 \\ ac' - a'c = 0 \\ bc' - b'c = 0 \end{cases}$$

That means that $\mathrm{ad}(P)|_{\mathfrak{g}_{\overline{1}}}$ and $\mathrm{ad}(Q)|_{\mathfrak{g}_{\overline{1}}}$ are not linearly independent and then $1 \leq \dim(V) \leq 2$. We consider the two following cases:

(1) If $\dim(V) = 1$ then $\mathrm{ad}(P)|_{\mathfrak{g}_{\overline{1}}} = \alpha\, \mathrm{ad}(X)|_{\mathfrak{g}_{\overline{1}}}$ and $\mathrm{ad}(Q)|_{\mathfrak{g}_{\overline{1}}} = \beta\, \mathrm{ad}(X)|_{\mathfrak{g}_{\overline{1}}}$, where α, $\beta \in \mathbb{C}$. Since $\mathrm{ad}(P) = [\mathrm{ad}(X), \mathrm{ad}(P)]$ and $\mathrm{ad}(Q) = -[\mathrm{ad}(X), \mathrm{ad}(Q)]$, one has $\alpha = \beta = 0$. As a consequence, we obtain $B([\mathfrak{g}_{\overline{0}}, \mathfrak{g}_{\overline{0}}], [\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]) = B([[\mathfrak{g}_{\overline{0}}, \mathfrak{g}_{\overline{0}}], \mathfrak{g}_{\overline{1}}], \mathfrak{g}_{\overline{1}}) = 0$. This implies $[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}] \subset \mathbb{C}Z$ since $\mathbb{C}Z$ is the orthogonal complementary of $[\mathfrak{g}_{\overline{0}}, \mathfrak{g}_{\overline{0}}]$ in $\mathfrak{g}_{\overline{0}}$. It is easy to see that $\mathfrak{g}$ is the double extension of the quadratic $\mathbb{Z}_2$-graded vector space $\mathfrak{q} = (\mathbb{C}X \oplus \mathbb{C}Z)^\perp$ of

dimension 4 by the map $C = \mathrm{ad}(X)|_{\mathfrak{q}}$. Write $\mathfrak{q} = \mathfrak{q}_{\bar{0}} \oplus \mathfrak{q}_{\bar{1}}$ and then C is a linear map in $\mathfrak{o}(\mathfrak{q}_{\bar{0}}) \oplus \mathfrak{sp}(\mathfrak{q}_{\bar{1}})$ defined by:

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & a \\ 0 & 0 & b & -\lambda \end{pmatrix}$$

where $C_{\bar{0}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{o}(\mathfrak{q}_{\bar{0}})$ and $C_{\bar{1}} = \begin{pmatrix} \lambda & a \\ b & -\lambda \end{pmatrix} \in \mathfrak{sp}(\mathfrak{q}_{\bar{1}})$. By [DU14], let $\mathfrak{g}$ and $\mathfrak{g}'$ be two double extensions of $\mathfrak{q}$ by C and C' then $\mathfrak{g}$ and $\mathfrak{g}'$ is i-isomorphic if and only if $C_{\bar{0}}$ and $\mu C'_{\bar{0}}$ are on the same $\mathrm{O}(\mathfrak{q}_{\bar{0}})$-orbit of $\mathfrak{o}(\mathfrak{q}_{\bar{0}})$, $C_{\bar{1}}$ and $\mu C'_{\bar{1}}$ are on the same $\mathrm{Sp}(\mathfrak{q}_{\bar{1}})$-orbit of $\mathfrak{sp}(\mathfrak{q}_{\bar{1}})$ for some nonzero $\mu \in \mathbb{C}$. Therefore, we have only two non i-isomorphic families of Lie superalgebras corresponding to matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}, \quad \lambda \neq 0.$$

As a consequence, we obtain Lie superalgebras as follows:

(i) $\mathfrak{g}^s_{6,1}$: $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_4$, $[X, Y_{\bar{1}}] = X_{\bar{1}}$, $[Y_{\bar{1}}, Y_{\bar{1}}] = Z$,

(ii) $\mathfrak{g}^s_{6,2}(\lambda)$: $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_4$, $[X, X_{\bar{1}}] = \lambda X_{\bar{1}}$, $[X, Y_{\bar{1}}] = -\lambda Y_{\bar{1}}$ and $[X_{\bar{1}}, Y_{\bar{1}}] = \lambda Z$,

where $\mathfrak{g}_{\bar{1}} = \mathrm{span}\{X_{\bar{1}}, Y_{\bar{1}}\}$ and $B(X_{\bar{1}}, Y_{\bar{1}}) = 1$. For (ii), $\mathfrak{g}^s_{6,2}(\lambda_1)$ and $\mathfrak{g}^s_{6,2}(\lambda_2)$ are i-isomorphic if and only if $\lambda_1 = \lambda_2$. Moreover, two isomorphic and i-isomorphic notions are equivalent.

(2) If $\dim(V) = 2$, assume that $\mathrm{ad}(X)|_{\mathfrak{g}_{\bar{1}}} = x\,\mathrm{ad}(P)|_{\mathfrak{g}_{\bar{1}}} + y\,\mathrm{ad}(Q)|_{\mathfrak{g}_{\bar{1}}}$, where $x, y \in \mathbb{C}$, then $\mathrm{ad}(P)|_{\mathfrak{g}_{\bar{1}}} = [\mathrm{ad}(X), \mathrm{ad}(P)]|_{\mathfrak{g}_{\bar{1}}} = 0$. Similarly, $\mathrm{ad}(Q)|_{\mathfrak{g}_{\bar{1}}} = 0$. This implies that $\mathrm{ad}(X)|_{\mathfrak{g}_{\bar{1}}} = 0$. This is a contradiction. Therefore, we can assume that $\mathrm{ad}(Q)|_{\mathfrak{g}_{\bar{1}}} = x\,\mathrm{ad}(X)|_{\mathfrak{g}_{\bar{1}}} + y\,\mathrm{ad}(P)|_{\mathfrak{g}_{\bar{1}}}$ (since $\mathrm{ad}(P)|_{\mathfrak{g}_{\bar{1}}}$ and $\mathrm{ad}(Q)|_{\mathfrak{g}_{\bar{1}}}$ play the same role). One has:

$$0 = [\mathrm{ad}(P)|_{\mathfrak{g}_{\bar{1}}}, \mathrm{ad}(Q)|_{\mathfrak{g}_{\bar{1}}}] = x[\mathrm{ad}(P)|_{\mathfrak{g}_{\bar{1}}}, \mathrm{ad}(X)|_{\mathfrak{g}_{\bar{1}}}] = -x\,\mathrm{ad}(P)|_{\mathfrak{g}_{\bar{1}}}.$$

Hence, $x = 0$ and $\mathrm{ad}(Q)|_{\mathfrak{g}_{\bar{1}}} = y\,\mathrm{ad}(P)|_{\mathfrak{g}_{\bar{1}}}$.

On the other hand, we have

$$\mathrm{ad}(Q)|_{\mathfrak{g}_{\bar{1}}} = [\mathrm{ad}(Q)|_{\mathfrak{g}_{\bar{1}}}, \mathrm{ad}(X)|_{\mathfrak{g}_{\bar{1}}}] = -y\,\mathrm{ad}(P)|_{\mathfrak{g}_{\bar{1}}}.$$

That means that $y = 0$ and then $\mathrm{ad}(Q)|_{\mathfrak{g}_{\bar{1}}} = 0$.

We need the following lemma. Its proof is lengthy but straightforward so we omit it.

Lemma 2.9. *Let A, B be two nonzero linear operators in $\mathfrak{sp}(V)$, where $\dim(V) = 2$, such that $[A, B] = B$. Then A is semi-simple and B is nilpotent.*

Applying this lemma, we choose a basis $\{X_{\overline{1}}, Y_{\overline{1}}\}$ of $\mathfrak{g}_{\overline{1}}$ such that $B(X_{\overline{1}}, Y_{\overline{1}}) = 1$; then $\mathrm{ad}(X)|_{\mathfrak{g}_{\overline{1}}} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$ and $\mathrm{ad}(P)|_{\mathfrak{g}_{\overline{1}}} = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}$, where $\mu \neq 0$.

By the invariancy of B, we have $[X_{\overline{1}}, Y_{\overline{1}}] = \frac{1}{2}Z$, $[Y_{\overline{1}}, Y_{\overline{1}}] = \mu Q$. Set $P' := \frac{P}{\mu}$ and $Q' := \mu Q$; then we have the following Lie superalgebra $\mathfrak{g}^s_{6,3}$: $\mathfrak{g}_{\overline{0}} = \mathfrak{g}_4$, $\mathrm{ad}(X)|_{\mathfrak{g}_{\overline{1}}} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, $\mathrm{ad}(P)|_{\mathfrak{g}_{\overline{1}}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $[X_{\overline{1}}, Y_{\overline{1}}] = \frac{1}{2}Z$ and $[Y_{\overline{1}}, Y_{\overline{1}}] = Q$.

Remark 2.10. It is easy to see that the Lie superalgebra $\mathfrak{g}^s_{6,1}$ is not isomorphic to $\mathfrak{g}^s_{6,2}(\lambda)$ and $\mathfrak{g}^s_{6,3}$ by checking the dimension of $[\mathfrak{g}, \mathfrak{g}]$. Also, the dimension of $[\mathfrak{g}_{\overline{1}}, \mathfrak{g}_{\overline{1}}]$ shows that $\mathfrak{g}^s_{6,3}$ is not isomorphic to $\mathfrak{g}^s_{6,2}(\lambda)$.

2.3.2. $\dim(\mathfrak{g}_{\overline{1}}) = 4$. In this case, $\dim(\mathfrak{g}_{\overline{0}}) = 2$ and if $\mathfrak{g}$ is non-Abelian then $\mathfrak{g}$ is a double extension [DU14]. Assume $\mathfrak{g}$ indecomposable. We can choose a basis $\{X_{\overline{0}}, Y_{\overline{0}}\}$ of $\mathfrak{g}_{\overline{0}}$ and a canonical basis $\{X_1, X_2, Y_1, Y_2\}$ of $\mathfrak{g}_{\overline{1}}$ such that $B(X_i, X_j) = B(Y_i, Y_j) = 0$, $B(X_i, Y_j) = \delta_{ij}$, $i, j = 1, 2$, such that $X_{\overline{0}} \in \mathscr{Z}(\mathfrak{g})$ and $C := \mathrm{ad}(Y_{\overline{0}})|_{\mathfrak{g}_{\overline{1}}} \in \mathfrak{sp}(\mathfrak{g}_{\overline{1}}, B|_{\mathfrak{g}_{\overline{1}} \times \mathfrak{g}_{\overline{1}}}) = \mathfrak{sp}(4)$. Moreover, the isomorphic classification of $\mathfrak{g}$ reduces to the classification of $\mathrm{Sp}(\mathfrak{g}_{\overline{1}})$-orbits of $\mathbb{P}^1(\mathfrak{sp}(\mathfrak{g}_{\overline{1}}))$. In other words, the isomorphic classification of $\mathfrak{g}$ follows the classification of linear maps $C \in \mathfrak{sp}(\mathfrak{g}_{\overline{1}})$ up to conjugation and up to non-zero multiples. In particular, if C is nilpotent then:

- $C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$; this corresponds to the partition $[2^2]$ of 4 (see details in [CM93]).

If C is not nilpotent, applying the classification of $\mathrm{Sp}(4)$-orbits of $\mathbb{P}^1(\mathfrak{sp}(4))$ in [DU14] one has three cases:

- $C = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$,
- $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}$, $\lambda \neq 0$,
- $C = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$

in a canonical basis $\{X_1, X_2, Y_1, Y_2\}$ of $\mathfrak{g}_{\overline{1}}$.

We obtain corresponding Lie superalgebras:

- $\mathfrak{g}^s_{6,4}$: $[Y_{\overline{0}}, X_2] = X_1$, $[Y_{\overline{0}}, Y_1] = -Y_2$ and $[X_2, Y_1] = X_{\overline{0}}$.

- $\mathfrak{g}^s_{6,5}$: $[Y_{\bar{0}}, X_2] = X_2$, $[Y_{\bar{0}}, Y_1] = X_1$, $[Y_{\bar{0}}, Y_2] = -Y_2$ and $[Y_1, Y_1] = [X_2, Y_2] = X_{\bar{0}}$.
- $\mathfrak{g}^s_{6,6}(\lambda)$: $[Y_{\bar{0}}, X_1] = X_1$, $[Y_{\bar{0}}, X_2] = \lambda X_2$, $[Y_{\bar{0}}, Y_1] = -Y_1$, $[Y_{\bar{0}}, Y_2] = -\lambda Y_2$, $[X_1, Y_1] = X_{\bar{0}}$ and $[X_2, Y_2] = \lambda X_{\bar{0}}$. In this case, $\mathfrak{g}^s_{6,6}(\lambda_1)$ is i-isomorphic to $\mathfrak{g}^s_{6,6}(\lambda_2)$ if and only if there exists $\mu \in \mathbb{C}$ nonzero such that $C(\lambda_1)$ is in the $\mathrm{Sp}(\mathfrak{g}_{\bar{1}})$-adjoint orbit through $\mu C(\lambda_2)$. That happens if and only if $\lambda_1 = \pm\lambda_2$ or $\lambda_2 = \pm{\lambda_1}^{-1}$.
- $\mathfrak{g}^s_{6,7}$: $[Y_{\bar{0}}, X_1] = X_1$, $[Y_{\bar{0}}, X_2] = X_2 + X_1$, $[Y_{\bar{0}}, Y_1] = -Y_1 - Y_2$, $[Y_{\bar{0}}, Y_2] = -Y_1$ and $[X_1, Y_1] = [X_2, Y_1] = [X_2, Y_2] = X_{\bar{0}}$.

We remark that the quadratic Lie superalgebras $\mathfrak{g}^s_{4,1}$ and $\mathfrak{g}^s_{6,4}$ have been mentioned in [ABB09] as examples of quadratic Lie superalgebras with a reductive even part and the action of the even part on the odd part not completely reducible.

3. Odd quadratic Lie superalgebras

Definition 3.1. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. If there is a non-degenerate supersymmetric bilinear form B on $\mathfrak{g}$ such that B is odd and invariant then the pair $(\mathfrak{g}, B)$ is called an *odd quadratic Lie superalgebra.*

We have the following proposition.

Proposition 3.2. *Let $\mathfrak{g}$ be a Lie algebra and $\phi : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}$ a symmetric bilinear map satisfying two conditions:*

(1) $\mathrm{ad}(X)\left(\phi(f,g)\right) + \phi\left(f, g \circ \mathrm{ad}(X)\right) + \phi\left(g, f \circ \mathrm{ad}(X)\right) = 0,$

(2) $f \circ \mathrm{ad}\left(\phi(g,h)\right) + \mathrm{cycle}(f,g,h) = 0$ *for all* $X \in \mathfrak{g}$ *and* $f, g \in \mathfrak{g}^*$.

Then the vector space $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{g}^*$ with the bracket

$$[X+f, Y+g] = [X,Y] + \mathrm{ad}^*(X)(g) - \mathrm{ad}^*(Y)(f) + \phi(f,g)$$

for all $X, Y \in \mathfrak{g}$, $f, g \in \mathfrak{g}^*$ is a Lie superalgebra and it is called the T^*_s-extension of $\mathfrak{g}$ by means of ϕ. *Moreover, if ϕ satisfies the* **cyclic** *condition:* $h\left(\phi(f,g)\right) = f\left(\phi(g,h)\right)$ *for all $f, g, h \in \mathfrak{g}^*$, then $\mathfrak{g}$ is odd quadratic with the bilinear form*

$$\overline{B}(X+f, Y+g) = f(Y) + g(X), \quad \forall\, X, Y \in \mathfrak{g},\ f, g \in \mathfrak{g}^*.$$

Example 3.3. Let $\mathfrak{g}$ be a two-dimensional Lie algebra spanned by $\{X, Y\}$ with $[X, Y] = Y$. Assume that ϕ is a symmetric bilinear form satisfying the conditions as in Proposition 3.2. It is easy to compute that in this case ϕ must be zero and then the T^*_s-extension of $\mathfrak{g}$ coincides with the semidirect product of $\mathfrak{g}$ and $\mathfrak{g}^*$ with the Lie bracket defined by $[X, Y] = Y$, $[X, Y^*] = -Y^*$ and $[Y, Y^*] = X^*$. This is regarded as a "superalgebra" type of the diamond Lie algebra.

Example 3.4. If $\mathfrak{g}$ spanned by $\{X, Y\}$ is the two-dimensional Abelian Lie algebra, we can check easily that every ϕ having the above properties must be defined as follows:

$$\phi(X^*, X^*) = \alpha X + \beta Y, \quad \phi(X^*, Y^*) = \beta X + \gamma Y, \quad \phi(Y^*, Y^*) = \gamma X + \lambda Y,$$

where $\alpha,\ \beta,\ \gamma,\ \lambda \in \mathbb{C}$. In this case, the T^*_s-extension of $\mathfrak{g}$ by means of ϕ is defined by $[X^*, X^*] = \alpha X + \beta Y$, $[X^*, Y^*] = \beta X + \gamma Y$ and $[Y^*, Y^*] = \gamma X + \lambda Y$.

Actually, this is a "Lie superalgebra" type of 2-nilpotent commutative algebras of dimension 4. Such algebras will be completely classified in Subsection 3.2.

Proposition 3.5. *Let* $\mathfrak{g}$ *be an odd quadratic Lie superalgebra; then* $\mathfrak{g}$ *is i-isomorphic to a* T_s^**-extension of* $\mathfrak{g}_{\bar{0}}$.

Proof. Assume that $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is an odd quadratic Lie superalgebra with the bilinear form B. We can identify $\mathfrak{g}_{\bar{1}}$ with $\mathfrak{g}_{\bar{0}}^*$ by the bilinear form B. Set the symmetric bilinear map $\phi : \mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}} \to \mathfrak{g}_{\bar{0}}$ by $\phi(f,g) := [f,g]$ for all $f,g \in \mathfrak{g}_{\bar{1}}$. Then it is easy to check that ϕ satisfies the conditions in Proposition 3.2 and $\mathfrak{g}$ is the T_s^*-extension of $\mathfrak{g}_{\bar{0}}$ by means of ϕ. □

Note that we can approach to odd quadratic Lie superalgebras by other constructions, namely *odd double extensions* and *generalized odd double extensions* of odd quadratic Lie superalgebras. For more details, the reader is referred to [ABB10].

3.1. **Odd quadratic Lie superalgebras of dimension 2.** We recall the classification of odd quadratic Lie superalgebras of dimension 2 in [ABB10] as follows. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = \mathbb{C}X_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}} = \mathbb{C}X_{\bar{1}}$. Define an odd bilinear form on $\mathfrak{g}$ by $B(X_{\bar{0}}, X_{\bar{1}}) = 1$. Then $\mathfrak{g}$ is Abelian or isomorphic to $\mathfrak{g}_2^o(\lambda)$, where the nonzero Lie bracket on $\mathfrak{g}_2^o(\lambda)$ is given by: $[X_{\bar{1}}, X_{\bar{1}}] = \lambda X_{\bar{1}}$.

By a straightforward checking, we can see that $\mathfrak{g}_2^o(1)$ is isomorphic to $\mathfrak{g}_2^o(\lambda)$. Moreover, they are also i-isomorphic by the following i-isomorphism: $A(X_{\bar{0}}) = \lambda^{\frac{1}{3}} X_{\bar{0}}$ and $A(X_{\bar{1}}) = \lambda^{-\frac{1}{3}} X_{\bar{1}}$. Actually, these algebras were classified in [DU10], Example 3.18 in terms of T^*-extension of the one-dimensional Abelian algebra.

3.2. **Odd quadratic Lie superalgebras of dimension 4.** Now, let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be an odd quadratic Lie superalgebra, where $\mathfrak{g}_{\bar{0}} = \mathbb{C}X_{\bar{0}} \oplus \mathbb{C}Y_{\bar{0}}$ and $\mathfrak{g}_{\bar{1}} = \mathbb{C}X_{\bar{1}} \oplus \mathbb{C}Y_{\bar{1}}$. If $\mathfrak{g}_{\bar{0}}$ is Abelian then $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \{0\}$ by the invariance of B. This case has just been classified in [DU10]. In particular, if $\mathfrak{g}$ is non-Abelian then it is i-isomorphic to each of the following algebras:

- $\mathfrak{g}_{4,1}^o$: $[X_{\bar{1}}, X_{\bar{1}}] = Y_{\bar{0}}$, $[X_{\bar{1}}, Y_{\bar{1}}] = X_{\bar{0}}$,
- $\mathfrak{g}_{4,2}^o$: $[X_{\bar{1}}, X_{\bar{1}}] = Y_{\bar{0}}$, $[X_{\bar{1}}, Y_{\bar{1}}] = X_{\bar{0}} + Y_{\bar{0}}$ and $[Y_{\bar{1}}, Y_{\bar{1}}] = X_{\bar{0}}$.

Note that $\mathfrak{g}_{4,1}^o$ and $\mathfrak{g}_{4,2}^o$ are not isomorphic.

If $\mathfrak{g}_{\bar{0}}$ is non-Abelian then we can assume $[X_{\bar{0}}, Y_{\bar{0}}] = Y_{\bar{0}}$. By $B(X_{\bar{0}}, [X_{\bar{0}}, \mathfrak{g}_{\bar{1}}]) = 0$, $[X_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subset \mathbb{C}Y_{\bar{1}}$. Moreover, $B(X_{\bar{1}}, [\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]) = 0$ implies that $[X_{\bar{1}}, \mathfrak{g}_{\bar{0}}] = \{0\}$. If $[X_{\bar{0}}, Y_{\bar{1}}] = \alpha Y_{\bar{1}}$ then $\alpha = B(Y_{\bar{0}}, [X_{\bar{0}}, Y_{\bar{1}}]) = B([Y_{\bar{0}}, X_{\bar{0}}], Y_{\bar{1}}) = -1$. Similarly, one obtains $[Y_{\bar{0}}, Y_{\bar{1}}] = X_{\bar{1}}$.

Now, we continue considering the Lie bracket on $\mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}}$ as follows. Assume that $[X_{\bar{1}}, X_{\bar{1}}] = \beta_1 X_{\bar{0}} + \gamma_1 Y_{\bar{0}}$. By the Jacobi identity and by $[X_{\bar{1}}, \mathfrak{g}_{\bar{0}}] = \{0\}$, $[X_0, [X_{\bar{1}}, X_{\bar{1}}]] = 0$ and $[Y_0, [X_{\bar{1}}, X_{\bar{1}}]] = 0$. Therefore, $\beta_1 = \gamma_1 = 0$. Similarly, $[X_{\bar{1}}, Y_{\bar{1}}] = [X_{\bar{1}}, Y_{\bar{1}}]] = 0$.

Finally, we obtain the nonzero Lie bracket on $\mathfrak{g}$ when $\mathfrak{g}_{\bar{0}}$ is non-Abelian that $[X_{\bar{0}}, Y_{\bar{0}}] = Y_{\bar{0}}$, $[X_{\bar{0}}, Y_{\bar{1}}] = -Y_{\bar{1}}$ and $[Y_{\bar{0}}, Y_{\bar{1}}] = X_{\bar{1}}$. It is easy to see that $\mathfrak{g}$ is the diamond Lie algebra $\mathfrak{g}_4$.

3.3. **Solvable odd quadratic Lie superalgebras of dimension 6.** Let $\mathfrak{g}$ be a solvable odd quadratic Lie superalgebra of dimension 6, where $\mathfrak{g}_{\bar{0}} = \operatorname{span}\{X_{\bar{0}}, Y_{\bar{0}}, Z_{\bar{0}}\}$ and $\mathfrak{g}_{\bar{1}} = \operatorname{span}\{X_{\bar{1}}, Y_{\bar{1}}, Z_{\bar{1}}\}$ such that $B(X_{\bar{0}}, X_{\bar{1}}) = B(Y_{\bar{0}}, Y_{\bar{1}}) = B(Z_{\bar{0}}, Z_{\bar{1}}) = 1$, the others are zero. If $\mathfrak{g}_{\bar{0}}$ is Abelian then $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \{0\}$. In this case, $\mathfrak{g}$ can be seen as a commutative algebra and the classification can be reduced to the classification of ternary cubic forms [DU10]. We will consider the case of $\mathfrak{g}_{\bar{0}}$ non-Abelian. Recall here three complex solvable Lie algebras of dimension 3 as follows:

- $\mathfrak{g}_{3,1}$: $[X_{\bar{0}}, Y_{\bar{0}}] = Z_{\bar{0}}$,
- $\mathfrak{g}_{3,2}$: $[X_{\bar{0}}, Y_{\bar{0}}] = Y_{\bar{0}}$, $[X_{\bar{0}}, Z_{\bar{0}}] = Y_{\bar{0}} + Z_{\bar{0}}$,
- $\mathfrak{g}_{3,3}$: $[X_{\bar{0}}, Y_{\bar{0}}] = Y_{\bar{0}}$, $[X_{\bar{0}}, Z_{\bar{0}}] = \mu Z_{\bar{0}}$, where $|\mu| \leq 1$.

3.3.1. *Case 1:* $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_{3,1}$. Since $[Z_{\bar{0}}, \mathfrak{g}_{\bar{0}}] = \{0\}$, one has $[Z_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \{0\}$ and $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] \subset \mathbb{C}X_{\bar{1}} \oplus \mathbb{C}Y_{\bar{1}}$ by the invariance of the bilinear form B. Moreover, $B([\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}], \mathbb{C}X_{\bar{1}} \oplus \mathbb{C}Y_{\bar{1}}) = 0$ implies that $[\mathfrak{g}_{\bar{0}}, \mathbb{C}X_{\bar{1}} \oplus \mathbb{C}Y_{\bar{1}}] = \{0\}$. As a consequence of the Jacobi identity, one has

$$[\mathbb{C}X_{\bar{1}} \oplus \mathbb{C}Y_{\bar{1}}, \mathbb{C}X_{\bar{1}} \oplus \mathbb{C}Y_{\bar{1}}] \subset \mathscr{Z}(\mathfrak{g}_{\bar{0}}) = \mathbb{C}Z_{\bar{0}}.$$

By a straightforward computation, $[Y_{\bar{0}}, Z_{\bar{1}}] = X_{\bar{1}}$ and $[X_{\bar{0}}, Z_{\bar{1}}] = -Y_{\bar{1}}$. Now, we assume that $[X_{\bar{1}}, X_{\bar{1}}] = \alpha Z_{\bar{0}}$, $[X_{\bar{1}}, Y_{\bar{1}}] = \beta Z_{\bar{0}}$, $[Y_{\bar{1}}, Y_{\bar{1}}] = \gamma Z_{\bar{0}}$ and $[Z_{\bar{1}}, Z_{\bar{1}}] = xX_{\bar{0}} + yY_{\bar{0}} + zZ_{\bar{0}}$. By the Jacobi identity, one has $[X_{\bar{0}}, [Z_{\bar{1}}, Z_{\bar{1}}]] + [Z_{\bar{1}}, [Z_{\bar{1}}, X_{\bar{0}}]] - [Z_{\bar{1}}, [X_{\bar{0}}, Z_{\bar{1}}]] = 0$. Therefore, $[Z_{\bar{1}}, Y_{\bar{1}}] = -\frac{y}{2} Z_{\bar{0}}$. Similarly, $[Z_{\bar{1}}, X_{\bar{1}}] = -\frac{x}{2} Z_{\bar{0}}$. By the invariance of B, we obtain $\alpha = \beta = \gamma = 0$ and $x = y = 0$.

It results that $[X_{\bar{0}}, Y_{\bar{0}}] = Z_{\bar{0}}$, $[Y_{\bar{0}}, Z_{\bar{1}}] = X_{\bar{1}}$, $[X_{\bar{0}}, Z_{\bar{1}}] = -Y_{\bar{1}}$ and $[Z_{\bar{1}}, Z_{\bar{1}}] = \lambda Z_{\bar{0}}$.

Remark 3.6. If $\lambda = 0$ then the Lie superalgebra $\mathfrak{g}$ is the six-dimensional quadratic Lie algebra $\mathfrak{g}_{6,1}$ given in Proposition 1.17. It is easy to check that if $\lambda \neq 0$ then $\mathfrak{g}(\lambda)$ is isomorphic to $\mathfrak{g}(1)$ by the following isomorphism A:

$$A(X_{\bar{0}}) = X_{\bar{0}}, \quad A\left(\frac{Y_{\bar{0}}}{\lambda}\right) = Y_{\bar{0}}, \quad A\left(\frac{Z_{\bar{0}}}{\lambda}\right) = Z_{\bar{0}},$$

$$A\left(\frac{X_{\bar{1}}}{\lambda^2}\right) = X_{\bar{1}}, \quad A\left(\frac{Y_{\bar{1}}}{\lambda}\right) = Y_{\bar{1}}, \quad A\left(\frac{Z_{\bar{1}}}{\lambda}\right) = Z_{\bar{1}}.$$

However, we have a stronger result that $\mathfrak{g}(\lambda)$ is i-isomorphic to $\mathfrak{g}(\lambda')$ by the i-isomorphism defined by $C(X_{\bar{0}}) = aX_{\bar{0}} + Y_{\bar{0}}$, $C(Y_{\bar{0}}) = aX_{\bar{0}} + 2Y_{\bar{0}}$, $C(Z_{\bar{0}}) = aZ_{\bar{0}}$, $C(X_{\bar{1}}) = \frac{2}{a} X_{\bar{1}} - Y_{\bar{1}}$, $C(Y_{\bar{1}}) = -\frac{1}{a} X_{\bar{1}} + Y_{\bar{1}}$ and $C(Z_{\bar{1}}) = \frac{1}{a} Z_{\bar{1}}$, where $a = \left(\frac{\lambda'}{\lambda}\right)^{1/3}$.

3.3.2. *Case 2:* $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_{3,2}$. By the invariance of B, one has $[\mathfrak{g}_{\bar{0}}, X_{\bar{1}}] = \{0\}$, $[X_{\bar{0}}, Y_{\bar{1}}] = -Y_{\bar{1}} - Z_{\bar{1}}$, $[Y_{\bar{0}}, Y_{\bar{1}}] = X_{\bar{1}}$, $[Z_{\bar{0}}, Y_{\bar{1}}] = X_{\bar{1}}$, $[X_{\bar{0}}, Z_{\bar{1}}] = -Z_{\bar{1}}$, $[Y_{\bar{0}}, Z_{\bar{1}}] = 0$ and $[Z_{\bar{0}}, Z_{\bar{1}}] = X_{\bar{1}}$.

By the Jacobi identity and $[\mathfrak{g}_{\bar{0}}, X_{\bar{1}}] = \{0\}$, one has $[X_{\bar{1}}, X_{\bar{1}}] = 0$. Assume that $[X_{\bar{1}}, Y_{\bar{1}}] = aX_{\bar{0}} + bY_{\bar{0}} + cZ_{\bar{0}}$. By a straightforward computation, $[Y_{\bar{0}}, [X_{\bar{1}}, Y_{\bar{1}}]] + [X_{\bar{1}}, [Y_{\bar{1}}, Y_{\bar{0}}]] - [Y_{\bar{1}}, [Y_{\bar{0}}, X_{\bar{1}}]] = 0$ implies $a = 0$, and $[X_{\bar{0}}, [X_{\bar{1}}, Y_{\bar{1}}]] + [X_{\bar{1}}, [Y_{\bar{1}}, X_{\bar{0}}]] - [Y_{\bar{1}}, [X_{\bar{0}}, X_{\bar{1}}]] = 0$ implies $[X_{\bar{1}}, Z_{\bar{1}}] = -(2b + c)Y_{\bar{0}} - 2cZ_{\bar{0}}$. As a consequence, $[X_{\bar{0}}, [X_{\bar{1}}, Z_{\bar{1}}]] + [X_{\bar{1}}, [Z_{\bar{1}}, X_{\bar{0}}]] - [Z_{\bar{1}}, [X_{\bar{0}}, X_{\bar{1}}]] = 0$ reduces to $-4(b+c)Y_{\bar{0}} - 4cZ_{\bar{0}} = 0$. Thus, $b = c = 0$.

In a similar way using the Jacobi identity, one has $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}$.

3.3.3. *Case 3:* $\mathfrak{g}_{\bar 0} = \mathfrak{g}_{3,3}$. Since $B([\mathfrak{g}_{\bar 0}, \mathfrak{g}_{\bar 0}], X_{\bar 1}) = 0$, one has $[\mathfrak{g}_{\bar 0}, X_{\bar 1}] = \{0\}$. By the invariance of B, it is easy to obtain the nonzero Lie brackets on $\mathfrak{g}_{\bar 0} \times \mathfrak{g}_{\bar 1}$ as follows:

$$[X_{\bar 0}, Y_{\bar 1}] = -Y_{\bar 1}, \quad [Y_{\bar 0}, Y_{\bar 1}] = X_{\bar 1}, \quad [X_{\bar 0}, Z_{\bar 1}] = -\mu Z_{\bar 1}, \quad \text{and} \quad [Z_{\bar 0}, Z_{\bar 1}] = \mu X_{\bar 1}.$$

(1) If $\mu = 0$ then $[\mathfrak{g}_{\bar 0}, \mathbb{C}X_{\bar 1} \oplus \mathbb{C}Z_{\bar 1}] = \{0\}$. By the Jacobi identity, $[\mathbb{C}X_{\bar 1} \oplus \mathbb{C}Z_{\bar 1}, \mathbb{C}X_{\bar 1} \oplus \mathbb{C}Z_{\bar 1}] \subset \mathscr{Z}(\mathfrak{g}_{\bar 0}) = \mathbb{C}Z_{\bar 0}$. We can assume that $[X_{\bar 1}, X_{\bar 1}] = \alpha Z_{\bar 0}$, $[X_{\bar 1}, Z_{\bar 1}] = \beta Z_{\bar 0}$, $[Z_{\bar 1}, Z_{\bar 1}] = \gamma Z_{\bar 0}$ and $[Y_{\bar 1}, Y_{\bar 1}] = xX_{\bar 0} + yY_{\bar 0} + zZ_{\bar 0}$. Since $[X_{\bar 0}, [Y_{\bar 1}, Y_{\bar 1}]] + [Y_{\bar 1}, [Y_{\bar 1}, X_{\bar 0}]] - [Y_{\bar 1}, [X_{\bar 0}, Y_{\bar 1}]] = 0$, one has $[Y_{\bar 1}, Y_{\bar 1}] = 0$. As a consequence, $[Y_{\bar 0}, [Y_{\bar 1}, Y_{\bar 1}]] + [Y_{\bar 1}, [Y_{\bar 1}, Y_{\bar 0}]] - [Y_{\bar 1}, [Y_{\bar 0}, Y_{\bar 1}]] = 0$ implies $[X_{\bar 1}, Y_{\bar 1}] = 0$. Similarly, by using the Jacobi identity for $X_{\bar 0}$, $Z_{\bar 1}$ and $Y_{\bar 1}$, we obtain $[Z_{\bar 1}, Y_{\bar 1}] = 0$.

Moreover, $[Y_{\bar 0}, [Y_{\bar 1}, Z_{\bar 1}]] + [Y_{\bar 1}, [Z_{\bar 1}, Y_{\bar 0}]] - [Z_{\bar 1}, [Y_{\bar 0}, Y_{\bar 1}]] = 0$ and $[X_{\bar 1}, [Y_{\bar 0}, Y_{\bar 1}]] + [Y_{\bar 0}, [Y_{\bar 1}, X_{\bar 1}]] + [Y_{\bar 1}, [Z_{\bar 1}, Y_{\bar 0}]] = 0$ imply that $\alpha = \beta = 0$. Summarily,

$$[X_{\bar 0}, Y_{\bar 0}] = Y_{\bar 0},\ [X_{\bar 0}, Y_{\bar 1}] = -Y_{\bar 1},\ [Y_{\bar 0}, Y_{\bar 1}] = X_{\bar 1} \text{ and } [Z_{\bar 1}, Z_{\bar 1}] = \gamma Z_{\bar 0}.$$

In this case, $\mathfrak{g} = \mathfrak{g}_{4,3}^{o} \overset{\perp}{\oplus} \mathfrak{g}_{2}^{o}(\gamma)$ decomposable.

(2) If $\mu \neq 0$: By the Jacobi identity and $[\mathfrak{g}_{\bar 0}, X_{\bar 1}] = \{0\}$, one has $[X_{\bar 1}, X_{\bar 1}] = 0$. Assume $[X_{\bar 1}, Y_{\bar 1}] = aX_{\bar 0} + bY_{\bar 0} + cZ_{\bar 0}$; then $[X_{\bar 0}, [X_{\bar 1}, Y_{\bar 1}]] + [X_{\bar 1}, [Y_{\bar 1}, X_{\bar 0}]] - [Y_{\bar 1}, [X_{\bar 0}, X_{\bar 1}]] = 0$ implies $aX_{\bar 0} + 2bY_{\bar 0} + c(\mu+1)Z_{\bar 0} = 0$. Therefore, $a = b = 0$ and $c(\mu+1) = 0$. Also, $[Y_{\bar 0}, [Y_{\bar 1}, Y_{\bar 1}]] + [Y_{\bar 1}, [Y_{\bar 1}, Y_{\bar 0}]] - [Y_{\bar 1}, [Y_{\bar 0}, Y_{\bar 1}]] = 0$ reduces to $[Y_{\bar 0}, [Y_{\bar 1}, Y_{\bar 1}]] = 2cZ_{\bar 0}$. Combining this with $[Y_{\bar 0}, \mathfrak{g}_{\bar 0}] = \mathbb{C}Y_{\bar 0}$, we obtain $c = 0$ and then $[Y_{\bar 1}, Y_{\bar 1}] = xY_{\bar 0} + yZ_{\bar 0}$. Similarly, $[X_{\bar 0}, [Y_{\bar 1}, Y_{\bar 1}]] + [Y_{\bar 1}, [Y_{\bar 1}, X_{\bar 0}]] - [Y_{\bar 1}, [X_{\bar 0}, Y_{\bar 1}]] = 0$ implies $3xY_{\bar 0} + y(\mu+2)Z_{\bar 0} = 0$. Since $|\mu| \leq 1$, we get $x = y = 0$.

Assume $[Z_{\bar 1}, Z_{\bar 1}] = uX_{\bar 0} + vY_{\bar 0} + wZ_{\bar 0}$. Since $[X_{\bar 0}, [Z_{\bar 1}, Z_{\bar 1}]] + [Z_{\bar 1}, [Z_{\bar 1}, X_{\bar 0}]] - [Z_{\bar 1}, [X_{\bar 0}, Z_{\bar 1}]] = 0$, we have $u = w = 0$ and $v(1+2\mu) = 0$.

If $[Z_{\bar 1}, Y_{\bar 1}] = \alpha X_{\bar 0} + \beta Y_{\bar 0} + \gamma Z_{\bar 0}$ then by applying the Jacobi identity for $X_{\bar 0}$, $Y_{\bar 1}$ and $Z_{\bar 1}$ we obtain $(\mu+1)\alpha = (\mu+2)\beta = \gamma(2\mu+1) = 0$. Therefore, $\beta = 0$ since $|\mu| \leq 1$.

By the Jacobi identity for $Y_{\bar 0}$, $Y_{\bar 1}$ and $Z_{\bar 1}$ we get $[Z_{\bar 1}, X_{\bar 1}] = -\alpha Y_{\bar 0}$ and then $[Z_{\bar 0}, [Z_{\bar 1}, Z_{\bar 1}]] + [Z_{\bar 1}, [Z_{\bar 1}, Z_{\bar 0}]] - [Z_{\bar 1}, [Z_{\bar 0}, Z_{\bar 1}]] = 0$ reduces to $\alpha = 0$. Since $[Z_{\bar 1}, [Z_{\bar 1}, Y_{\bar 1}]] + [Z_{\bar 1}, [Y_{\bar 1}, Z_{\bar 1}]] + [Y_{\bar 1}, [Z_{\bar 1}, Z_{\bar 1}]] = 0$, one has $v = -2\gamma\mu$. Combining this with $\gamma(2\mu+1) = 0$, one has $v = \gamma$. Therefore, $[Z_{\bar 1}, Z_{\bar 1}] = \gamma Y_{\bar 0}$ and $[Z_{\bar 1}, Y_{\bar 1}] = \gamma Z_{\bar 0}$ where $\gamma(2\mu+1) = 0$.

We have the two following cases:

- If $\gamma = 0$ then the nonzero Lie bracket is defined $\mathfrak{g}$ by $[X_{\bar 0}, Y_{\bar 0}] = Y_{\bar 0}$, $[X_{\bar 0}, Z_{\bar 0}] = \mu Z_{\bar 0}$, $[X_{\bar 0}, Y_{\bar 1}] = -Y_{\bar 1}$, $[Y_{\bar 0}, Y_{\bar 1}] = X_{\bar 1}$, $[X_{\bar 0}, Z_{\bar 1}] = -\mu Z_{\bar 1}$ and $[Z_{\bar 0}, Z_{\bar 1}] = \mu X_{\bar 1}$.
- If $\gamma \neq 0$ then $\mu = -\frac{1}{2}$ and $[X_{\bar 0}, Y_{\bar 0}] = Y_{\bar 0}$, $[X_{\bar 0}, Z_{\bar 0}] = -\frac{1}{2}Z_{\bar 0}$, $[X_{\bar 0}, Y_{\bar 1}] = -Y_{\bar 1}$, $[Y_{\bar 0}, Y_{\bar 1}] = X_{\bar 1}$, $[X_{\bar 0}, Z_{\bar 1}] = \frac{1}{2}Z_{\bar 1}$, $[Z_{\bar 0}, Z_{\bar 1}] = -\frac{1}{2}X_{\bar 1}$, $[Z_{\bar 1}, Z_{\bar 1}] = \gamma Y_{\bar 0}$, $[Z_{\bar 1}, Y_{\bar 1}] = \gamma Z_{\bar 0}$. Replacing $Y_{\bar 0}$ by $\gamma Y_{\bar 0}$ and $Y_{\bar 1}$ by $\gamma^{-1} Y_{\bar 1}$, we obtain the result that $[X_{\bar 0}, Y_{\bar 0}] = Y_{\bar 0}$, $[X_{\bar 0}, Z_{\bar 0}] = -\frac{1}{2}Z_{\bar 0}$, $[X_{\bar 0}, Y_{\bar 1}] = -Y_{\bar 1}$, $[Y_{\bar 0}, Y_{\bar 1}] = X_{\bar 1}$, $[X_{\bar 0}, Z_{\bar 1}] = \frac{1}{2}Z_{\bar 1}$, $[Z_{\bar 0}, Z_{\bar 1}] = -\frac{1}{2}X_{\bar 1}$, $[Z_{\bar 1}, Z_{\bar 1}] = Y_{\bar 0}$ and $[Z_{\bar 1}, Y_{\bar 1}] = Z_{\bar 0}$.

Summarily, solvable odd quadratic Lie superalgebras of dimension 6 are listed in Table 1.

$\mathfrak{g}$	$\mathfrak{g}_{\bar 0}$	Other nonzero brackets
$\mathfrak{g}^o_{6,0}$	Abelian	
$\mathfrak{g}^o_{6,1}$	Abelian	$[\mathfrak{g}_{\bar 1}, \mathfrak{g}_{\bar 1}] \subset \mathfrak{g}_{\bar 0}$
$\mathfrak{g}^o_{6,2}$	$[X_{\bar 0}, Y_{\bar 0}] = Z_{\bar 0}$	$[Y_{\bar 0}, Z_{\bar 1}] = X_{\bar 1}$, $[X_{\bar 0}, Z_{\bar 1}] = -Y_{\bar 1}$
$\mathfrak{g}^o_{6,3}$	$[X_{\bar 0}, Y_{\bar 0}] = Z_{\bar 0}$	$[Y_{\bar 0}, Z_{\bar 1}] = X_{\bar 1}$, $[X_{\bar 0}, Z_{\bar 1}] = -Y_{\bar 1}$, $[Z_{\bar 1}, Z_{\bar 1}] = Z_{\bar 0}$
$\mathfrak{g}^o_{6,4}$	$[X_{\bar 0}, Y_{\bar 0}] = Y_{\bar 0}$, $[X_{\bar 0}, Z_{\bar 0}] = Y_{\bar 0} + Z_{\bar 0}$	$[X_{\bar 0}, Y_{\bar 1}] = -Y_{\bar 1} - Z_{\bar 1}$, $[Y_{\bar 0}, Y_{\bar 1}] = X_{\bar 1}$, $[Z_{\bar 0}, Y_{\bar 1}] = X_{\bar 1}$, $[X_{\bar 0}, Z_{\bar 1}] = -Z_{\bar 1}$, $[Z_{\bar 0}, Z_{\bar 1}] = X_{\bar 1}$
$\mathfrak{g}^o_{6,5}$	$[X_{\bar 0}, Y_{\bar 0}] = Y_{\bar 0}$	$\mathfrak{g} = \mathfrak{g}^o_{4,3} \overset{\perp}{\oplus} \mathfrak{g}^o_2(\gamma)$
$\mathfrak{g}^o_{6,6}$	$[X_{\bar 0}, Y_{\bar 0}] = Y_{\bar 0}$, $[X_{\bar 0}, Z_{\bar 0}] = \mu Z_{\bar 0}$	$[X_{\bar 0}, Y_{\bar 1}] = -Y_{\bar 1}$, $[Y_{\bar 0}, Y_{\bar 1}] = X_{\bar 1}$, $[X_{\bar 0}, Z_{\bar 1}] = -\mu Z_{\bar 1}$, $[Z_{\bar 0}, Z_{\bar 1}] = \mu X_{\bar 1}$, $(\mu \neq 0,\ \lvert\mu\rvert \leq 1)$
$\mathfrak{g}^o_{6,7}$	$[X_{\bar 0}, Y_{\bar 0}] = Y_{\bar 0}$, $[X_{\bar 0}, Z_{\bar 0}] = -\frac{1}{2} Z_{\bar 0}$	$[X_{\bar 0}, Y_{\bar 1}] = -Y_{\bar 1}$, $[Y_{\bar 0}, Y_{\bar 1}] = X_{\bar 1}$, $[X_{\bar 0}, Z_{\bar 1}] = \frac{1}{2} Z_{\bar 1}$, $[Z_{\bar 0}, Z_{\bar 1}] = -\frac{1}{2} X_{\bar 1}$, $[Z_{\bar 1}, Z_{\bar 1}] = Y_{\bar 0}$, $[Z_{\bar 1}, Y_{\bar 1}] = Z_{\bar 0}$.

Table 1. **Complex solvable six-dimensional odd quadratic Lie superalgebras**.

Remark 3.7.

(1) The Lie superalgebras $\mathfrak{g}^o_{6,i}$, $2 \leq i \leq 7$, $i \neq 5$, are indecomposable.

(2) Note that a Lie superalgebra isomorphism is necessarily even, so if two Lie superalgebras have even parts not isomorphic then obviously they are not isomorphic. Therefore, we can easily see that Lie superalgebras $\mathfrak{g}^o_{6,i}$, $2 \leq i \leq 7$ are not isomorphic.

(3) For the non-solvable case, an example can be found in [ABB10] where $\mathfrak{g}_{\bar 0} = \mathfrak{sl}(2)$.

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