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PROJECTIVE SPACES IN THE ALGEBRAIC SETS OF PLANAR NORMAL SECTIONS OF HOMOGENEOUS ISOPARAMETRIC SUBMANIFOLDS

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ABSTRACT. The present paper is devoted to studying the algebraic sets of planar normal sections of homogeneous isoparametric submanifolds. The main objective is to describe the presence of real projective spaces in these algebraic sets. This indicates an important connection between homogeneous isoparametric submanifolds and the family of symmetric spaces of the corresponding group.

1. Introduction

The present paper is devoted to studying an interesting property of the *algebraic sets* of unit tangent vectors which define *planar normal sections* of *homogeneous isoparametric submanifolds*.

By definition, *normal sections* are the curves cut out of a submanifold M^n of $\mathbb{R}^{n+k}$ by the affine subspace generated by a unit tangent vector and the normal space, at a given point p of M^n. A normal section γ at *any* point p of M^n $(p = \gamma(0))$ is called *planar* at p if its first three derivatives $\gamma'(0)$, $\gamma''(0)$, $\gamma'''(0)$ are linearly dependent. The unit tangent vectors defining planar normal sections at $p \in M^n$, form a *real algebraic set* $\widehat{X}_p[M^n]$ (see (1) in Section 2 for a proper definition) which has notable interest in the study of the geometry of submanifolds of Euclidean spaces, [3, 4].

We restrict our considerations to *homogeneous isoparametric submanifolds* which, due to a celebrated theorem of G. Thorbergsson [13], is no restriction if our submanifold has codimension greater than or equal to three. The reason for this restriction is made clear in the next Section.

The main objective of the present paper is to describe the presence of certain *real projective spaces* in the algebraic sets $X_p[M^n]$ (see Section 2) which indicates an important connection between homogeneous isoparametric submanifolds and the family of *symmetric spaces* of the "corresponding" group. This connection has been observed in [5] for the *manifolds of complete flags* of a compact simple Lie

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group appearing there, as a property of this particular set of homogeneous isoparametric submanifolds involving only the *inner symmetric spaces* of the group. In the present paper we show that this is in fact a property of *all homogeneous isoparametric submanifolds* manifested in the "nature" of the corresponding algebraic sets $\widehat{X}_p[M^n]$ of unit tangent vectors defining *planar normal sections* and establishing an unexpected connection with *all* the symmetric spaces of the group.

The paper is organized as follows. In the next section we present the basic facts and definitions and in Section 3 we recall the definition of isoparametric submanifold that we shall use. Section 4 contains the basic definition of R-spaces and its first two subsections 4.1 and 4.2 contain the well known information concerning roots and restricted roots that we are going to need. In subsection 4.3 we introduce some extra notation and some required lemmata. In Section 5 we obtain the formula (23) which characterizes the unit vectors defining planar normal section for our submanifold M^n. In Section 6, which constitutes the core of the paper, we present the method to construct the *projective spaces* in the algebraic set $X[M^n]$ when M^n is a principal R-space (isoparametric submanifold), and finally in Section 7 we present (for the sake of briefness) a couple of examples of the use of the method described. The method could be used to obtain all the projective subspaces *associated* to the symmetric spaces corresponding to a given compact simple Lie group.

The results contained in the present paper are part of the Thesis of the first author, written under the direction of the second one.

2. Basic facts

We start recalling the basic definitions. Let M^n be a compact connected n-dimensional Riemannian manifold (that from now on call M) and $I : M \to \mathbb{R}^{n+k}$ an isometric embedding into the Euclidean space $\mathbb{R}^{n+k}$. We identify M with its image by I. The submanifold is usually called *full* if it is not included in any affine hyperplane.

We denote by $\langle *, * \rangle$ the inner product in $\mathbb{R}^{n+k}$. Let ∇^E be the Euclidean covariant derivative in $\mathbb{R}^{n+k}$ and ∇ the Levi-Civita connection in M associated to the induced metric.

We shall say that the submanifold M is *spherical* if it is contained in a sphere of radius r in $\mathbb{R}^{n+k}$ which we may think centered at the origin. Let α denote the second fundamental form of the embedding in $\mathbb{R}^{n+k}$. M is called *extrinsically homogeneous* [2, p. 35] if for any two points $p, q \in M$ there is an isometry g of $\mathbb{R}^{n+k}$ such that $g(M) = M$ and $g(p) = q$. We denote by $T_p(M)$ and $T_p(M)^{\perp}$ the tangent and normal spaces to M at p, respectively.

Let p be a point in M and consider, in the tangent space $T_p(M)$, a **unit vector** Y.

Define an affine subspace of $\mathbb{R}^{n+k}$ by: $S(p, Y) = p + \mathrm{Span}\left\{Y, T_p(M)^{\perp}\right\}$. If U is a small enough neighborhood of p in M, then the intersection $U \cap S(p, Y)$ can be considered the image of a C^{∞} **regular** curve $\gamma(s)$, parametrized by arc-length, such that $\gamma(0) = p$, $\gamma'(0) = Y$. Two of these curves coincide in a neigborhood of p. This curve is called a **normal section of M at p in the direction of Y**. These curves are **normal sections only at the point** $p = \gamma(0)$ because as soon as we leave p and move to $\gamma(s)$, the curve is, in general, no longer a normal section at $\gamma(s)$.

Definition 1. *A curve $\gamma(s)$ parametrized by arc-length in the submanifold $M^n \subset \mathbb{R}^{n+k}$ is said to be planar at p if $p = \gamma(0)$ and its first three derivatives $\gamma'(0)$, $\gamma''(0)$ and $\gamma'''(0)$ are* linearly dependent in $T_p\left(\mathbb{R}^{n+k}\right)$.

One can prove the following basic fact:

Lemma 2. [17] *Let M^n be a compact submanifold of the sphere $S^{n+k-1} \subset \mathbb{R}^{n+k}$. The normal section γ of M at p in the direction of $X \in T_p(M)$ is planar at p if and only if the covariant derivative of the second fundamental form vanishes on the vector $X = \gamma'(0)$. That is, if and only if X satisfies the equation $\left(\overline{\nabla}_X \alpha\right)(X, X) = 0$.* □

Given $p \in M$ we shall denote

$$\widehat{X}_p[M] = \left\{Y \in T_p(M) : \|Y\| = 1, \left(\overline{\nabla}_Y \alpha\right)(Y, Y) = 0\right\}. \tag{1}$$

This is then a *real algebraic set* in the unit sphere $S(T_p(M))$. We call this the *algebraic set of planar normal sections* of M.

If $X \in \widehat{X}_p[M]$ then clearly $(-X)$ defines a planar normal section (the same curve in opposite direction); then one may identify X with $(-X)$ and so obtain an *algebraic set* in the real projective space $\mathbb{RP}(T_p(M))$ which we denote by $X_p[M]$. If M is extrinsically homogeneous, then, $\widehat{X}_p[M]$ does not "depend" on the point p. In fact if p and q are two points in M there is an isometry g of $\mathbb{R}^{n+k}$ such that $g(M) = M$ and $g(p) = q$, then

$$\left(\overline{\nabla}_{\left(g_*|_p X\right)} \alpha\right)\left(g_*|_p X, g_*|_p X\right) = g_*|_p \left(\overline{\nabla}_X \alpha\right)(X, X),$$

and we clearly have that: $\widehat{X}_q[M] = \widehat{X}_{g(p)}[M] = g_*|_p\left(\widehat{X}_p[M]\right)$, and we may free ourselves from the point p. This isomorphism obviously goes to the projective space $\mathbb{RP}(T_p(M))$.

We have another simple basic fact [17]:

Lemma 3. *Let M^n be a compact submanifold of the sphere $S^{n+k-1} \subset \mathbb{R}^{n+k}$ and p a point in M^n. Then, for $X \in T_p(M)$ we have $\left\langle\left(\overline{\nabla}_X \alpha\right)(X, X), p\right\rangle = 0$.* □

3. Isoparametric submanifolds of $\mathbb{R}^{n+k}$.

The embedded submanifold $M^n \subset \mathbb{R}^{n+k}$ as above, is said to have *constant principal curvatures* if, for any parallel normal field $\xi(t)$ along any piecewise differentiable curve in M, the eigenvalues of the shape operator $A_{\xi(t)}$ are constant. Furthermore, it is called an *isoparametric submanifold* if it also has *flat normal bundle*. It is known that the submanifolds with constant principal curvatures are either isoparametric or one of their focal manifolds [2, 5.3.3]. The submanifold is called *irreducible* if it is not the product of two lower dimensional isoparametric submanifolds. For a full isoparametric submanifold of M of $\mathbb{R}^{n+k}$ the *rank* is its codimension, namely k.

Now let M be a compact rank k isoparametric submanifold of $\mathbb{R}^{n+k}$; then M is spherical ([12, 6.3.11 p. 123], [2, 5.2.10]) and we may think that the sphere has center $0 \in \mathbb{R}^{n+k}$ and radius 1. M is a regular level set of an isoparametric polynomial map $f : \mathbb{R}^{n+k} \to \mathbb{R}^k$ which has components $f = (h_1, h_2, \dots, h_k)$; usually one takes $M = f^{-1}(0)$. Let p be a point in M; as above, we denote by $S(T_p(M))$ the unit sphere in the tangent space at p.

We want to study here (extrinsic) *homogeneous isoparametric submanifolds* of $\mathbb{R}^{n+k}$. Many reasons can be adduced for this restriction. A very important one is a celebrated Theorem due to G. Thorbergsson ([13], [2, p. 162]) asserting that all *full irreducible isoparametric submanifolds* of rank at least three in $\mathbb{R}^m$ are orbits of an s-representation (also called R-spaces) and hence extrinsically homogeneous. However, the class of isoparametric submanifolds of rank two in $\mathbb{R}^m$, which coincides with that of isoparametric hypersurfaces in spheres, has members which are not homogeneous [7], [11] and other well known homogeneous ones.

Recall that the so called "s-representations" are the tangential representations of the symmetric spaces (compact or non-compact) and their orbits are the so called R-spaces. However not all R-spaces are *isoparametric submanifolds* because only the "principal" orbits have flat normal bundle. Therefore in the present article we study the normal sections of what we may call "*principal R-spaces*".

We start recalling basic facts about R-spaces. We shall try to keep the requirements from the theory of Lie algebras to a minimum.

4. R-spaces

Let $\mathfrak{g}$ be a real *simple* Lie algebra with Killing form B. We assume that $\mathfrak{g}$ is not itself a complex Lie algebra (which means that $\mathfrak{g} \neq \mathfrak{k}^{\mathbb{R}}$ for every *complex* Lie algebra $\mathfrak{k}$).

A decomposition of $\mathfrak{g}$ into a direct sum of two subspaces as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is called a *Cartan decomposition* if the following two conditions are satisfied

$$\begin{aligned} &(I)\ \ [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k} \\ &(II)\ B \text{ is } \begin{cases} \text{negative definite on } \mathfrak{k}, \\ \text{positive definite on } \mathfrak{p}. \end{cases} \end{aligned} \tag{2}$$

It is important to notice that it follows from (I) in (2) that $\mathfrak{k}$ is orthogonal to $\mathfrak{p}$ with respect to B. The function

$$\theta : \mathfrak{k} \oplus \mathfrak{p} \longrightarrow \mathfrak{k} \oplus \mathfrak{p}, \qquad \theta(X+Y) = X - Y, \quad \forall X \in \mathfrak{k}, Y \in \mathfrak{p} \tag{3}$$

is easily seen to be an involutive automorphism (called Cartan involution) of $\mathfrak{g}$ that satisfies that $B_\theta(X,Y) := -B(X,\theta Y)$ is strictly positive definite on $\mathfrak{g}$. This decomposition of $\mathfrak{g}$ is orthogonal with respect to B and B_θ. We take in $\mathfrak{p}$ the inner product defined by $\langle X, Y\rangle = B(X,Y) = B_\theta(X,Y)$. Then $(\mathfrak{p}, \langle *, * \rangle)$ is a Euclidean space. If these conditions are satisfied $\mathfrak{k}$ is a *maximal compactly embedded subalgebra* of $\mathfrak{g}$, which means that the analytic subgroup$K$ of $Int\,(\mathfrak{g})$ (corresponding to the subalgebra $ad_{\mathfrak{g}}\,(\mathfrak{k})$ of $ad_{\mathfrak{g}}\,(\mathfrak{g}) = Lie\,(Int\,(\mathfrak{g})) = \mathfrak{g}$) is compact.

Let $E \in \mathfrak{p}$ and consider the orbit $M = Ad\,(K)\,E \subset \mathfrak{p}$. The submanifold M of the Euclidean space $\mathfrak{p}$ is a so called *R-space.*

4.1. **Roots.** Let $\mathfrak{g}^{\mathbb{C}}$ be the *complexification* of our *simple* Lie algebra $\mathfrak{g}$; then (by our hypothesis on $\mathfrak{g}$) $\mathfrak{g}^{\mathbb{C}}$ is a complex *simple* Lie algebra. It is usual notation to say that $\mathfrak{g}$ is a *real form* of $\mathfrak{g}^{\mathbb{C}}$. Let $\mathfrak{h}^{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We may consider the set of *roots* of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$ which we denote by $\Delta = \Delta\left(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}\right)$.

Recall that if α is a linear functional on the complex vector space $\mathfrak{h}^{\mathbb{C}}$ one calls $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\alpha}$ to the linear subspace

$$\left(\mathfrak{g}^{\mathbb{C}}\right)_{\alpha} = \left\{X \in \mathfrak{g}^{\mathbb{C}} : [H, X] = \alpha\,(H)\,X, \forall H \in \mathfrak{h}^{\mathbb{C}}\right\}. \tag{4}$$

If $\left(\mathfrak{g}^{\mathbb{C}}\right)_{\alpha} \neq \{0\}$ and $\alpha \neq 0$, the linear functional is called a root (of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$).

The set of roots $\Delta = \Delta\left(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}\right)$ is a finite subset of $\left(\mathfrak{h}^{\mathbb{C}}\right)^{*}$ the dual space. If $H \in \mathfrak{h}^{\mathbb{C}}$ is an arbitrary element then the *complex numbers* $\alpha\,(H)$ $(\forall \alpha \in \Delta)$ are the nonzero eigenvalues of the linear transformation $ad_{\mathfrak{g}^{\mathbb{C}}}\,(H) : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$.

We have the following facts that we get for instance from [20, p. 3–5].

i) One may write (as a direct sum)

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta} \left(\mathfrak{g}^{\mathbb{C}}\right)_{\alpha}. \tag{5}$$

ii) $\dim_{\mathbb{C}} \left(\mathfrak{g}^{\mathbb{C}}\right)_{\alpha} = 1$.

iii) The decomposition in (i) is orthogonal relative to $B^{\mathbb{C}}$.

iv) If $\alpha \in \Delta$ then $c\alpha \in \Delta$ $(c \in \mathbb{C}) \iff c = \pm 1$.

v) The restriction of $B^{\mathbb{C}}$ to $\mathfrak{h}^{\mathbb{C}} \times \mathfrak{h}^{\mathbb{C}}$ is *nondegenerate* ([14, p. 166]) so for each $\alpha \in \Delta$ there exists a unique element $H_{\alpha} \in \mathfrak{h}^{\mathbb{C}}$ (root vector) such that $B^{\mathbb{C}}\,(H, H_{\alpha}) = \alpha\,(H)\,, \forall H \in \mathfrak{h}^{\mathbb{C}},\ \alpha\,(H_{\alpha}) \neq 0$; we have then an isomorphism $\alpha \longmapsto H_{\alpha}$ from $\left(\mathfrak{h}^{\mathbb{C}}\right)^{*}$ to $\mathfrak{h}^{\mathbb{C}}$ and this defines a bilinear form $\langle \alpha, \beta \rangle = B^{\mathbb{C}}\,(H_{\alpha}, H_{\beta})$ on $\left(\mathfrak{h}^{\mathbb{C}}\right)^{*}$.

vi) For each $\alpha \in \Delta$ a vector $X_{\alpha} \in \left(\mathfrak{g}^{\mathbb{C}}\right)_{\alpha}$ can be chosen so that for all α, β in Δ we have

$$\begin{aligned} &[X_{\alpha}, X_{-\alpha}] = H_{\alpha}, \qquad [H, X_{\alpha}] = \alpha\,(H)\,X_{\alpha}, \qquad \forall H \in \mathfrak{h}^{\mathbb{C}} \\ &[X_{\alpha}, X_{\beta}] = 0 \qquad \text{if } \alpha + \beta \neq 0 \text{ and } \alpha + \beta \notin \Delta \\ &[X_{\alpha}, X_{\beta}] = N_{\alpha,\beta} X_{\alpha+\beta} \qquad \text{if } \alpha + \beta \in \Delta \end{aligned}$$

where the *real* constants $N_{\alpha,\beta}$ satisfy: $N_{\alpha,\beta} = -N_{-\alpha,-\beta} = -N_{\beta,\alpha}$ and $N_{-\alpha,(\alpha+\beta)} = N_{(\alpha+\beta),-\beta} = N_{-\beta,-\alpha}$ [14, p. 176, 5.5]. The set $\{H_a : \alpha \in \Delta\} \cup \{X_a : \alpha \in \Delta\}$ is called a *Weyl basis of* $\mathfrak{g}^{\mathbb{C}}$ *modulo* $\mathfrak{h}^{\mathbb{C}}$.

Furthermore, setting

$$\mathfrak{h}_{\mathbb{R}} = \sum_{\alpha \in \Delta} \mathbb{R} H_{\alpha} \tag{6}$$

we also have:

1) The restriction of the Killing form $B^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ to $\mathfrak{h}_{\mathbb{R}}$ is *real and strictly positive definite* on $\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}$.

2) The Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ splits as $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$ and

$$\mathfrak{h}_{\mathbb{R}} = \left\{ H \in \mathfrak{h}^{\mathbb{C}} : \alpha(H) \in \mathbb{R}, \forall \alpha \in \Delta \right\}.$$

4.2. **Restricted roots.** Let $\mathfrak{a} \subset \mathfrak{p}$ be a *maximal abelian subalgebra* of $\mathfrak{p}$ and let $\mathfrak{a}^*$ be the dual of $\mathfrak{a}$. For $\lambda \in \mathfrak{a}^*$ define

$$\mathfrak{g}_{\lambda} = \{ U \in \mathfrak{g} : [H, U] = \lambda(H) U, \ \forall H \in \mathfrak{a} \}. \tag{7}$$

The elements in $ad_{\mathfrak{g}}(\mathfrak{a})$ form a commutative set of self-adjoint linear transformations of $\mathfrak{g}$.

Let us set also $\mathfrak{g}_0 = \{ X \in \mathfrak{g} : ad(H) X = 0, \ \forall H \in \mathfrak{a} \}$; $\mathfrak{g}_0$ is *the centralizer* of $\mathfrak{a}$ in $\mathfrak{g}$, and if $\lambda \in \mathfrak{a}^*$ is $\lambda \neq 0$ and $\mathfrak{g}_{\lambda} \neq \{0\}$ then λ is called a *restricted root* of $\mathfrak{g}$ *with respect to* $\mathfrak{a}$ and the subspace $\mathfrak{g}_{\lambda}$ a *restricted root space.* We denote by

$$\Delta_R = \Delta_R(\mathfrak{g}, \mathfrak{a}) \tag{8}$$

the set of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. It is important to notice that the set of restricted roots $\Delta_R \subset (\mathfrak{a}^* - \{0\})$ may be *non-reduced.*

Then $\mathfrak{g}$ can be written as an *orthogonal* direct sum of the common eigenspaces, that is

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Delta_R} \mathfrak{g}_{\lambda}, \tag{9}$$

and this is usually called the *restricted root space decomposition* of $\mathfrak{g}$ with respect to $\mathfrak{a}$.

The subspaces $\mathfrak{g}_{\lambda}$ satisfy: $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\lambda}] \subset \mathfrak{g}_{\alpha+\lambda}$, and $\theta(\mathfrak{g}_{\lambda}) = \mathfrak{g}_{-\lambda}$, $\forall \lambda, \alpha \in \Delta_R$, where θ is the involution (3).

In particular $\mathfrak{g}_0$ is a subalgebra θ-invariant and so we have the orthogonal decomposition $\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus (\mathfrak{g}_0 \cap \mathfrak{p}) = \mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

It is convenient, for our purpose, to introduce some extra notation associated to the restricted roots $\Delta_R(\mathfrak{g}, \mathfrak{a})$ and the Cartan decomposition of $\mathfrak{g}$. Clearly, as a consequence of (2), if $H \in \mathfrak{a} \subset \mathfrak{p}$ then $ad(H)$ interchanges $\mathfrak{k}$ and $\mathfrak{p}$. In fact $ad(H)\mathfrak{k} \subset \mathfrak{p}$ and $ad(H)\mathfrak{p} \subset \mathfrak{k}$. Then it is convenient to define, for $\lambda \in \Delta_R(\mathfrak{g}, \mathfrak{a})$, the subspaces

$$\mathfrak{k}_{\lambda} = \left\{ X \in \mathfrak{k} : (ad(H))^2 X = \lambda^2(H) X, \quad \forall H \in \mathfrak{a} \right\}$$
$$\mathfrak{p}_{\lambda} = \left\{ X \in \mathfrak{p} : (ad(H))^2 X = \lambda^2(H) X, \quad \forall H \in \mathfrak{a} \right\}$$

and observe that obviously $\mathfrak{k}_{\lambda} = \mathfrak{k}_{-\lambda}$ and $\mathfrak{p}_{\lambda} = \mathfrak{p}_{-\lambda}$. Furthermore one can prove that

$$\mathfrak{g}_{\lambda} \oplus \mathfrak{g}_{-\lambda} = \mathfrak{k}_{\lambda} \oplus \mathfrak{p}_{\lambda}. \tag{10}$$

We have now:

Proposition 4. *Let as above $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. If $\mathfrak{t} \subset \mathfrak{m}$ is a maximal abelian subspace of $\mathfrak{m}$ then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ (i.e. $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$) and $\mathfrak{h}$ is stable under θ.*

The dimension of $\mathfrak{a}$ is called the *split rank* of $\mathfrak{g}$. Since the complexification $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, we may consider the set of roots of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$ which we denote by $\Delta = \Delta\left(\mathfrak{g}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}\right)$ as above. Let us consider again (6) the *real vector space* $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{h}^{\mathbb{C}}$ generated by the vectors H_α for $\alpha \in \Delta$ and where all roots are real valued. Then we have $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{t} \oplus \mathfrak{a}$.

Furthermore $i\mathfrak{h}_{\mathbb{R}} = \mathfrak{t} \oplus i\mathfrak{a}$ is a *Cartan subalgebra of the compact real form* $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}^{\mathbb{C}}$.

We have then the decompositions (5) and (9) in terms of (4) and (7). Hence we notice that $\mathfrak{g}_\lambda = \mathfrak{g} \cap \left\{\sum_{\alpha\in\Delta,\, \alpha|\mathfrak{a}=\lambda} \left(\mathfrak{g}^{\mathbb{C}}\right)_\alpha\right\}$; $\mathfrak{m}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \sum_{\alpha\in\Delta,\, \alpha|\mathfrak{a}=o} \left(\mathfrak{g}^{\mathbb{C}}\right)_\alpha$ and if $\mathfrak{m} = \{0\}$ then $\mathfrak{g}$ is a *split real form* of $\mathfrak{g}^{\mathbb{C}}$.

4.3. **Notation and some lemmata.** We indicate now some lemmata and notation that are needed below. The references here are [2] and also [14] as usual. Others are indicated when used.

We have the Cartan decomposition of our simple real Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We also have the corresponding Cartan involution, $\theta : \mathfrak{g} \longrightarrow \mathfrak{g}$, and the symmetric, *positive definite*, bilinear form $B_\theta(X, Y) = -B(X, \theta Y)$.

We consider the real Lie group $Int(\mathfrak{g})$ and the analytic subgroup K mentioned above. Let us take $E \in \mathfrak{p}$ and consider, as above, the orbit $M = Ad(K)E \subset \mathfrak{p}$, which is a so called R-space.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing E. Then we have the elements and notation developed above.

It is usual to say that the element $E \in \mathfrak{a}$ is *regular* if $\lambda(E) \neq 0\ \forall\ \lambda \in \Delta_R = \Delta_R(\mathfrak{g}, \mathfrak{a})$. Since the generic elements in $\mathfrak{a}$ are regular they generate the *principal orbits* of the action of K on $\mathfrak{p}$ and these are the orbits which are *isoparametric submanifolds* of the Euclidean space $(\mathfrak{p}, \langle *, * \rangle)$ and are the object of our main interest. It may happen that some orbit (of a not necessarily regular element) of the action of $Ad(K)$ on $\mathfrak{p}$ is a symmetric space and in that case that orbit is called a *symmetric R-space*. These particular orbits are very important geometric objects whose very nice properties have been deeply studied. In particular they have several characterizations one of which is useful in this work.

Lemma 5. [2, p. 310] *Let $V \in \mathfrak{p}$, then the R-space $N = Ad(K)V$ is symmetric if and only if the eigenvalues of $ad(V)$ in $\mathfrak{g}$ are $\{-1, 0, +1\}$.*

Lemma 6. *If E is regular in $\mathfrak{a}$ (that is, for every restricted root $\lambda \in \Delta_R$ we have $\lambda(E) \neq 0$) then:*

$$Z_{\mathfrak{g}}(E) = \mathfrak{m} \oplus \mathfrak{a}, \qquad Z_{\mathfrak{p}}(E) = \mathfrak{a}, \qquad Z_{\mathfrak{k}}(E) = \mathfrak{m}.$$

Proof. Let us take $X \in Z_{\mathfrak{g}}(E)$. Since $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \sum_{\lambda\in\Delta_R} \mathfrak{g}_\lambda$, we may write:

$$X = H + Y + \sum_{\lambda\in\Delta_R} Y_\lambda, \qquad Y \in \mathfrak{m},\ H \in \mathfrak{a},\ Y_\lambda \in \mathfrak{g}_\lambda.$$

Then

$$0 = [E, X] = \sum_{\lambda \in \Delta_R} [E, Y_\lambda] = \sum_{\lambda \in \Delta_R} \lambda(E) Y_\lambda$$
$$\Longrightarrow \lambda(E) Y_\lambda = 0,\ \forall \lambda \in \Delta_R \quad \Longrightarrow Y_\lambda = 0,\ \forall \lambda \in \Delta_R \quad \Longrightarrow X = H + Y.$$

Then we have $Z_{\mathfrak{g}}(E) \subset \mathfrak{m} \oplus \mathfrak{a}$. Since the other inclusion is clear we have $Z_{\mathfrak{g}}(E) = \mathfrak{m} \oplus \mathfrak{a}$. So in fact we have the equalities $Z_{\mathfrak{p}}(E) = Z_{\mathfrak{g}}(E) \cap \mathfrak{p} = \mathfrak{a}$, and $Z_{\mathfrak{k}}(E) = Z_{\mathfrak{g}}(E) \cap \mathfrak{k} = \mathfrak{m}$. □

Now from [19] (also [15]) we have the following:

Lemma 7. *The following facts hold:*

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu}. \tag{11}$$

$$[\mathfrak{k}_\lambda, \mathfrak{a}] = \mathfrak{p}_\lambda. \tag{12}$$

If $\lambda + \mu \in \Delta_R \cup \{0\}$ *or* $\lambda - \mu \in \Delta_R \cup \{0\}$ *then*

$$[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \neq \{0\}. \tag{13}$$

Lemma 8. *If* $\lambda \in \Delta_R$*, then* $\mathfrak{p}_\lambda \neq \{0\}$ *and* $\mathfrak{k}_\lambda \neq \{0\}$*.*

Proof. By definition, $\lambda \in \Delta_R$ if and only if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$. We have $\theta(\mathfrak{g}_\lambda) = \mathfrak{g}_{-\lambda}$ and so $\mathfrak{g}_{-\lambda} \neq \{0\}$ too. By (10) we have then $\mathfrak{k}_\lambda \oplus \mathfrak{p}_\lambda \neq \{0\}$. However we may have $\mathfrak{p}_\lambda = \{0\}$ and $\mathfrak{k}_\lambda \neq \{0\}$ (or the other way around, i.e. $\mathfrak{p}_\lambda \neq \{0\}$ and $\mathfrak{k}_\lambda = \{0\}$).

We start by showing that the assumption $\mathfrak{p}_\lambda = \{0\}$ leads to a contradiction. In fact, if this were the case, we have $\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda} = \mathfrak{k}_\lambda \neq \{0\}$. Then, by definition (7), there is an $U \neq 0$ in $\mathfrak{g}_\lambda$ such that $[H, U] = \lambda(H) U$, $\forall H \in \mathfrak{a}$. Then $U \in \mathfrak{k}_\lambda$ and by (12) in Lemma 7, we have $[U, \mathfrak{a}] \neq \{0\}$, $[U, \mathfrak{a}] \subset [\mathfrak{k}_\lambda, \mathfrak{a}] = \mathfrak{p}_\lambda = \{0\}$. This contradiction shows that the first alternative is not possible.

Let us assume now that $\mathfrak{p}_\lambda \neq \{0\}, \mathfrak{k}_\lambda = \{0\}$. We have $\mathfrak{g}_\lambda \oplus \mathfrak{g}_{-\lambda} = \mathfrak{p}_\lambda$ and again by (12) in Lemma 7, we reach a contradiction which proves the lemma. □

Let us set the notation: $\mathfrak{k}_+ = \sum_{\lambda \in \Delta_R} \mathfrak{k}_\lambda$, $\mathfrak{p}_+ = \sum_{\lambda \in \Delta_R} \mathfrak{p}_\lambda$, $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{k}_+$, and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_+$.

If $X \in \mathfrak{k}_\lambda$ and $\lambda \in \Delta_R$ we have $[\mathfrak{k}_\lambda, E] \subset \mathfrak{p}_\lambda$, but in fact the function $Ad(E)$ which is $ad(E) : \mathfrak{k}_\lambda \longrightarrow \mathfrak{p}_\lambda$, and $ad(E) : \mathfrak{p}_\lambda \longrightarrow \mathfrak{k}_\lambda$, is an isomorphism for each $\lambda \in \Delta_R$ because in any of the two spaces we have $Ad(E)^2 = \lambda(E)^2 Id$. Let us notice that in $(\mathfrak{p}, \langle *, * \rangle)$ we have $\langle \mathfrak{p}_+, \mathfrak{a} \rangle = 0$. Then we may identify the tangent and normal spaces to M at the regular element E as:

$$T_E(M) = [\mathfrak{k}_+, E] = \mathfrak{p}_+, \qquad T_E(M)^\perp = \mathfrak{a}. \tag{14}$$

5. The planar normal sections of M

We study now the planar normal sections of the orbit of the regular element E by the group K that is $M = Ad(K) E \subset \mathfrak{p}$, and consider on this submanifold **the induced metric** from $\langle *, * \rangle$ on $\mathfrak{p}$. We have the Euclidean covariant derivative ∇^E and the Riemannian (Levi-Civita) connection ∇ on M.

We consider the second fundamental form α of M on $\mathfrak{p}$ and the Gauss formula

$$\nabla^E_U W = \nabla_U W + \alpha\left(U, W\right), \tag{15}$$

where $\nabla^E_U W$ is the usual covariant derivative in Euclidean space $(\mathfrak{p}, \langle *, * \rangle)$.

Let us take, for some $X \in \mathfrak{k}$, the curve in M of the form

$$\gamma\left(t\right) = \left(Ad\left(\exp\left(tX\right)\right)E\right). \tag{16}$$

Its tangent vector at E is : $\gamma'\left(0\right) = [X, E]$. If we take $t_1 > 0$ then we may compute the derivative of γ in $t_1$by

$$\begin{aligned}
\gamma'\left(t_1\right) &= \left.\frac{d}{dt}\right|_{t=0} \left(Ad\left(\exp\left(\left(t_1 + t\right)X\right)\right)E\right) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(Ad\left(\exp\left(\left(t_1\right)X\right)\right)Ad\left(\exp\left(\left(t\right)X\right)\right)E\right) \\
&= Ad\left(\exp\left(\left(t_1\right)X\right)\right)\left.\frac{d}{dt}\right|_{t=0} Ad\left(\exp\left(\left(t\right)X\right)\right)E \\
&= Ad\left(\exp\left(\left(t_1\right)X\right)\right)[X, E],
\end{aligned} \tag{17}$$

and so this gives the tangent field along $\gamma\left(t\right)$.

We are going to consider also on M the *canonical connection* determined by the decomposition $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{k}_+$, which we shall denote by ∇^c.

It is known that the curves (16) are ∇^c-geodesics and that the *parallel translation* along these geodesics is precisely given by (17).

Continuing with our computation we take a tangent vector at E

$$[Y, E] \in T_E\left(M\right) = [\mathfrak{k}, E] = [\mathfrak{k}_+, E]$$

and we extend it to a field along γ by: $[Y, E]^* = Ad\left(\exp\left(tX\right)\right)[Y, E]$.

Now we compute

$$\nabla^E_{[X,E]}\left([Y, E]^*\right) = \left.\frac{d}{dt}\right|_{t=0} \left(Ad\left(\exp\left(tX\right)\right)[Y, E]\right) = [X, [Y, E]] \in \mathfrak{p}, \qquad X, Y \in \mathfrak{k},$$

where we may in fact take $X, Y \in \mathfrak{k}_+$.

Now using (15) we have $\nabla^E_{[X,E]}[Y, E]^* = \nabla_{[X,E]}[Y, E]^* + \alpha\left([X, E], [Y, E]\right)$.

Then clearly

$$\begin{aligned}
\nabla_{[X,E]}[Y, E]^* &= \left([X, [Y, E]]\right)_{\mathfrak{p}_+} \\
\alpha\left([X, E], [Y, E]\right) &= \left([X, [Y, E]]\right)_{\mathfrak{a}}
\end{aligned} \tag{18}$$

because we have (14) and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_+$.

Now we proceed to compute the covariant derivative of α. By definition that is

$$\begin{aligned}
\left(\overline{\nabla}_{[X,E]}\alpha\right)\left([Y, E], [Z, E]\right) = \nabla^\perp_{[X,E]}\alpha\left([Y, E], [Z, E]\right) &- \alpha\left(\nabla_{[X,E]}[Y, E], [Z, E]\right) \\
&- \alpha\left([Y, E], \nabla_{[X,E]}[Z, E]\right)
\end{aligned} \tag{19}$$

But now we may avoid computing the derivative $\nabla^\perp_{[X,E]}\alpha\left([Y, E], [Z, E]\right)$.

By [9] we know that, since M is a R-space, $\left(\nabla^c_{[X,E]}\alpha\right)([Y,E],[Z,E]) = 0$ (in fact this identity *characterizes* R-spaces). Recalling now the definition of $(\nabla^c_U\alpha)$ in [9] which is

$$\begin{aligned} 0 &= \left(\nabla^c_{[X,E]}\alpha\right)([Y,E],[Z,E]) \\ &= \nabla^{\perp}_{[X,E]}\alpha([Y,E],[Z,E]) - \alpha\left(\nabla^c_{[X,E]}[Y,E],[Z,E]\right) - \alpha\left([Y,E],\nabla^c_{[X,E]}[Z,E]\right) \end{aligned} \tag{20}$$

and subtracting (20) from (19) we get

$$\begin{aligned} \left(\overline{\nabla}_{[X,E]}\alpha\right)([Y,E],[Z,E]) = &-\alpha\left(D\left([X,E],[Y,E]\right),[Z,E]\right) \\ &-\alpha\left([Y,E],D\left([X,E],[Z,E]\right)\right), \end{aligned} \tag{21}$$

where $D = \nabla - \nabla^c$ is the *difference tensor* of the two connections. Then setting $X = Y = Z$ we have an important equivalence:

$$\left(\overline{\nabla}_{[X,E]}\alpha\right)([X,E],[X,E]) = 0 \iff \alpha\left(D\left([X,E],[X,E]\right),[X,E]\right) = 0. \tag{22}$$

So we need to compute the difference tensor $D([X,E],[Y,E])$. Using the property that the tangent field $[Y,E]^*$ is *parallel with respect to the canonical connection* ∇^c *along* γ. We have

$$D([X,E],[Y,E]) = \nabla_{[X,E]}[Y,E]^* - \nabla^c_{[X,E]}[Y,E]^* = \nabla_{[X,E]}[Y,E]^*,$$

and going back to (18) we get

$$D([X,E],[Y,E]) = \nabla_{[X,E]}[Y,E]^* = Ta([X,[Y,E]]) = ([X,[Y,E]])_{\mathfrak{p}_+}.$$

We have then that the condition $\alpha\left(D\left([X,E],[X,E]\right),[X,E]\right) = 0$ indicated in (22) becomes here:

The normal section generated by $[X,E]$ (or by X) for $M = Ad(K)\ E \subset \mathfrak{p} \subset \mathfrak{g}$ for our regular element $E \in \mathfrak{a}$ *is planar if and only if:*

$$\left(\left[X,([X,[X,E]])_{\mathfrak{p}_+}\right]\right)_{\mathfrak{a}} = 0. \tag{23}$$

In this formula $X \in \mathfrak{k} = \mathfrak{m}\oplus\mathfrak{k}_+$, but we may take in fact $X \in \mathfrak{k}_+$. Here $\mathfrak{k} = \mathfrak{m}\oplus\mathfrak{k}_+$ is the decomposition associated to the Riemannian homogeneous space K/K_E and $\mathfrak{k}_+$ is the tangent space at the base point $[K_E]$.

6. Projective subvarieties

In general, the presence of projective spaces in an algebraic set such as $X[M]$, is an interesting property of the set. So we want to address this problem here in the R-spaces. As we indicated in the Introduction this is one of the objectives of the present paper.

We are going to study the presence of *real projective spaces* in the algebraic set $X[M]$ when M is a principal R-space (isoparametric submanifold). So we always assume that M is the orbit $M = Ad(K)\,E$, where E is a regular element in a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ corresponding to a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

6.1. **Construction.** We describe a way to construct a real projective space $\mathbb{RP}^n$ contained in the real algebraic set $X[M]$ for our principal R-space M. The method relies on the *structure* of the restricted root system $\Delta_R = \Delta_R(\mathfrak{g}, \mathfrak{a})$.

Let us choose a subset $\Omega \subset \Delta_R$ and define

$$\begin{array}{ll} \mathfrak{k}_+(\Omega) = \sum_{\lambda\in\Omega} \mathfrak{k}_\lambda & \mathfrak{p}_+(\Omega) = \sum_{\lambda\in\Omega} \mathfrak{p}_\lambda \\ \mathfrak{k}_0(\Omega) = \mathfrak{m} \oplus \sum_{\lambda\in(\Delta_R-\Omega)} \mathfrak{k}_\lambda & \mathfrak{p}_0(\Omega) = \mathfrak{a} \oplus \sum_{\lambda\in(\Delta_R-\Omega)} \mathfrak{p}_\lambda, \end{array} \tag{24}$$

and, as above, we have associated to E the subspaces $\mathfrak{k}_+$ and $\mathfrak{p}_+$ with $\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{k}_+$ and $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_+$.

Since $\Omega \subset \Delta_R$, it is clear that $\mathfrak{k}_+(\Omega) \subset \mathfrak{k}_+$ and $\mathfrak{p}_+(\Omega) \subset \mathfrak{p}_+$.

Proposition 9. *Let us assume that the set Ω has the property that*

$$[\mathfrak{k}_+(\Omega), \mathfrak{p}_+(\Omega)] \subset \mathfrak{p}_0(\Omega), \tag{25}$$

$$[\mathfrak{k}_+(\Omega), \mathfrak{p}_0(\Omega)] \subset \mathfrak{p}_+. \tag{26}$$

Then the vector subspace $\mathfrak{k}_+(\Omega)$ has the property: $\mathbb{RP}(\mathfrak{k}_+(\Omega)) \subset X[M]$, where M is the orbit of E. Let us notice that, as vector spaces over $\mathbb{R}$, $\mathfrak{k}_+(\Omega) \simeq \mathfrak{p}_+(\Omega)$.

Proof. We take $X \in \mathfrak{k}_+(\Omega) \subset \mathfrak{k}_+$. In order to prove that $X \in \widehat{X}[M]$, we need to show that $\left(\left[X, ([X,[X,E]])_{\mathfrak{p}_+}\right]\right)_{\mathfrak{a}} = 0$.

By (12) in Lemma 7 we have: $[X, E] \in [\mathfrak{k}_+(\Omega), \mathfrak{a}] = \mathfrak{p}_+(\Omega)$, hence by hypothesis (25) we have

$$[X,[X,E]] \in [\mathfrak{k}_+(\Omega), \mathfrak{p}_+(\Omega)] \subset \mathfrak{p}_0(\Omega) \tag{27}$$

Let us set, for convenience, $\mathfrak{p}_0(\Omega) \cap \mathfrak{p}_+ = \mathfrak{h}(\Omega)$; then clearly (27) yields:

$$([X,[X,E]])_{\mathfrak{p}_+} = ([X,[X,E]])_{\mathfrak{h}(\Omega)}. \tag{28}$$

Then we have to consider $\left(\left[X, ([X,[X,E]])_{\mathfrak{p}_+}\right]\right)_{\mathfrak{a}} = \left(\left[X, ([X,[X,E]])_{\mathfrak{h}(\Omega)}\right]\right)_{\mathfrak{a}}$. Now, by hypothesis (26), we have $\left[X, ([X,[X,E]])_{\mathfrak{h}(\Omega)}\right] \in \mathfrak{p}_+$ and since $\mathfrak{p}_+$ and $\mathfrak{a}$ are orthogonal we get

$$\left(\left[X, ([X,[X,E]])_{\mathfrak{p}_+}\right]\right)_{\mathfrak{a}} = \left(\left[X, ([X,[X,E]])_{\mathfrak{h}(\Omega)}\right]\right)_{\mathfrak{a}} = 0.$$

Then $X \in \widehat{X}[M]$ as was to be proven. □

This leads us to define:

Definition 10. *Let $\Omega \subset \Delta_R$. We shall say that the set Ω is a* ***pre-symmetric set*** *if it satisfies*

$$\lambda, \gamma \in \Omega \implies \lambda + \gamma \notin \Omega.$$

Notice that, if Ω is pre-symmetric, then neither $\lambda+\gamma$ nor $\lambda-\gamma$ are in Ω. In fact, if for a given pair $\lambda, \gamma \in \Omega$ it happens that $\lambda - \gamma \in \Omega$ then $\gamma + (\lambda - \gamma) = \lambda \in \Omega$, which contradicts the definition of Ω.

Now we prove:

Theorem 11. *A subset $\Omega \subset \Delta_R$ satisfies the inclusions (25) and (26) if and only if it is a* ***pre-symmetric set.***

Proof. Let us assume first that $\Omega \subset \Delta_R$ is a *pre-symmetric set* and consider the subspaces defined above $\mathfrak{k}_+(\Omega)$, $\mathfrak{p}_+(\Omega)$, $\mathfrak{k}_0(\Omega)$ and $\mathfrak{p}_0(\Omega)$. Then $[\mathfrak{k}_+(\Omega), \mathfrak{p}_+(\Omega)] = \sum_{\lambda,\gamma\in\Omega} [\mathfrak{k}_\lambda, \mathfrak{p}_\gamma]$.

Now by (11) in Lemma 7 we have, for the pair $\lambda, \gamma \in \Omega$, $[\mathfrak{k}_\lambda, \mathfrak{p}_\gamma] \subset \mathfrak{p}_{\lambda+\gamma} + \mathfrak{p}_{\lambda-\gamma}$, but since Ω is a pre-symmetric set then either $\lambda+\gamma$ and $\lambda-\gamma$ are not roots or they belong to $(\Delta_R - \Omega)$ (recall that zero is not a root). Then $[\mathfrak{k}_\lambda, \mathfrak{p}_\gamma] \subset \mathfrak{p}_0(\Omega)$ $\forall \lambda, \gamma \in \Omega$, and therefore $[\mathfrak{k}_+(\Omega), \mathfrak{p}_+(\Omega)] \subset \mathfrak{p}_0(\Omega)$, which is the first inclusion.

Now we study the other one. By definition of $\mathfrak{p}_0(\Omega)$ (24) we have

$$\begin{aligned}[\mathfrak{k}_+(\Omega), \mathfrak{p}_0(\Omega)] &= \left[\left(\sum_{\lambda\in\Omega} \mathfrak{k}_\lambda\right), \left(\mathfrak{a} \oplus \sum_{\mu\in(\Delta_R-\Omega)} \mathfrak{p}_\mu\right)\right] \\ &= \sum_{\lambda\in\Omega} [\mathfrak{k}_\lambda, \mathfrak{a}] + \sum_{\lambda\in\Omega} \sum_{\mu\in(\Delta_R-\Omega)} [\mathfrak{k}_\lambda, \mathfrak{p}_\mu],\end{aligned}$$

and we may consider each of the terms separately.

By (12) we have $[\mathfrak{k}_\lambda, \mathfrak{a}] = \mathfrak{p}_\lambda$, then $\sum_{\lambda\in\Omega} [\mathfrak{k}_\lambda, \mathfrak{a}] = \sum_{\lambda\in\Omega} \mathfrak{p}_\lambda \subset \mathfrak{p}_+$, so this takes care of the first term.

Now, by (11), for the second sum we have

$$\sum_{\lambda\in\Omega} \sum_{\mu\in(\Delta_R-\Omega)} [\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \sum_{\lambda\in\Omega} \sum_{\mu\in(\Delta_R-\Omega)} \mathfrak{p}_{\lambda+\mu} + \sum_{\lambda\in\Omega} \sum_{\mu\in(\Delta_R-\Omega)} \mathfrak{p}_{\lambda-\mu}.$$

Here we have then $\lambda \in \Omega$, $\mu \in (\Delta_R - \Omega)$. If *neither* $\lambda+\mu$ *nor* $\lambda-\mu$ is a root then $\mathfrak{p}_{\lambda+\mu} = \mathfrak{p}_{\lambda-\mu} = \{0\}$.

But it may happen however that either $\lambda+\mu$ or $\lambda-\mu$ is a root, or that they are both roots. In any case they belong to Δ_R. This yields

$$\sum_{\lambda\in\Omega} \sum_{\mu\in(\Delta_R-\Omega)} \mathfrak{p}_{\lambda+\mu} + \sum_{\lambda\in\Omega} \sum_{\mu\in(\Delta_R-\Omega)} \mathfrak{p}_{\lambda-\mu} \subset \sum_{\lambda\in\Delta_R} \mathfrak{p}_\lambda = \mathfrak{p}_+.$$

and then the second term is also in $\mathfrak{p}_+$, and so we have the second indicated inclusion.

Let us see the converse. Assume now that the inclusions (25) and (26) hold for our chosen Ω and let λ and μ be elements in Ω. If $\lambda+\mu$ and $\lambda-\mu$ are not roots then there is nothing to prove. Then we have to consider the following three possibilities, namely:

(1) Both of them are roots (that is both $\lambda+\mu$ and $\lambda-\mu \in \Delta_R$).
(2) $\lambda+\mu$ is a root and $\lambda-\mu$ is not.
(3) $\lambda-\mu$ is a root and $\lambda+\mu$ is not.

Let us consider separately each possibility.

Case (1). In this case, by Lemma 8, we have:

$$\begin{gathered}\mathfrak{k}_\lambda, \mathfrak{p}_\lambda \neq \{0\}, \quad \mathfrak{k}_\mu, \mathfrak{p}_\mu \neq \{0\}, \\ \mathfrak{k}_{\lambda+\mu}, \mathfrak{p}_{\lambda+\mu} \neq \{0\}, \quad \mathfrak{k}_{\lambda-\mu}, \mathfrak{p}_{\lambda-\mu} \neq \{0\},\end{gathered}$$

and by (13) and (11) in Lemma 7, we have $[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \neq 0$, $[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset (\mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu})$, but *by assumption* $[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_0(\Omega) = \mathfrak{a} \oplus \sum_{\lambda \in (\Delta_R - \Omega)} \mathfrak{p}_\lambda$, and then, since *a non-trivial part* of $(\mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu})$ is in $\mathfrak{p}_0(\Omega)$, then a *non-trivial part* of $(\mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu})$ is outside $\mathfrak{p}_+(\Omega)$. This implies $(\mathfrak{p}_{\lambda+\mu} + \mathfrak{p}_{\lambda-\mu}) \nsubseteq \sum_{\lambda \in \Omega} \mathfrak{p}_\lambda = \mathfrak{p}_+(\Omega)$, and then in the case (1) we conclude that neither $\lambda + \mu$ nor $\lambda - \mu \in \Omega$.

Case (2). We have in the present case

$$\mathfrak{k}_\lambda, \mathfrak{p}_\lambda \neq \{0\}, \quad \mathfrak{k}_\mu, \mathfrak{p}_\mu \neq \{0\},$$
$$\mathfrak{k}_{\lambda+\mu}, \mathfrak{p}_{\lambda+\mu} \neq \{0\}, \quad \mathfrak{k}_{\lambda-\mu}, \mathfrak{p}_{\lambda-\mu} = \{0\},$$

and by (13) and (11) in Lemma 7, we have $[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \neq 0$, $[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_{\lambda+\mu}$, but *by assumption* $[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_0(\Omega) = \mathfrak{a} \oplus \sum_{\lambda \in (\Delta_R - \Omega)} \mathfrak{p}_\lambda$, and this implies $\mathfrak{p}_{\lambda+\mu} \nsubseteq \sum_{\lambda \in \Omega} \mathfrak{p}_\lambda = \mathfrak{p}_+(\Omega)$. Then again $\lambda + \mu \notin \Omega$, and also $\lambda - \mu \notin \Omega$ since $\lambda - \mu$ is not a root.

Case (3). We have

$$\mathfrak{k}_\lambda, \mathfrak{p}_\lambda \neq \{0\}, \quad \mathfrak{k}_\mu, \mathfrak{p}_\mu \neq \{0\},$$
$$\mathfrak{k}_{\lambda+\mu}, \mathfrak{p}_{\lambda+\mu} = \{0\}, \quad \mathfrak{k}_{\lambda-\mu}, \mathfrak{p}_{\lambda-\mu} \neq \{0\},$$

and, once more, (13) and (11) in Lemma 7, indicate that $[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \neq \{0\}$ and $[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_{\lambda-\mu}$, but *by assumption* $[\mathfrak{k}_\lambda, \mathfrak{p}_\mu] \subset \mathfrak{p}_0(\Omega) = \mathfrak{a} \oplus \sum_{\lambda \in (\Delta_R - \Omega)} \mathfrak{p}_\lambda$, which yields $\mathfrak{p}_{\lambda-\mu} \nsubseteq \sum_{\lambda \in \Omega} \mathfrak{p}_\lambda = \mathfrak{p}_+(\Omega)$, and then $\lambda - \mu \notin \Omega$. Also $\lambda + \mu \notin \Omega$ since $\lambda + \mu$ is not a root.

Having considered the three alternatives we conclude that neither $\lambda + \mu$ nor $\lambda - \mu \in \Omega$, and so Ω is a *pre-symmetric* set ☐

7. Examples

To construct examples we just have to find pre-symmetric sets in the set of restricted roots $\Delta_R(\mathfrak{g}, \mathfrak{a})$; but this does not seem to be an easy task. However this can be facilitated by the knowledge of the structure of the set $\Delta_R(\mathfrak{g}, \mathfrak{a})$ obtained by Araki in [1], because using this and the classification of the symmetric spaces, it is simple to find pre-symmetric sets in $\Delta_R(\mathfrak{g}, \mathfrak{a})$. It is possible to give a complete description of the pre-symmetric sets for all the restricted root system, but since our intention is to describe the procedure we will indicate only two examples. In the determination of the dimension of the projective subspace one has to use the *multiplicities* of the restricted roots that participate.

7.1. **First example.** In some cases, the adjoint representation of K on $\mathfrak{p}$ has an orbit $S = Ad(K)V$ $(V \in \mathfrak{a})$ which happens to be a *symmetric space.* In this case one has:

Proposition 12. *If $S = Ad(K)V$ is a symmetric R-space then all normal sections at V (and hence at every point) are planar. That is, $X[S] = \mathbb{RP}^{s-1}$, $s = \dim(S)$.*

Proof. This follows from [6] where it is proved that for symmetric R-spaces the second fundamental form is parallel.

If $S = Ad\,(K)\,V$ is a symmetric R-space and $M = Ad\,(K)\,E$ is a principal orbit then the set $\Omega_S \subset \Delta_R$ defined by $\Omega_S = \{\lambda \in \Delta_R\,(\mathfrak{g}, \mathfrak{a}) : \lambda\,(V) \neq 0\}$ (recall that $V \in \mathfrak{a}$), is a pre-symmetric set. In fact, it is indicated in Lemma 5 that an R-space such as $S = Ad\,(K)\,V$ ($V \in \mathfrak{a}$) is *symmetric* if and only if the eigenvalues of $ad\,(V)$ on $\mathfrak{g}$ are $\{-1, 0, +1\}$. Therefore $\lambda \in \Omega_S$ if and only if $\lambda\,(V) = \pm 1$, then for λ and μ in Ω_S we have:

$$(\lambda + \mu)\,(V) = \begin{cases} 2 & \text{if } \lambda\,(V) = \mu\,(V) = 1 \\ 0 & \text{if } \lambda\,(V) = -\mu\,(V) \\ -2 & \text{if } \lambda\,(V) = \mu\,(V) = -1, \end{cases}$$

and the same thing happens to $\lambda - \mu$. Then the only possible value for $\lambda + \mu$ and $\lambda - \mu$ on V is 0 and therefore Ω_S is pre-symmetric. □

As a consequence we have:

Corollary 13. *If $S = Ad\,(K)\,V$ is a symmetric R-space and $M = Ad\,(K)\,E$ is a principal orbit then $\widehat{X}\,[M]$ contains $\widehat{X}\,[S]$. Hence $X\,[S] = \mathbb{RP}^{s-1} \subset X\,[M]$, $s = \dim\,(S)$.* □

7.2. **Second example.** We take now the case of the symmetric space EII. See [14, p. 518] and [18, p. 365].

That is: EII $\quad E_6/\,(SU\,(6)\,Sp\,(1))$ $\quad \dim\,(EII) = 40$ $\quad$ rank $= 4$ $\quad \Delta_R = F_4$

This is an inner symmetric space. We have in this space roots with *multiplicity* 1 and 2.

$$\begin{array}{ccc} \dim\,(F_4) = 52 & \#(\text{roots}) = 48 & |\Delta_R| = 24 \\ \dim\,(\mathfrak{p}) = \dim\,(EII) = 40 & \mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_+ & \dim\,(\mathfrak{p}_+) = 36 \end{array}$$

In $\Delta_R = F_4$ there are 24 positive roots. Of these 24 we have that 12 roots have multiplicity 1 and 12 have multiplicity 2.

The dimension of $FII = F_4/Spin\,(9)$ is 16. By [1] we have in Δ_R two simple roots with $m = 1$ and two others with $m = 2$.

$$\underset{\lambda_1}{\circ} - \underset{\lambda_2}{\circ} \Longrightarrow \underset{\lambda_3}{\circ} - \underset{\lambda_4}{\circ}, \qquad m\,(\lambda_i) = 1, i = 1, 2, \qquad m\,(\lambda_i) = 2,\ i = 3, 4.$$

We consider the set Ω defined by (the order is that in [16, p. 58]):

$$\begin{array}{ll} \beta_1 = \lambda_4 = (0,0,0,1) & \beta_5 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = (1,1,1,1) \\ \beta_2 = \lambda_3 + \lambda_4 = (0,0,1,1) & \beta_6 = \lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_4 = (1,1,2,1) \\ \beta_3 = \lambda_2 + \lambda_3 + \lambda_4 = (0,1,1,1) & \beta_7 = \lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 = (1,2,2,1) \\ \beta_4 = \lambda_2 + 2\lambda_3 + \lambda_4 = (0,1,2,1) & \beta_8 = \lambda_1 + 2\lambda_2 + 3\lambda_3 + \lambda_4 = (1,2,3,1)\,. \end{array}$$

This is a subset of the set of 24 short roots of F_4. These 8 roots are in fact the *positive* ones in the set of 16 roots defining the tangent space to the inner symmetric space $F_4/Spin\,(9)$ at the origin. In other words this subspace is the $\mathfrak{m}$ corresponding to the decomposition $\mathfrak{f}_4 = \mathfrak{so}\,(9) \oplus \mathfrak{m}$, and $\dim\,(\mathfrak{m}) = 16$. It is clear that this Ω is a pre-symmetric set (the sum of any two $\beta_i + \beta_j$ would be of the form $a\lambda_1 + b\lambda_2 + c\lambda_3 + 2\lambda_4$).

Finally, we would like to add a comment on the Satake diagrams in [1] and [14, p. 532–534].

The *black circles* in them represent those simple roots of the complexification, that *vanish* on restriction, while those with *white circles* are the simple roots that *do not vanish* on restriction. The *white circles joined by a curved arrow* are those *simple* roots that have *the same restriction*. If one takes all the roots of the complex algebra then it is possible to obtain the restrictions and then find the multiplicities.

Looking at the table of the roots of $\mathfrak{e}_6$, we see that *all the roots* $\beta_j \in \Omega$ ***have multiplicity*** 2.

Then the $\mathbb{R}$-vector subspace $\mathfrak{k}_+(\Omega) \simeq \mathfrak{p}_+(\Omega)$ (of dimension 16) yields a projective subspace of dimension 15. $\mathbb{R}P(\mathfrak{p}_+(\Omega)) = \mathbb{R}P(\mathfrak{k}_+(\Omega)) \subset X[M]$, where M is the orbit of E (an *arbitrary* regular element).

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