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ACCURATE APPROXIMATION OF A GENERALIZED MATHIEU SERIES

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ABSTRACT. New, accurate lower and upper bounds for the sum of the generalized Mathieu series

$$\sum_{k=1}^{\infty} \frac{2k}{(k^2+x^2)^{p+1}}, \quad p>0,$$

are obtained.

1. Introduction

Our goal is to estimate the sum $S(p,x)$ of the series

$$S(p,x) \equiv \sum_{k=1}^{\infty} \frac{2k}{(k^2+x^2)^{p+1}}\,, \quad p>0, \tag{1}$$

which is a mild generalization of $S(1,x)$, the sum of Mathieu's series [12]. This series and its generalizations have been the subject of research for a long period, e.g. [2, 4, 5, 13, 14, 15, 16, 17, 18, 19, 20]. For example, Alzer [2] obtained the inequality

$$\frac{1}{x^2+1/(2\zeta(3))} < S(1,x) < \frac{1}{x^2+1/6}\,, \quad \text{for } x\in\mathbb{R}\,. \tag{2}$$

Lately, F. Qi et al. [16, Th. 4.2, p. 2550] presented the estimate

$$s_1(x) \le S(1,x) \le s_2(x), \quad \text{for } x>0\,, \tag{3}$$

where

$$s_1(x) \equiv \frac{\left(1+4x^2\right)\left(e^{-\pi/x}+e^{-\pi/(2x)}\right)-4x^2-1}{\left(e^{-\pi/x}-1\right)\left(1+x^2\right)\left(1+4x^2\right)},$$

$$s_2(x) \equiv \frac{\left(1+4x^2\right)\left(e^{-\pi/x}-e^{-\pi/(2x)}\right)-4\left(x^2+1\right)}{\left(e^{-\pi/x}-1\right)\left(1+x^2\right)\left(1+4x^2\right)}.$$

The estimates (2) and (3) are relatively rough as has been shown several years ago in the paper [11, pp. 2269–2276, Figure 1], where more accurate bounds for

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$S(1,x)$ were given. For example, referring to [11, relations (17), (20) and (21)] there was presented the double inequality

$$a_L(m,x) < S(1,x) < b_L(m,x) \qquad (m \in \mathbb{N},\ x \in \mathbb{R}), \tag{4}$$

where

$$\sigma(m,x) \equiv \sum_{k=1}^{m-1} \frac{2k}{(k^2+x^2)^2} + \frac{1}{m^2+x^2} + \frac{m}{(m^2+x^2)^2} + \frac{3m^2-x^2}{6\,(m^2+x^2)^3}, \tag{5}$$

$$a_L(m,x) \equiv \left(1 - \frac{1}{2\,(m^2+x^2)^2 + 2m^3 + m^2}\right) \cdot \sigma(m,x), \tag{6}$$

$$b_L(m,x) \equiv \left(1 + \frac{1}{2\,(m^2+x^2)^2 + 2m^3 + m^2}\right) \cdot \sigma(m,x). \tag{7}$$

In the recently published paper [13] further bounds for $S(1,x)$ were given:

$$a_M(x) < S(1,x) < b_M(x) \qquad (x \in \mathbb{R}), \tag{8}$$

with

$$a_M(x) \equiv \frac{5\left(42x^6+341x^4+885x^2+814\right)}{6(x^2+1)(x^2+4)(35x^4+115x^2+72)}, \tag{9}$$

$$b_M(x) \equiv \frac{1680x^{10}+22460x^8+130092x^6+403017x^4+665570x^2+499305}{6(x^2+1)^2(280x^8+3230x^6+15583x^4+36627x^2+34614)}. \tag{10}$$

However, it turned out that the new bounds (8) are not more accurate than (4). This is illustrated in Figure 1, where the graphs of the functions $x \mapsto a_L(m,x)/a_M(x)$ (the upper lines) and $x \mapsto b_L(m,x)/b_M(x)$ (the lower lines) are plotted, using `Mathematica` [21], for several values of the parameter m.

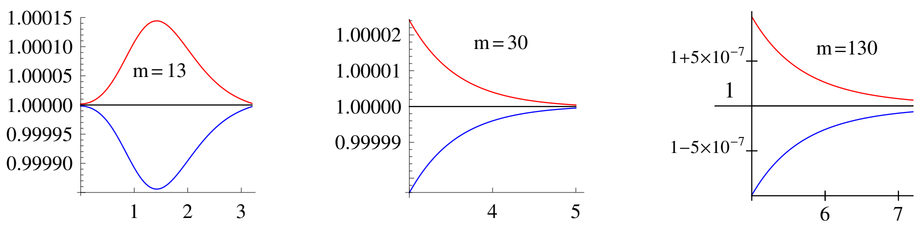

FIGURE 1. The graphs of the functions $x \mapsto a_L(m,x)/a_M(x)$ (the upper lines) and $x \mapsto b_L(m,x)/b_M(x)$ (the lower curves) for several values of the parameter m.

Additionally, for example, we estimate, also using `Mathematica` [21],

$$a_L(130,10)/a_M(10) = 1.000\,000\,028\,868\,\ldots > 1,$$
$$b_L(130,10)/b_M(10) = 0.999\,999\,996\,495\,\ldots < 1,$$
$$a_L(400,20)/a_M(20) = 1.000\,000\,000\,108\,\ldots > 1,$$
$$b_L(400,20)/b_M(20) = 0.999\,999\,999\,997\,\ldots < 1.$$

Unfortunately, it appears that Mortici [13] was not aware of the work published in [11].

In the present paper we extend our study of Mathieu's series $S(1,x)$ by inclusion of the generalized Mathieu's p-series $S(p,x)$. Our main result is the estimate

$$|S(p,x)-\sigma(m,p,x)| < \frac{p^3+10p^2+22p+10}{24\,(m^2+x^2)^{p+2}},$$

where $m \in \mathbb{N}$ is a parameter, and

$$\sigma(m,p,x) \equiv \sum_{k=1}^{m-1} \frac{2k}{(k^2+x^2)^{p+1}} + \frac{1}{p\,(m^2+x^2)^{p}} + \frac{m}{(m^2+x^2)^{p+1}} + \frac{(2p+1)m^2-x^2}{6\,(m^2+x^2)^{p+2}}.$$

As an instrument for this computation we use the Euler-Maclaurin summation formula which is the subject of the next section.

2. Summation of infinite series

We shall use the Euler-Maclaurin formula of order four, i.e. the Hermite rule, which produces a theorem comparing the convergence of a series and an integral. Referring to [10, Theorem, p. 320] or[1] [9, Theorem 2, p. 119] we have the following lemma.

Lemma 1 ([10, 9])**.** *If $f \in C^4[1,\infty)$, $\int_1^\infty \left|f^{(4)}(t)\right| \mathrm{d}t$ converges, and the finite limits $\lambda_0 := \lim_{\substack{n\in\mathbb{N}\\ n\to\infty}} f(n)$ and $\lambda_1 := \lim_{\substack{n\in\mathbb{N}\\ n\to\infty}} f'(n)$ exist, then:*

(a) *The series $\sum_{k=1}^\infty f(k)$ converges. $\iff$ The sequence $n \mapsto \int_1^n f(t)\,\mathrm{d}t$ converges.*

(b) *If the series $\sum_{k=1}^\infty f(k)$ converges, then $\lambda_0 = 0$, and*

$$\sum_{k=1}^{\infty} f(k) = \sum_{k=1}^{m-1} f(k) + \lim_{\substack{n\in\mathbb{N}\\ n\to\infty}} \int_m^n f(t)\,\mathrm{d}t + \frac{f(m)}{2} + \frac{\lambda_1}{12} - \frac{f'(m)}{12} + \rho(m), \tag{11}$$

for every integer $m \geq 1$, where $\rho(m) = -\int_m^\infty w(t) f^{(4)}(t)\,\mathrm{d}t$ with $w(t) = \frac{1}{24}\left[t(1-t)\right]^2$ for $t \in [0,1]$ and $w(t+1) \equiv w(t)$ (w is a 1-periodic function); consequently

$$\max_{0\leq t\leq 1} w(t) = \frac{1}{384} \qquad \text{and} \qquad |\rho(m)| \leq \frac{1}{384} \int_m^\infty \left|f^{(4)}(t)\right| \mathrm{d}t. \tag{12}$$

3. Approximating the sum $S(p,x)$

Referring to the previous section, the generalized Mathieu series (1) results from the function

$$f_{p,x}(t) \equiv \frac{2t}{(t^2+x^2)^{p+1}}, \tag{13}$$

where $p \in \mathbb{R}^+$ and $x \in \mathbb{R}$ are parameters.

[1] Supplementary literature: [1, 3, 6, 7, 8].

We have the integral,

$$\int_m^\infty f_{p,x}(t)\,\mathrm{d}t = \frac{1}{p\,(m^2+x^2)^p} \tag{14}$$

and the derivatives,

$$f'_{p,x}(t) \equiv -2\frac{(1+2p)t^2-x^2}{(t^2+x^2)^{p+2}}, \tag{15}$$

$$f^{(4)}_{p,x}(t) \equiv 8\left(p^2+3p+2\right)\frac{\left(4p^2+8p+3\right)t^5-10(2p+3)x^2t^3+15x^4t}{(t^2+x^2)^{p+5}}\,. \tag{16}$$

Thus, referring to (12) and (16), we consider the integral $R(t,p,x)$, defined as

$$\begin{aligned}
-R(t,p,x) &:= \frac{8\left(p^2+3p+2\right)}{384}\int\frac{\left(4p^2+8p+3\right)t^5+10(2p+3)x^2t^3+15x^4t}{(t^2+x^2)^{p+5}}\,\mathrm{d}t \\
&= \frac{p^2+3p+2}{48}\cdot\frac{1}{2\left(p^3+9p^2+26p+24\right)(t^2+x^2)^{p+4}} \\
&\quad\cdot\Big[\left(4p^4+36p^3+107p^2+117p+36\right)t^4 \\
&\qquad+2\left(14p^3+99p^2+205p+132\right)t^2x^2+\left(\underline{43p^2+161p+156}\right)x^4\Big] \\
&< \frac{(p+1)\cancel{(p+2)}}{96\cancel{(p+2)}(p+3)(p+4)(m^2+x^2)^{p+4}} \\
&\quad\cdot\Big[\left(4p^4+\dots\right)t^4+2\left(14p^3+\dots\right)t^2x^2+\left(\underline{44p^2+220p+176}\right)x^4\Big] \\
&= \frac{p+1}{96(p+3)\cancel{(p+4)}(t^2+x^2)^{p+\underline{\underline{4}}}}\cdot\Big[(p+3)\cancel{(p+4)}(2p+1)(2p+3)\underline{\underline{t^4}} \\
&\qquad+2\cancel{(p+4)}(2p+3)(7p+11)\underline{\underline{t^2x^2}}+44(p+1)\cancel{(p+4)}\,\underline{\underline{x^4}}\Big] \\
&\le \frac{(p+1)\big[(p+3)(2p+1)(2p+3)+2\,(2p+3)(7p+11)+44(p+1)\big]}{96(p+3)\,(t^2+x^2)^{p+\underline{\underline{2}}}} \\
&= \frac{4p^3+40p^2+85p+21+56/(p+3)}{96\,(t^2+x^2)^{p+2}} \\
&< \frac{4p^3+40p^2+88p+40}{96\,(t^2+x^2)^{p+2}} = \frac{p^3+10p^2+22p+10}{24\,(t^2+x^2)^{p+2}}\,.
\end{aligned} \tag{17}$$

Considering the above derivations and using (11) and (12), we can formulate the following theorem.

Theorem 1. *For any integer $m \ge 1$, for every positive p, and any real x, we have*

$$S(p,x) = \sigma(m,p,x) + \rho(m,p,x), \tag{18}$$

where

$$\sigma(m,p,x) \equiv \sum_{k=1}^{m-1} \frac{2k}{(k^2+x^2)^{p+1}} + \frac{1}{p\,(m^2+x^2)^p} + \frac{m}{(m^2+x^2)^{p+1}} + \frac{(2p+1)m^2-x^2}{6\,(m^2+x^2)^{p+2}}\,, \tag{19}$$

and where the error term is estimated as

$$|\rho(m,p,x)| < \frac{p^3+10p^2+22p+10}{24\,(m^2+x^2)^{p+2}}\,. \tag{20}$$

The relative error

$$r(m,p,x) := \frac{S(p,x)-\sigma(m,p,x)}{\sigma(m,p,x)} \tag{21}$$

of the approximation $S(p,x) \approx \sigma(m,p,x)$ is estimated, referring to (18) and (19), as

$$|r(m,p,x)| < R(m,p,x) := \frac{p^3+10p^2+22p+10}{24(m^2+x^2)^{p+2}\cdot\sigma(m,p,x)}\,, \tag{22}$$

for $m \geq 1$, $p > 0$ and $x \in \mathbb{R}$. This “a posteriori” estimate is rather good as can be seen in Figure 2, where, using `Mathematica` [21], the graphs of the sequences

$$m \mapsto r\big(m, \tfrac{1}{2}, 0\big) \equiv \frac{2\zeta(2)-\sigma\big(m,\frac{1}{2},x\big)}{\sigma\big(m,\frac{1}{2},x\big)} = \frac{\frac{\pi}{3}-\sigma\big(m,\frac{1}{2},x\big)}{\sigma\big(m,\frac{1}{2},x\big)}$$

and $m \mapsto R\left(m, \frac{1}{2}, 0\right)$, are depicted.

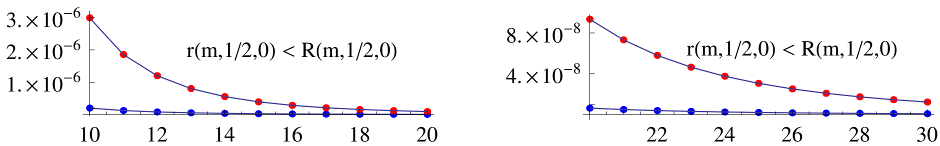

FIGURE 2. The graphs of the sequences $m \mapsto r\big(m, \frac{1}{2}, 0\big)$ and $m \mapsto R\big(m, \frac{1}{2}, 0\big)$.

The sequence $m \mapsto Q_m := \left|\frac{R(m,1/2,0)}{r(m,1/2,0)}\right|$ is nearly constant as is shown in Figure 3.

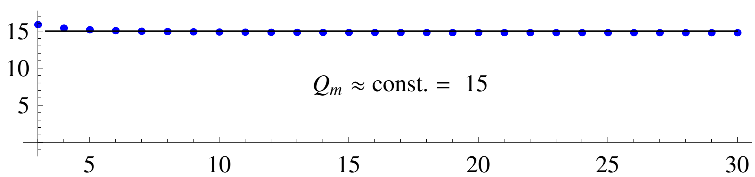

FIGURE 3. The graph of the sequence $m \mapsto Q_m$.

We can obtain also some “a priori” estimates for $r(m,p,x)$ using the modified relative error

$$r^*(m,p,x) := \frac{S(p,x)-\sigma(m,p,x)}{S(p,x)}\,. \tag{23}$$

Obviously, we have

$$r(m,p,x) \equiv \frac{r^*(m,p,x)}{1-r^*(m,p,x)}, \quad \text{or, equivalently,} \quad r^*(m,p,x) \equiv \frac{r(m,p,x)}{1+r(m,p,x)}\,. \tag{24}$$

The sum $S(p,x)$ we bound from below considering the monotonicity of the function $t \mapsto f_{p,x}(t)$ on certain domains. Referring to (15), this function is strictly decreasing on the interval $[t^*,\infty)$, where $t^* := |x|/\sqrt{2p+1} \geq 0$. We have two possibilities: $t^* \leq 1$ or $t^* > 1$.

In case $t^* \leq 1$, i.e. that $|x| \leq \sqrt{2p+1}$, the function $t \mapsto f_{p,x}(t)$ is strictly decreasing on the interval $[1,\infty)$. Consequently, in this case we have

$$S(p,x) = \sum_{k=1}^{\infty} f_{p,x}(k) > \sum_{k=1}^{\infty} \int_k^{k+1} f_{p,x}(t)\,\mathrm{d}t = \int_1^{\infty} f_{p,x}(t)\,\mathrm{d}t = \frac{1}{p(1+x^2)^p}\,. \tag{25}$$

In case $t^* > 1$, i.e. when $|x| > \sqrt{2p+1}$, the function $t \mapsto f_{p,x}(t)$ is strictly decreasing on the interval $[k^*+1,\infty)$, where $k^* = \lfloor t^* \rfloor$, the integer part of t^* ($k^* \in \mathbb{N}$ and $t^*-1 < k^* \leq t^*$). Therefore the following inequalities hold:

$$\begin{aligned} S(p,x) &> \sum_{k=k^*+1}^{\infty} f_{p,x}(k) > \sum_{k=k^*+1}^{\infty} \int_k^{k+1} f_{p,x}(t)\,\mathrm{d}t = \int_{k^*+1}^{\infty} f_{p,x}(t)\,\mathrm{d}t \\ &= \frac{1}{p\big((k^*+1)^2+x^2\big)^p} > \frac{1}{p\big((t^*+1)^2+x^2\big)^p}\,. \end{aligned} \tag{26}$$

As $t^* > 1$, we have $(t^*+1)^2 = t^{*2}+2t^*+1 < t^{*2}+2t^{*2}+t^{*2} = 4t^{*2} = 4x^2/(2p+1)$. Hence, referring to (26), we have the estimates

$$S(p,x) > \frac{1}{p\left(\frac{4x^2}{2p+1}+x^2\right)^p} = \frac{1}{p\,x^{2p}\left(\frac{4}{2p+1}+1\right)^p} > \frac{1}{p\,x^{2p}\,e^2}\,. \tag{27}$$

We summarize these calculations in the following lemma.

Lemma 2. *For $p>0$ and $x \in \mathbb{R}$ we have the estimate*

$$S(p,x) > \begin{cases} \dfrac{1}{p(1+x^2)^p}\,, & \text{if } |x| \leq \sqrt{2p+1}, \\[2ex] \dfrac{1}{p\cdot x^{2p}\cdot e^2}\,, & \text{if } |x| > \sqrt{2p+1}\,. \end{cases} \tag{28}$$

From (18), (20) and (28) we obtain the next theorem concerning the modified relative error $r^*(m,p,x) = \rho(m,p,x)/S(p,x)$.

Theorem 2. *For given $p>0$ and $x \in \mathbb{R}$, and for the modified relative error $r^*(m,p,x)$ there holds the estimate $|r^*(m,p,x)| < R^*(m,p,x)$, where*

$$R^*(m,p,x) := \begin{cases} \dfrac{p^4+10p^3+22p^2+10p}{24\,(m^2+x^2)^2}\left(1+\dfrac{m^2-1}{x^2+1}\right)^{-p}, & \text{if } |x| \leq \sqrt{2p+1}\,, \\[3ex] e^2\cdot\dfrac{(p^4+10p^3+22p^2+10p)}{24\,(m^2+x^2)^2}\left(1+\left(\dfrac{m}{x}\right)^2\right)^{-p}, & \text{if } |x| > \sqrt{2p+1}\,. \end{cases}$$

Obviously, we have $\lim\limits_{p\to\infty} R^*(m,p,x) = \lim\limits_{p\downarrow 0} R^*(m,p,x) = 0$, for $m \geq 2$ and $x \in \mathbb{R}$, and $\lim\limits_{|x|\to\infty} R^*(m,p,x) = 0$, for $m \geq 1$ and $p > 0$. In Figures 4 and 5 the graphs of the functions $p \mapsto R^*\big(m,p,x\big)$ are plotted.

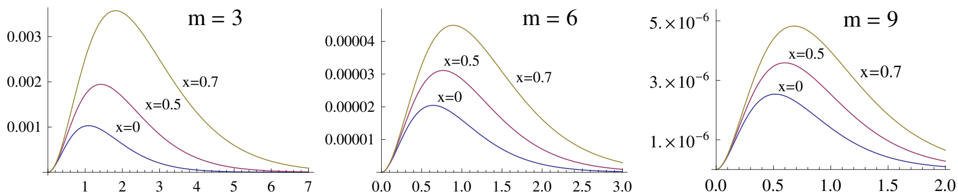

FIGURE 4. The graphs of the functions $p \mapsto R^*\big(m,p,x\big)$.

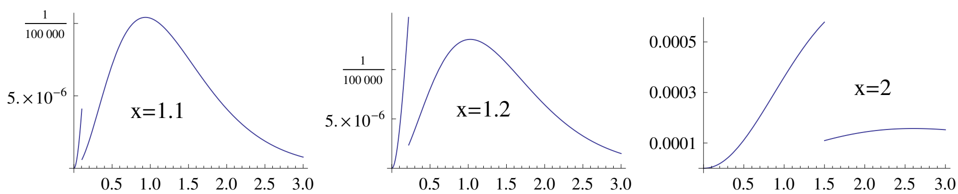

FIGURE 5. The graphs of the functions $p \mapsto R^*\big(9,p,x\big)$.

Example 1. *Using an integer $m \geq 4$ and $|x| \leq \frac{1}{2}m$ we have the inequalities $\left(\frac{m}{x}\right)^2 \geq 4$ and*

$$\frac{m^2-1}{x^2+1} \geq \frac{m^2-1}{\frac{1}{4}m^2+1} \geq \frac{m^2-1}{\frac{1}{4}m^2+\frac{1}{4}m} = 4\left(1-\frac{1}{m}\right) \geq 3.$$

Therefore, according to Theorem 2, we estimate

$$R^*(m,p,x) \leq \begin{cases} \dfrac{p^4+10p^3+22p^2+10p}{24m^4} \cdot 4^{-p}, & \text{for } |x| \leq \sqrt{2p+1}, \\ e^2 \cdot \dfrac{p^4+10p^3+22p^2+10p}{24m^4} \cdot 5^{-p}, & \text{for } |x| > \sqrt{2p+1}. \end{cases}$$

Moreover, using the rather rough estimates $a^p > \frac{\left(p\ln(a)\right)^k}{k!}$, i.e. $p^k a^{-p} < \frac{k!}{\left(\ln(a)\right)^k}$, true for $a > 1$, $k \in \mathbb{N}$ and $p > 0$, we obtain

$$\left(p^4+10p^3+22p^2+10p\right)4^{-p} < \frac{24}{\big(\ln(4)\big)^4} + 10\frac{6}{\big(\ln(4)\big)^3} + 22\frac{2}{\big(\ln(4)\big)^2} + \frac{10}{\ln(4)} < 60$$

and similarly

$$\left(p^4+10p^3+22p^2+10p\right)5^{-p} < \frac{24}{\big(\ln(5)\big)^4} + 10\frac{6}{\big(\ln(5)\big)^3} + 22\frac{2}{\big(\ln(5)\big)^2} + \frac{10}{\ln(5)} < 41.2.$$

Consequently, $R^*(m,p,x) < \frac{60}{24m^4}$*, for* $|x| \le \sqrt{2p+1}$*, and* $R^*(m,p,x) < \frac{41.2\,e^2}{24m^4} < \frac{305}{24m^4}$*, for* $|x| > \sqrt{2p+1}$*. Hence,*

$$|r^*(m,p,x)| \le R^*(m,p,x) < \frac{305}{24m^4}, \tag{29}$$

for $m \ge 4$*,* $p > 0$ *and* $|x| \le \frac{1}{2}m$*. Particularly, we have* $|r^*(20,p,x)| < 7.93 \cdot 10^{-5}$ *and using* (24) *also* $|r(20,p,x)| < 8 \cdot 10^{-5}$*, for* $p > 0$ *and* $|x| \le 10$*.*

Thus, according to (21) *we obtain the double uniform inequality,*

$$\boxed{0.99992 \cdot \sigma(20,p,x) < S(p,x) < 1.00008 \cdot \sigma(20,p,x)\,,} \tag{30}$$

true for $p > 0$ *and* $|x| \le 10$*. Of course, this uniform estimate is rather rough in comparison with the local estimate at a given point* (p,x)*.*

The graphs of the function $(p,x) \mapsto S(p,x)$, shown in Figures 6–8, were obtained by plotting $\sigma(20,p,x)$, using the estimate (30).

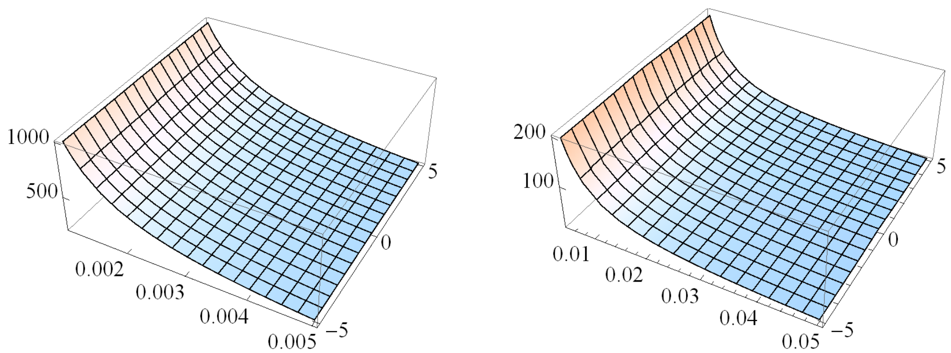

FIGURE 6. The graph of the function $(p,x) \mapsto S(p,x)$: left for $0.001 \le p \le 0.005$ and right for $0.005 \le p \le 0.05$.

Note that, according to (20), we have

$$\lim_{p\to\infty} \rho(2,p,x) = 0,$$

for every real x. Also, thanks to (19),

$$\lim_{p\to\infty} \sigma(2,p,x) = \begin{cases} 2, & \text{for } x = 0, \\ 0, & \text{for } x \ne 0. \end{cases}$$

Therefore, using (18) with $m = 2$, we obtain the δ-effect

$$\lim_{p\to\infty} S(p,x) = \begin{cases} 2, & \text{for } x = 0, \\ 0, & \text{for } x \ne 0. \end{cases}$$

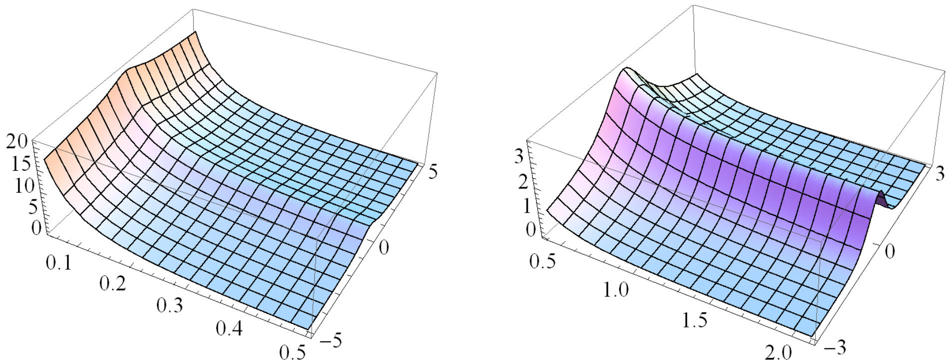

FIGURE 7. The graph of the function $(p, x) \mapsto S(p, x)$: left for $0.05 \leq p \leq 0.5$ and right for $0.4 \leq p \leq 2.1$.

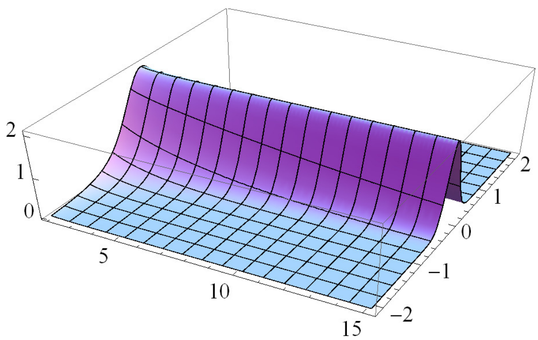

FIGURE 8. The graph of the function $(p, x) \mapsto S(p, x)$ for $2 \leq p \leq 15.2$.

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