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ON THE RATE OF CONVERGENCE FOR MODIFIED GAMMA OPERATORS

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Abstract. We give direct approximation theorems for some linear operators in certain weighted spaces. The results are given in terms of some Ditzian-Totik moduli of smoothness.

1. Introduction

Let A_n, $n \in \mathbb{N} := \{1, 2, \ldots\}$, be the gamma type operators given by the formula

$$A_n(f;x) = \int_0^\infty \frac{(2n+3)!\, x^{n+3} t^n}{n!(n+2)!(x+t)^{2n+4}} f(t)\, dt, \quad x, t \in \mathbb{R}_+ := (0, \infty) \tag{1.1}$$

and defined for any f for which the above integral is convergent.

The operators (1.1) are linear and positive, and preserve the functions $e_0(t) = 1$, $e_2(t) = t^2$ (see [5, 9, 11]). Basic facts on positive linear operators, their generalizations and applications, can be found in [2, 3].

Approximation properties of A_n were examined in [5]–[8]. Recently, İzgi [4] obtained the following result.

Lemma 1.1 ([4]). *For any $r \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$,*

$$A_n^{(r)}(f;x) = \frac{(2n+3)!}{n!(n+2)!} x^{n+3-r} \int_0^\infty \frac{t^{n+r}}{(x+t)^{2n+4}} f^{(r)}(t)\, dt, \quad x \in \mathbb{R}_+,$$

provided that the r-th derivative $f^{(r)}$ $(r = 0, 1, 2, \ldots)$ exists continuously.

The local rate of convergence and the Voronovskaya type theorem for operators $A_n^{(r)}$ were given in [4, 10].

In this paper we shall study approximation properties of $A_n^{(r)}$ for functions $f \in C_p$, $p \in \mathbb{N}_0$, where C_p is a polynomial weighted space with the weight function w_p,

$$w_0(x) = 1 \qquad \text{and} \qquad w_p(x) = \frac{1}{1+x^p}, \quad p \in \mathbb{N}, \tag{1.2}$$

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and C_p is the set of all real-valued functions f for which fw_p is uniformly continuous and bounded on $\mathbb{R}_0 = [0, \infty)$. The norm on C_p is defined by

$$\|f\|_p = \sup_{x \in \mathbb{R}_0} w_p(x)|f(x)|.$$

We shall prove approximation theorems which are similar to some results given in [8] for the operators (1.1).

We consider the modulus of continuity of $f \in C_p$,

$$\omega_p(f, \delta) = \sup_{h \in [0,\delta]} \|\Delta_h f\|_p,$$

and the modulus of smoothness of $f \in C_p$,

$$\omega_p^2(f, \delta) = \sup_{h \in [0,\delta]} \left\|\Delta_h^2 f\right\|_p,$$

where

$$\Delta_h f(x) = f(x+h) - f(x), \qquad \Delta_h^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

for $x, h \in \mathbb{R}_0$.

Throughout this paper we shall denote by $M_{\alpha,\beta}$ positive constants depending only on indicated parameters α, β, and point out that they are not the same at each occurrence.

2. Auxiliary results

In this section we give some preliminary results which will be used in the sequel.

Let

$$a(n,r) = \frac{(2n+3)!}{n!(n+2)!} x^{n+3-r} \int_0^\infty \frac{t^{n+r}}{(x+t)^{2n+4}} dt = \frac{(n+r)!(n+2-r)!}{n!(n+2)!}.$$

We consider the sequence of positive operators $\{A_{n,r}^*\}$, $n \in \mathbb{N}$, given by the formula

$$A_{n,r}^*(g;x) = \frac{(2n+3)! x^{n+3-r}}{(n+r)!(n+2-r)!} \int_0^\infty \frac{t^{n+r}}{(x+t)^{2n+4}} g(t)\, dt, \tag{2.1}$$

$x, t \in \mathbb{R}_+$, $r \in \mathbb{N}_0$ (see [4]).

If f is right side continuous at $x = 0$, we define

$$A_n(f, 0) = f(0), \quad A_{n,r}^*(f, 0) = f(0), \quad n \in \mathbb{N}.$$

In the sequel the following functions will be meaningful:

$$e_s(t) = t^s, \quad \phi_{x,s}(t) = (t-x)^s, \qquad s \in \mathbb{N}_0,\ x \in \mathbb{R}_0.$$

The s-th moments $A_{n,r}^*(e_s; x)$ were characterized in [4].

Lemma 2.1. ([4]) *For any $s \in \mathbb{N}$, $s \le n+2-r$, and $r \le n+2$ we have*

$$A^*_{n,r}(e_s;x) = \frac{(n+r+s)!(n+2-r-s)!}{(n+r)!(n+2-r)!}x^s,$$

$$A^*_{n,r}(\phi_{x,1};x) = \frac{2r-1}{n+2-r}x, \tag{2.2}$$

$$A^*_{n,r}(\phi_{x,2};x) = \frac{2(n+2r^2+1)}{(n+2-r)(n+1-r)}x^2, \tag{2.3}$$

$$A^*_{n,r}(\phi_{x,4};x) = \frac{c_{n,r}}{(n+2-r)(n+1-r)(n-r)(n-1-r)}x^4, \tag{2.4}$$

where $c_{n,r} = 12(n^2+6n+5)+4(n^2+23n+32)r+4(11n+32)r^2+64r^3+16r^4$ for each $x \in \mathbb{R}_0$.

Let $p, r \in \mathbb{N}_0$. By C_p^r we denote the space of all functions $f \in C_p$ on $\mathbb{R}_0$ such that $f', \ldots, f^{(r)} \in C_p$ on $\mathbb{R}_+$.

With the representations of Lemma 2.1 we may now determine the fundamental properties of the operators $A^*_{n,r}$ and $A_n^{(r)}$ necessary for characterizing their approximation properties in the next section.

Theorem 2.1. *Let $p \in \mathbb{N}_0$. The operator $A_n^{(r)}$ maps C_p^r into C_p^r and*

$$\left\| \frac{1}{a(n,r)} A_n^{(r)}(f;\cdot) \right\|_p \le M_{p,r} \left\| f^{(r)} \right\|_p \tag{2.5}$$

for $f \in C_p^r$ and $n > p+r-2$, where $M_{p,r}$ is some positive constant.

Proof. Let $p \in \mathbb{N}$. First, we remark that

$$\begin{aligned} w_p(x) A^*_{n,r}\left(\frac{1}{w_p};x\right) &= w_p(x)\left\{A^*_{n,r}(e_0;x) + A^*_{n,r}(e_p;x)\right\} \\ &= w_p(x)\left\{1 + \frac{(n+r+p)!(n+2-r-p)!}{(n+r)!(n+2-r)!}x^p\right\} \\ &\le M_{p,r} w_p(x)\{1+x^p\} = M_{p,r}, \end{aligned} \tag{2.6}$$

where

$$M_{p,r} = \max\left\{\sup_n \frac{(n+r+p)!(n+2-r-p)!}{(n+r)!(n+2-r)!}, 1\right\}.$$

Observe that for all $f \in C_p^r$ and every $x \in \mathbb{R}_+$ we get

$$\begin{aligned} & w_p(x)\left|\frac{1}{a(n,r)} A_n^{(r)}(f;x)\right| \\ &\le w_p(x) \frac{\alpha_n x^{n+3-r}}{a(n,r)} \int_0^\infty \frac{t^{n+r}}{(x+t)^{2n+4}} \left|f^{(r)}(t)\right| w_p(t) \frac{1}{w_p(t)}\, dt \\ &\le \left\|f^{(r)}\right\|_p w_p(x) A^*_{n,r}\left(\frac{1}{w_p};x\right). \end{aligned}$$

From (2.6) we obtain

$$w_p(x)\left|\frac{1}{a(n,r)}A_n^{(r)}(f;x)\right| \le M_{p,r}\left\|f^{(r)}\right\|_p$$

and we have the assertion for $p \in \mathbb{N}$.

In the case $p = 0$ the proof follows immediately. □

Lemma 2.2. *Let $p, r \in \mathbb{N}_0$. For the operators $A^*_{n,r}$, there exists a constant $M_{p,r} > 0$ such that*

$$w_p(x)A^*_{n,r}\left(\frac{\phi_{x,2}}{w_p};x\right) \le M_{p,r}\frac{x^2}{n}$$

for all $x \in \mathbb{R}_0$ and $n > p + r - 2$.

Proof. Using Lemma 2.1 we can write

$$\begin{aligned} w_0(x)A^*_{n,r}\left(\frac{\phi_{x,2}}{w_0};x\right) &= \frac{2(n+2r^2+1)}{(n+2-r)(n+1-r)}x^2 \\ &= \frac{2n(n+2r^2+1)}{(n+2-r)(n+1-r)}\cdot\frac{x^2}{n} \le M_r\frac{x^2}{n}, \end{aligned} \tag{2.7}$$

which gives the result for $p = 0$.

Let $p \ge 1$. Then, we get from Lemma 2.1

$$\begin{aligned} &A^*_{n,r}\left(\frac{\phi_{x,2}}{w_p};x\right) \\ &= A^*_{n,r}(e_{p+2};x) - 2xA^*_{n,r}(e_{p+1};x) + x^2A^*_{n,r}(e_p;x) + A^*_{n,r}(\phi_{x,2};x) \\ &= \frac{(n+r+p)!(n-r-p)!}{(n+r)!(n+2-r)!}x^{p+2}\left[(n+r+p+1)(n+r+p+2)\right. \\ &\quad \left. - 2(n+r+p+1)(n+1-r-p) + (n+1-r-p)(n+2-r-p)\right] \\ &\quad + \frac{2(n+2r^2+1)}{(n+2-r)(n+1-r)}x^2 \\ &= \left\{1 + \left[1 + \frac{4rp+2p^2}{n+2r^2+1}\right]\frac{(n+r+p)!(n-r-p)!}{(n+r)!(n-r)!}x^p\right\} \\ &\quad \times \frac{2x^2(n+2r^2+1)}{(n+2-r)(n+1-r)} \le M_{p,r}\frac{x^2}{n}(1+x^p), \end{aligned}$$

where $M_{p,r}$ is some positive constant. This completes the proof. □

3. Rate of convergence

The proof of the direct theorems will follow along standard lines using a Jackson type inequality, the Steklov means, and appropriate estimates of the moments of the operators.

Theorem 3.1. *Let $r, p \in \mathbb{N}_0$. If $g \in C_p^1$, then there exists a positive constant $M_{p,r}$ such that*

$$w_p(x)\left|A_{n,r}^*(g;x) - g(x)\right| \le M_{p,r}\,\|g'\|_p\,\frac{x}{\sqrt{n}}$$

for all $x \in \mathbb{R}_+$ and $n > p + r - 2$.

Proof. Let $x \in \mathbb{R}_+$. We have

$$g(t) - g(x) = \int_x^t g'(u)\,du, \quad t \ge 0.$$

Taking into account the fact that $A_{n,r}^*(1;x) = 1$ and using the linearity of $A_{n,r}^*$ we get

$$A_{n,r}^*(g;x) - g(x) = A_{n,r}^*\left(\int_x^t g'(u)\,du; x\right), \quad n \in \mathbb{N}. \tag{3.1}$$

Remark that

$$\left|\int_x^t g'(u)\,du\right| \le \|g'\|_p \left|\int_x^t \frac{du}{w_p(u)}\right| \le \|g'\|_p \left(\frac{1}{w_p(t)} + \frac{1}{w_p(x)}\right)|t - x|.$$

Hence and from (3.1) we obtain

$$w_p(x)\left|A_{n,r}^*(g;x) - g(x)\right| \le \|g'\|_p \left\{A_{n,r}^*\left(|\phi_{x,1}|;x\right) + w_p(x) A_{n,r}^*\left(\frac{|\phi_{x,1}|}{w_p};x\right)\right\}.$$

Applying the Cauchy–Schwarz inequality we can write

$$A_{n,r}^*\left(|\phi_{x,1}|;x\right) \le \left\{A_{n,r}^*(\phi_{x,2};x)\right\}^{1/2},$$

$$A_{n,r}^*\left(\frac{|\phi_{x,1}|}{w_p};x\right) \le \left\{A_{n,r}^*\left(\frac{1}{w_p};x\right)\right\}^{1/2}\left\{A_{n,r}^*\left(\frac{\phi_{x,2}}{w_p};x\right)\right\}^{1/2}.$$

Finally, using (2.3), (2.7), (2.6) and Lemma 2.2 we obtain

$$w_p(x)\left|A_{n,r}^*(g;x) - g(x)\right| \le M_{p,r}\,\|g'\|_p\,\frac{x}{\sqrt{n}}$$

for $n > p + r - 2$. □

Theorem 3.2. *Let $p, r \in \mathbb{N}_0$. If $f \in C_p^r$, then there exists a positive constant $M_{p,r}$ such that*

$$w_p(x)\left|\frac{1}{a(n,r)}A_n^{(r)}(f;x) - f^{(r)}(x)\right| \le M_{p,r}\,\omega_p\left(f^{(r)}, \frac{x}{\sqrt{n}}\right)$$

for all $x \in \mathbb{R}_+$ and $n > p + r - 2$.

Proof. Let $x \in \mathbb{R}_+$. We denote the Steklov means of $f^{(r)}$ by $f_h^{(r)}$, $h > 0$. Here we recall that

$$f_h^{(r)}(x) = \frac{1}{h}\int_0^h f^{(r)}(x+t)\,dt, \quad h, x \in \mathbb{R}_+.$$

It is obvious that

$$f_h^{(r)}(x) - f^{(r)}(x) = \frac{1}{h}\int_0^h \left[f^{(r)}(x+t) - f^{(r)}(x)\right]\,dt,$$

$$f_h^{(r+1)}(x) = \frac{1}{h}\left[f^{(r)}(x+h) - f^{(r)}(x)\right]$$

for $h, x \in \mathbb{R}_+$. Hence, if $f^{(r)} \in C_p$, then $f_h^{(r)} \in C_p^1$ for every fixed $h > 0$. Moreover we have

$$\left\|f_h^{(r)} - f^{(r)}\right\|_p \le \omega_p\left(f^{(r)}, h\right), \qquad \left\|f_h^{(r+1)}\right\|_p \le \frac{1}{h}\omega_p\left(f^{(r)}, h\right). \tag{3.2}$$

Observe that

$$\begin{aligned} w_p(x)&\left|A_{n,r}^*\left(f^{(r)}; x\right) - f^{(r)}(x)\right| \\ &\le w_p(x)\left|A_{n,r}^*\left(f^{(r)} - f_h^{(r)}; x\right)\right| + w_p(x)\left|A_{n,r}^*\left(f_h^{(r)}; x\right) - f_h^{(r)}(x)\right| \\ &\quad + w_p(x)\left|f_h^{(r)}(x) - f^{(r)}(x)\right|. \end{aligned}$$

Using Theorem 2.1 and (3.2) we obtain

$$\begin{aligned} w_p(x)&\left|A_{n,r}^*\left(f^{(r)} - f_h^{(r)}; x\right)\right| \\ &= w_p(x)\left|A_{n,r}^*\left((f - f_h)^{(r)}; x\right)\right| = w_p(x)\left|\frac{1}{a(n,r)}A_n^{(r)}\left(f - f_h; x\right)\right| \\ &\le M_{p,r}\left\|(f - f_h)^{(r)}\right\|_p = M_{p,r}\left\|f^{(r)} - f_h^{(r)}\right\|_p \le M_{p,r}\,\omega_p\left(f^{(r)}, h\right) \end{aligned}$$

for $h, x \in \mathbb{R}_+$ and $n > p + r - 2$. From Theorem 3.1 and (3.2) we get

$$\begin{aligned} w_p(x)\left|A_{n,r}^*\left(f_h^{(r)}; x\right) - f_h^{(r)}(x)\right| &\le M_{p,r}\left\|f_h^{(r+1)}\right\|_p \frac{x}{\sqrt{n}} \\ &\le M_{p,r}\frac{1}{h}\omega_p\left(f^{(r)}, h\right)\frac{x}{\sqrt{n}}. \end{aligned}$$

By (3.2) we can write

$$w_p(x)\left|f_h^{(r)}(x) - f^{(r)}(x)\right| \le \left\|f_h^{(r)} - f^{(r)}\right\|_p \le \omega_p\left(f^{(r)}, h\right)$$

for $h, x \in \mathbb{R}_+$ and $n > p + r - 2$. Finally we obtain

$$\begin{aligned} w_p(x)\left|\frac{1}{a(n,r)}A_n^{(r)}(f; x) - f^{(r)}(x)\right| &= w_p(x)\left|A_{n,r}^*\left(f^{(r)} - f_h^{(r)}; x\right)\right| \\ &\le \omega_p\left(f^{(r)}, h\right)\left[M_{p,r} + \frac{1}{h}M_{p,r}\frac{x}{\sqrt{n}} + 1\right] \end{aligned}$$

for $h, x \in \mathbb{R}_+$, $n > p + r - 2$. Thus, choosing $h = \frac{x}{\sqrt{n}}$, the proof is completed. □

Now, we establish the next auxiliary result.

Lemma 3.1. *Let $p, r \in \mathbb{N}_0$ and $n_0 = \max\{4 - 2r, p + r - 2\}$. If*

$$H_{n,r}(f; x) = A_{n,r}^*(f; x) - f\left(x + \frac{2r-1}{n+2-r}x\right) + f(x), \tag{3.3}$$

then there exists a positive constant $M_{p,r}$ such that, for all $x \in \mathbb{R}_+$ and $n > n_0$, we have

$$w_p(x)|H_{n,r}(g;x) - g(x)| \le M_{p,r} \left\|g''\right\|_p \frac{x^2}{n}$$

for any function $g \in C_p^2$.

Proof. By the Taylor formula one can write

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du, \quad t \in \mathbb{R}_+.$$

Observe that

$$H_{n,r}(e_0;x) = H_{n,r}(1;x) = 1, \quad H_{n,r}(\phi_{x,1};x) = 0.$$

Then,

$$\begin{aligned} |H_{n,r}(g;x) - g(x)| = |H_{n,r}(g - g(x);x)| &= \left| H_{n,r}\left(\int_x^t (t-u)g''(u)\,du; x \right) \right| \\ &= \left| A^*_{n,r}\left(\int_x^t (t-u)g''(u)\,du; x \right) \right. \\ &\quad \left. - \int_x^{x + \frac{2r-1}{n+2-r}x} \left(x + \frac{2r-1}{n+2-r}x - u \right) g''(u)\,du \right|. \end{aligned}$$

Since

$$\left| \int_x^t (t-u)g''(u)\,du \right| \le \frac{\left\|g''\right\|_p (t-x)^2}{2} \left(\frac{1}{w_p(x)} + \frac{1}{w_p(t)} \right)$$

and

$$\left| \int_x^{x + \frac{2r-1}{n+2-r}x} \left(x + \frac{2r-1}{n+2-r}x - u \right) g''(u)\,du \right| \le \frac{\left\|g''\right\|_p}{2w_p(x)} \left(\frac{2r-1}{n+2-r}x \right)^2,$$

we get

$$\begin{aligned} w_p(x)|H_{n,r}(g;x) - g(x)| &\le \frac{\left\|g''\right\|_p}{2} \left[A^*_{n,r}\left(\phi_{x,2};x\right) + w_p(x) A^*_{n,r}\left(\frac{\phi_{x,2}}{w_p}; x \right) \right] \\ &\quad + \frac{\left\|g''\right\|_p}{2} \left(\frac{2r-1}{n+2-r}x \right)^2. \end{aligned}$$

Hence, by (2.7) and Lemma 2.2 we obtain

$$w_p(x)|H_{n,r}(g;x) - g(x)| \le M_{p,r} \left\|g''\right\|_p \frac{x^2}{n}$$

for any function $g \in C_p^2$ and $n > n_0$, where $n_0 = \max\left\{4 - 2r, p + r - 2\right\}$. The lemma is proved. □

A further uniform estimate is indicated in the next theorem.

Theorem 3.3. *Let $p, r \in \mathbb{N}_0$ and $n_0 = \max\{4-2r, p+r-2\}$. If $f \in C_p^r$, then there exists a positive constant $M_{p,r}$ such that*

$$w_p(x)\left|\frac{1}{a(n,r)}A_n^{(r)}(f;x) - f^{(r)}(x)\right| \le M_{p,r}\,\omega_p^2\left(f^{(r)}, \frac{x}{\sqrt{n}}\right) + \omega_p\left(f^{(r)}, \frac{2r-1}{n+2-r}x\right)$$

for all $x \in \mathbb{R}_+$ and $n > n_0$.

Proof. Let $f \in C_p^r$. We consider the Steklov means $\tilde{f}_h^{(r)}$ of second order of $f^{(r)}$ given by the formula (see [1], p. 317)

$$\tilde{f}_h^{(r)}(x) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\left\{2f^{(r)}(x+s+t) - f^{(r)}(x+2(s+t))\right\}ds\,dt$$

for $h, x \in \mathbb{R}_+$. We have

$$f^{(r)}(x) - \tilde{f}_h^{(r)}(x) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\Delta_{s+t}^2 f^{(r)}(x)\,ds\,dt,$$

which gives

$$\left\|f^{(r)} - \tilde{f}_h^{(r)}\right\|_p \le \omega_p^2\left(f^{(r)}, h\right). \tag{3.4}$$

Remark that

$$\tilde{f}_h^{(r+2)}(x) = \frac{1}{h^2}\left(8\Delta_{h/2}^2 f^{(r)}(x) - \Delta_h^2 f^{(r)}(x)\right)$$

and

$$\left\|\tilde{f}_h^{(r+2)}\right\|_p \le \frac{9}{h^2}\omega_p^2(f^{(r)}, h). \tag{3.5}$$

From (3.4) and (3.5) we conclude that $\tilde{f}_h^{(r)} \in C_p^2$ if $f^{(r)} \in C_p$.

Observe that

$$\begin{aligned}
&\left|A_{n,r}^*(f^{(r)};x) - f^{(r)}(x)\right| \\
&\quad\le H_{n,r}\left(\left|f^{(r)} - \tilde{f}_h^{(r)}\right|;x\right) + \left|f^{(r)}(x) - \tilde{f}_h^{(r)}(x)\right| \\
&\quad\quad+ \left|H_{n,r}\left(\tilde{f}_h^{(r)};x\right) - \tilde{f}_h^{(r)}(x)\right| + \left|f^{(r)}\left(x + \frac{2r-1}{n+2-r}x\right) - f^{(r)}(x)\right|,
\end{aligned}$$

where $H_{n,r}$ is defined in (3.3). Since $\tilde{f}_h^{(r)} \in C_p^2$ by the above, it follows from Theorem 2.1 and Lemma 3.1 that

$$\begin{aligned}
w_p(x)\left|\frac{1}{a(n,r)}A_n^{(r)}(f;x) - f^{(r)}(x)\right| &= w_p(x)\left|A_{n,r}^*\left(f^{(r)};x\right) - f^{(r)}(x)\right| \\
&\le (M+3)\left\|f^{(r)} - \tilde{f}_h^{(r)}\right\|_p + M_{p,r}\left\|\tilde{f}_h^{(r+2)}\right\|_p\frac{x^2}{n} \\
&\quad+ w_p(x)\left|f^{(r)}\left(x + \frac{2r-1}{n+2-r}x\right) - f^{(r)}(x)\right|
\end{aligned}$$

for $n > n_0$, $n_0 = \max\{4-2r, p+r-2\}$. By (3.4) and (3.5), the last inequality yields that

$$w_p(x)\left|\frac{1}{a(n,r)}A_n^{(r)}(f;x)-f^{(r)}(x)\right| \\ \leq M_{p,r}\,\omega_p^2(f,h)\left\{1+\frac{1}{h^2}\frac{x^2}{n}\right\}+\omega_p\left(f^{(r)},\frac{2r-1}{n+2-r}x\right).$$

Thus, choosing $h=\frac{x}{\sqrt{n}}$ we get the result. □

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