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QUASILINEAR EIGENVALUES

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Abstract. We review and extend some well known results for the eigenvalues of the Dirichlet p-Laplace operator to a more general class of monotone quasilinear elliptic operators. As an application we obtain some homogenization results for nonlinear eigenvalues.

1. Introduction

In this work we review the eigenvalue problem associated to the p-Laplace operator,

$$\begin{cases} -\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda\rho u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

we describe its history and the main results obtained in the past years, and we extend those results to more general quasilinear problems.

To be precise, we consider the equation

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = \lambda\rho(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the function $a(x,\xi)$ has the same homogeneity of $|\xi|^{p-2}\xi$, and has precise hypotheses that we state below (see Section 3). The domain $\Omega \subset \mathbb{R}^N$ is assumed to be bounded, $N \geq 1$, and the weight function ρ is assumed to be bounded away from zero and infinity.

We denote the spectrum of (1.1) by Σ, i.e.

$$\Sigma := \{\lambda \in \mathbb{R} : \text{there exists a nontrivial weak solution to (1.1)}\},$$

and we focus our attention on the properties of the set Σ and the associated eigenfunctions.

The paper is organized as follows: in Section 2 we introduce the origins of the p-Laplace operator, and the history of the developments made for the eigenvalue problem. In Section 3 we introduce more general operators generalizing the p-Laplacian. Section 4 is devoted to the properties of eigenvalues and eigenfunctions.

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We define the variational spectrum (it has been a long-standing question to determine if Σ and the variational spectrum coincide; in some situations it is known that they differ, see [11, 24]), and we collect some necessary definitions and results. Finally we close the paper with some recent results on eigenvalue homogenization in Section 5.

2. A bit of history

The one dimensional p-Laplace ordinary differential equation,

$$-\left(|y'|^{p-2}y'\right)' = \rho(x)|y|^{p-2}y \tag{2.1}$$

was studied first by Leonhard Euler, in the work [26] appeared in 1732. Several cases were presented as an example of a nonlinear second order equation which cannot be integrated with known techniques.

Euler considered nonlinear equations of the general form

$$ax^m dx^p = y^n dy^{p-2} ddy,$$

where a is a constant, which correspond to equation (2.1) when $a = -(p-1)^{-1}$, $\rho = x^m$, and $n = 1-p$. He introduced in that work the exponential function in order to change variables, and reduced it to a first order equation. He used the substitution

$$\begin{cases} x^{\frac{1}{n+p-1}} = e^{\int z dt} \\ y = e^{(m+p)\int z dt} \end{cases}$$

in paragraphs 7-9, where "*dx constant ponatur*" means that x was chosen as the independent variable, and then $ddx = 0$.

Observe that, although this substitution enables us to work with Emden-Fowler like equations

$$-(|y'|^{p-2}y')' = |y|^{q-2}y,$$

a different one is needed when $p = q$ since $n = 1-p$. This case was included in paragraph 20, where he considered

$$dy^{m-1}ddy = P(x)y^m dx^{m+1} + Q(x)y^{m-b}dy^b dx^{m-b+1},$$

where we have interchanged x and y for readability. In modern notation, with $m = p-1$, it reads

$$|y'|^{p-2}y'' = P(x)|y|^{p-2}y + Q(x)y^{p-1-b}|y'|^b.$$

Euler emphasized the homogeneity of the three terms involved, and the fact that more similar terms can be added. For this equation he derived a generalized Riccati equation, rediscovered for the one dimensional p-Laplacian in the 20th century:

$$z^{p-2}z' + z^p = Q(y)z^b + P(y),$$

or, by calling $z^{p-1} = w$,

$$\frac{w'}{p-1} + w^{\frac{p}{p-1}} = Q(y)w^{\frac{b}{p-1}} + P(y). \tag{2.2}$$

When $Q \equiv 0$, the Riccati equation (2.2) was used by Beesack in 1961, see [9], connected with optimal constants in Hardy's inequality. Let us remark that Bihari in 1956 studied a related nonlinear equation in [10],

$$-y'' = Q(x)f(y, y'),$$

with $yf(y, y') > 0$, $f(cy, cy') = cf(y, y')$, and Lipschitz on every bounded domain of $\mathbb{R}$. However, this last condition excludes the p-Laplacian.

A few years later, Browder studied N-dimensional quasilinear equations, inspired in previous works of Višik, see [13, 14] and the references in this work. He introduced the so-called monotonicity methods (discovered almost simultaneously by Minty [39], and Vainberg and Kachurovski [41]). Since then, the study of quasilinear operators experimented an explosive growth, and both variational and non-variational techniques were introduced by Browder, Fučík, Ladyzhenskaya, Leray, Lions, Morrey, Nečas, Rabinowicz, Schauder, Serrin, and Trudinger, among several other mathematicians.

The eigenvalue problem for the p-Laplace operator started with the pioneering work of Browder [15, 16, 17, 18]. In those papers, he studied the nonlinear eigenvalue problem $A(u) = \lambda B(u)$, where λ is real parameter, and

$$A(u) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha F_{p^\alpha}(x, u, \dots, D^m u),$$

$$B(u) = \sum_{|\beta| \le m-1} (-1)^{|\beta|} G_{p^\beta}(x, u, \dots, D^{m-1} u).$$

This higher-order elliptic problem is the Euler-Lagrange equation of the variational problem

$$\min \left\{ \int_\Omega F(x, u, \dots, D^m u)\, dx : u \in V \text{ and } \int_\Omega G(x, u, \dots, D^{m-1} u)\, dx = c \right\},$$

where V is some closed subspace of $W^{m,p}(\Omega)$.

By introducing the variables $\zeta = \{\zeta_\alpha : |\alpha| = m\}$, $\psi = \{\psi_\xi : |\xi| \le m-1\}$, the functions F, G are measurable in x, and C^1 in the variables ψ, ζ, satisfying polynomial growth conditions which include the following particular case for the Dirichlet boundary value problem,

$$\begin{aligned} |F(x, \psi, \zeta)| &\le c(1 + |\zeta|^p + |\psi|^p), \\ |F_\alpha| + |F_\xi| &\le c(1 + |\zeta|^{p-1} + |\psi|^{p-1}), \\ G &= u^q, \end{aligned}$$

where $1 < p < \infty$, $q < np(n - mp)^{-1}$ for $n > mp$, and any q for $n < mp$.

With appropriate conditions of ellipticity and coercivity, the existence of an eigenvalue and a corresponding eigenfunction which is a weak solution of $A(u) = \lambda B(u)$ can be found in [15]. Moreover, for $p \ge 2$, and imposing more regularity on F and G (at least C^2 in the variables ψ and ζ), the existence of a sequence of eigenvalues was announced in [16] and proved in [17]. We can found in those

works the heavy (now standard) machinery of Palais-Smale sequences, deformation lemmas, Lyusternik-Schnirelman category, and monotonicity arguments.

Finally, a different approach can be found in [18], based on Galerkin approximations. Here, for higher-order quasilinear operators satisfying the same coercivity and polynomial growth conditions, the regularity conditions can be relaxed, and a sequence of eigenvalues is obtained for C^1 functions and $1 < p < \infty$.

Since then, several works devoted to this subject appeared. The interested reader can browse into the book of Fučík, Nečas, J. Souček, and V. Souček [33] for a survey up to the mid 1970s. It is worth noticing that several of the works cited therein were published in Russian, or in journals from Central and East Europe, so many results were rediscovered later. Nonlinear eigenvalue problems was an active area of research among Czech, German and Hungarian mathematicians in this decade (we mention Amann, Elbert, Fučík, Hess, Kufner, Nečas, and Zeidler, to cite only a few of them). See for example [3] for generalizations of Browder's results and applications to Hammerstein's equations; [32] for integro-differential equations; and [43] where two sequences of eigenvalues going to $\pm\infty$ were obtained for indefinite eigenvalue problems.

In the p-Laplacian case, i.e. when $a(x,\xi) = |\xi|^{p-2}\xi$, and for Dirichlet boundary conditions, the structure of Σ has been analyzed by several authors and it is known that

- $\Sigma \subset (0,\infty)$ is a closed set, see the work of Lindqvist [38].
- $\lambda_1 = \min \Sigma$ is the only eigenvalue that has a nonnegative associated eigenfunction, i.e., it is a principal eigenvalue. This principal eigenvalue has a variational characterization given by

$$\lambda_1 = \inf_{v \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |v|^p \, dx}.$$

  The above infimum is realized precisely at eigenfunctions associated to λ_1. See [4, 38].
- λ_1 is *isolated* and *simple*. That is, there exists $\delta > 0$ such that

$$(\lambda_1, \lambda_1 + \delta) \cap \Sigma = \emptyset,$$

  and if $u_1, u_2 \in W_0^{1,p}(\Omega)$ are two eigenfunctions associated to λ_1 then there exists $c \in \mathbb{R}$ such that $u_1 = cu_2$. See [2, 4, 38].
- There exists a sequence of *variational eigenvalues*, usually denoted by Σ_{var}, given by

$$\lambda_k := \inf_{C \in \mathcal{C}_k} \sup_{v \in C} \frac{\int_\Omega |\nabla v|^p \, dx}{\int_\Omega |v|^p \, dx},$$

  where $\mathcal{C}_k := \{C \subset W_0^{1,p}(\Omega) : C \text{ is closed}, C = -C, \gamma(C) \geq k\}$ and γ is the Krasnoselskii genus. This was the approach of Browder, by using the Lyusternik-Schnirelmann theory, see also [34, 35].

- There exist other possible ways to construct variational eigenvalues for this type of equations. Some authors prefer to call Σ_{var} the *Lyusternik-Schnirelmann* eigenvalues, although in this work we will use the more extended denomination and refer to these as the *variational eigenvalues*. See [24] for a comprehensive discussion on this topic.
- The sequence Σ_{var} has the asymptotic behavior given by Weyl's law
$$c\left(\frac{k}{|\Omega|}\right)^{\frac{N}{p}} \leq \lambda_k \leq C\left(\frac{k}{|\Omega|}\right)^{\frac{N}{p}},$$
for some (universal) constants $c, C > 0$ depending only on N and p. See [31, 35].
- As the first eigenvalue λ_1 is isolated in Σ which is a closed set, the second eigenvalue is well defined as
$$\Lambda_2 = \min\{\lambda \in \Sigma : \lambda > \lambda_1\} > \lambda_1.$$
It is known that Λ_2 coincides with the second variational eigenvalue λ_2. See [6, 22].
- For one dimensional problems, $\Omega = (a, b) \subset \mathbb{R}$, it is known that any eigenvalue is simple, the eigenfunction corresponding to λ_k has exactly $k + 1$ zeros counting the boundary points a and b, and this fact enables us to obtain them variationally. The eigenvalues can be computed explicitly, and the corresponding eigenfunctions are obtained in terms of the Gaussian hypergeometric function (see [8, 23, 28, 42]).
- A major open question is whether $\Sigma = \Sigma_{\text{var}}$ or not. An answer to this problem is only known in one space dimension. In this situation the question is answered positively, using that eigenvalues associated to λ_k have k nodal domains. See [28, 42]. A negative result is known for periodic boundary conditions, see [11, 24, 25].

The objective of this paper is the extension of all these facts to the more general problem (1.1). Let us observe that the first item follows by monotonicity arguments, and the second one was already generalized to (1.1) by [37]. So here we complete the program in performing the other extensions.

As a corollary of our results we obtain some alternative proofs of convergence theorems for nonlinear eigenvalue homogenization that were originally proved in [19].

3. Preliminary results

In this section we review some results gathered from the literature, enabling us to clearly state our results and making the paper self-contained.

3.1. **Monotone operators.** First, we give the precise hypotheses on the coefficient $a(x, \xi)$ in order to be able to treat the eigenvalue equation (1.1) variationally. The precise context is the assumption that the induced operator $\mathcal{A} : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ given by
$$\mathcal{A}u := -\operatorname{div}(a(x, \nabla u)),$$

defines a monotone operator.

So we assume that $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies, for every $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$, the following conditions:

(H0) *measurability:* $a(\cdot,\cdot)$ is a Carathéodory function, i.e. $a(x,\cdot)$ is continuous a.e. $x \in \Omega$, and $a(\cdot,\xi)$ is measurable for every $\xi \in \mathbb{R}^N$.
(H1) *monotonicity:* $0 \le (a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2)$.
(H2) *coercivity:* $\alpha|\xi|^p \le a(x,\xi)\xi$.
(H3) *continuity:* $a(x,\xi) \le \beta|\xi|^{p-1}$.
(H4) *p-homogeneity:* $a(x,t\xi) = t^{p-1}a(x,\xi)$ for every $t > 0$.
(H5) *oddness:* $a(x,-\xi) = -a(x,\xi)$.

Let us introduce $\Psi(x,\xi_1,\xi_2) = a(x,\xi_1)\xi_1 + a(x,\xi_2)\xi_2$ for all $\xi_1,\xi_2 \in \mathbb{R}^N$, and all $x \in \Omega$; and let $\delta = \min\{p/2,(p-1)\}$.

(H6) *equi-continuity:*

$$|a(x,\xi_1) - a(x,\xi_2)| \le c\Psi(x,\xi_1,\xi_2)^{(p-1-\delta)/p}(a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2)^{\delta/p}.$$

(H7) *cyclical monotonicity:* $\sum_{i=1}^k a(x,\xi_i)(\xi_{i+1} - \xi_i) \le 0$, for all $k \ge 1$, and $\xi_1,\dots,\xi_{k+1}$, with $\xi_1 = \xi_{k+1}$.
(H8) *strict monotonicity:* let $\gamma = \max(2,p)$, then

$$\alpha|\xi_1 - \xi_2|^\gamma \Psi(x,\xi_1,\xi_2)^{1-(\gamma/p)} \le (a(x,\xi_1) - a(x,\xi_2))(\xi_1 - \xi_2).$$

See [7, Section 3.4], where a detailed discussion on the relation and implications of every condition (H0)–(H8) is given.

In particular, under these conditions, we have the following Proposition:

Proposition 3.1 ([7], Lemma 3.3)**.** *Given $a(x,\xi)$ satisfying* (H0)–(H8) *there exists a unique Carathéodory function Φ which is even, p-homogeneous strictly convex and differentiable in the variable ξ satisfying*

$$\alpha|\xi|^p \le \Phi(x,\xi) \le \beta|\xi|^p \tag{3.1}$$

for all $\xi \in \mathbb{R}^N$ a.e. $x \in \Omega$ such that

$$\nabla_\xi \Phi(x,\xi) = p\, a(x,\xi)$$

and normalized such that $\Phi(x,0) = 0$.

Remark 3.2. In the one dimensional case, hypotheses (H4) and (H5) imply that

$$a(x,\xi) = a(x)|\xi|^{p-2}\xi,$$

with $a(x) := a(x,1)$. In this case, the potential function Φ is given by

$$\Phi(x,\xi) = a(x)|\xi|^p.$$

Remark 3.3. In dimension $N > 1$, the prototypical example for $a(x,\xi)$ is

$$a(x,\xi) = |A(x)\xi\cdot\xi|^{\frac{p-2}{2}}A(x)\xi.$$

In this case, the potential function $\Phi(x,\xi)$ of Proposition 3.1 is given by

$$\Phi(x,\xi) = 2|A(x)\xi\cdot\xi|^{\frac{p}{2}}.$$

3.2. **Definition of G-convergence.** For our application to homogenization, the concept of G-convergence of operators is needed. We review here the basic definitions and properties.

Definition 3.4. We say that the family of operators $\mathcal{A}_\varepsilon u := -\operatorname{div}(a_\varepsilon(x,\nabla u))$ G-converges to $\mathcal{A}u := -\operatorname{div}(a(x,\nabla u))$ if for every $f \in W^{-1,p'}(\Omega)$ and for every f_ε strongly convergent to f in $W^{-1,p'}(\Omega)$, the solutions u^ε of the problem

$$\begin{cases} -\operatorname{div}(a_\varepsilon(x,\nabla u^\varepsilon)) = f_\varepsilon & \text{in } \Omega \\ u^\varepsilon = 0 & \text{on } \partial\Omega \end{cases}$$

satisfy the following conditions

$$\begin{aligned} u^\varepsilon &\rightharpoonup u && \text{weakly in } W_0^{1,p}(\Omega), \\ a_\varepsilon(x,\nabla u^\varepsilon) &\rightharpoonup a(x,\nabla u) && \text{weakly in } (L^p(\Omega))^N, \end{aligned}$$

where u is the solution to the equation

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

For instance, in the linear periodic case, the family $-\operatorname{div}(A(\frac{x}{\varepsilon})\nabla u)$ G-converges to a limit operator $-\operatorname{div}(A^*\nabla u)$ where A^* is a constant matrix which can be characterized in terms of A and certain auxiliary functions. See for example [21].

It is shown in [7] that properties (H0)–(H8) are stable under G-convergence, i.e.

Theorem 3.5 ([7], Theorem 2.3)**.** *If $\mathcal{A}_\varepsilon u := -\operatorname{div}(a_\varepsilon(x,\nabla u))$ G-converges to $\mathcal{A}u := -\operatorname{div}(a(x,\nabla u))$ and $a_\varepsilon(x,\xi)$ satisfies* (H0)–(H8) *uniformly, then $a(x,\xi)$ also satisfies* (H0)–(H8)*.*

In the periodic case, i.e. when $a_\varepsilon(x,\xi) = a(\frac{x}{\varepsilon},\xi)$, and $a(\cdot,\xi)$ is Q-periodic for every $\xi \in \mathbb{R}^N$, one has that $\mathcal{A}_\varepsilon$ G-converges to the homogenized operator $\mathcal{A}_h$ given by $\mathcal{A}_h u = -\operatorname{div}(a_h(\nabla u))$, where $a_h : \mathbb{R}^N \to \mathbb{R}^N$ can be characterized by

$$a_h(\xi) = \lim_{s\to\infty} \frac{1}{s^N} \int_{Q_s(z_s)} a(x,\nabla\chi_s^\xi + \xi)dx, \tag{3.2}$$

where $\xi \in \mathbb{R}^N$, $Q_s(z_s)$ is the cube of side length s centered at z_s for any family $\{z_s\}_{s>0}$ in $\mathbb{R}^N$, and χ_s^ξ is the solution of the auxiliary problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla\chi_s^\xi + \xi)) = 0 & \text{in } Q_s(z_s) \\ \chi_s^\xi \in W_0^{1,p}(Q_s(z)), \end{cases} \tag{3.3}$$

see [12] for the proof.

In the general case, one has the following compactness result due to [20]

Proposition 3.6 ([20], Theorem 4.1)**.** *Assume that $a_\varepsilon(x,\xi)$ satisfies* (H1)–(H3)*; then, up to a subsequence, $\mathcal{A}_\varepsilon$ G-converges to a maximal monotone operator $\mathcal{A}$ whose coefficient $a(x,\xi)$ also satisfies* (H1)–(H3)*.*

In the one dimensional setting the G-limit is easily computed. In fact we have the following fairly easy proposition. For $p = 2$ this is well known (see [1]) and for general p the extension is straightforward

Proposition 3.7. *Let $\mathcal{A}_\varepsilon u := -(a_\varepsilon(x)|u'|^{p-2}u')'$ with $a_\varepsilon \in L^\infty(\mathbb{R})$ that satisfies*

$$\alpha \le a_\varepsilon(x) \le \beta, \tag{3.4}$$

for some constants $\alpha, \beta > 0$. Then, up to a subsequence, $\mathcal{A}_\varepsilon$ G-converges to $\mathcal{A}u := -(a_p^*(x)|u'|^{p-2}u')'$, with $a_p^* \in L^\infty(\mathbb{R})$ given by

$$a_p^* = \bar{a}_p^{-(p-1)} \qquad \text{and} \qquad a_\varepsilon^{-\frac{1}{p-1}} \overset{*}{\rightharpoonup} \bar{a}_p.$$

Proof. Let $f_\varepsilon \in W^{-1,p'}(I)$ be such that $f_\varepsilon \to f$ in $W^{-1,p'}(I)$.

Let $g_\varepsilon \in L^p(I)$ be such that $g_\varepsilon' = f_\varepsilon$ and $g_\varepsilon \to g$ in $L^p(I)$. Hence $g' = f$.

Let u_ε be the weak solution to

$$\begin{cases} -(a_\varepsilon(x)|u_\varepsilon'|^{p-2}u_\varepsilon')' = f_\varepsilon & \text{in } I, \\ u_\varepsilon(0) = u_\varepsilon(1) = 0. \end{cases}$$

Then, there exists a constant c_ε such that $a_\varepsilon(x)|u_\varepsilon'|^{p-2}u_\varepsilon' = c_\varepsilon - g_\varepsilon$.

Let $\varphi_p(x) = |x|^{p-2}x$. Then φ_p is invertible and so

$$u_\varepsilon' = \varphi_p^{-1}(c_\varepsilon - g_\varepsilon)a_\varepsilon(x)^{-\frac{1}{p-1}}. \tag{3.5}$$

Since $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $W_0^{1,p}(I)$, we can assume that it is weakly convergent to some $u \in W_0^{1,p}(I)$ and, since a_ε is bounded away from zero and infinity, so is $a_\varepsilon^{-\frac{1}{p-1}}$, so we can assume that there exists $\bar{a}_p \in L^\infty(I)$ such that

$$a_\varepsilon^{-\frac{1}{p-1}} \overset{*}{\rightharpoonup} \bar{a}_p.$$

Moreover, we can assume that $g_\varepsilon \to g$ in $L^p(I)$, and that $c_\varepsilon \to c$.

Now we can pass to the limit in (3.5) and obtain

$$u' = \varphi_p^{-1}(c - g)\bar{a}_p(x).$$

The proof is now complete. □

4. Properties of the eigenvalues and eigenfunctions

In this section we prove the main results of the paper, namely we study the properties of the spectrum Σ of the following (nonlinear) eigenvalue problem:

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = \lambda\rho|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $a(x, \xi)$ verifies (H0)–(H8) and

$$0 < \rho^- \le \rho(x) \le \rho^+ < \infty \qquad \text{a.e. in } \Omega. \tag{4.2}$$

As we mentioned in the introduction we extend here to (4.1) the results that are well-known for the p-Laplacian case.

The methods in the proofs here very much resemble the ones used for the p-Laplacian and we refer the reader to the articles [5, 6, 4, 36, 38].

We recall that the spectrum Σ is defined by

$$\Sigma := \{\lambda \in \mathbb{R} : \text{there exists } u \in W_0^{1,p}, \text{nontrivial solution to (4.1)}\}.$$

We begin with this proposition.

Proposition 4.1. *The spectrum Σ of (4.1) is closed and, moreover, $\Sigma \subset (0, \infty)$.*

Proof. First, observe that (H2) trivially implies that $\Sigma \subset (0, +\infty)$. In fact, if $\lambda \in \Sigma$ and $u \in W_0^{1,p}(\Omega)$ is an eigenfunction associated to λ, then we have, from (H2),

$$\alpha \int_\Omega |\nabla u|^p \, dx \leq \int_\Omega a(x, \nabla u) \nabla u \, dx = \lambda \int_\Omega \rho(x) |u|^p \, dx, \tag{4.3}$$

from where it follows that

$$\lambda \geq \frac{\alpha \int_\Omega |\nabla u|^p \, dx}{\int_\Omega \rho(x)|u|^p \, dx} > 0.$$

The fact that Σ is closed follows from the monotonicity of the operator $\mathcal{A}$. In fact, let $\lambda_j \in \Sigma$ be such that $\lambda_j \to \lambda$ and let $u_j \in W_0^{1,p}(\Omega)$ be an eigenfunction associated to λ_j. We can assume, from (H4), that u_j is chosen so that $\|u_j\|_{L^p(\Omega)} = 1$. Then, since $\{\lambda_j\}_{j \in \mathbb{N}}$ is bounded, from (4.3) it follows that $\{u_j\}_{j \in \mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$.

Therefore, taking a subsequence if necessary, we have that there exists $u \in W_0^{1,p}(\Omega)$ and, from (H3), that there exists $\eta \in (L^{p'}(\Omega))^N$ such that

$$\begin{aligned} u_j &\rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \\ u_j &\to u \quad \text{strongly in } L^p(\Omega) \text{ and a.e. in } \Omega, \\ a(x, \nabla u_j) &\rightharpoonup \eta \quad \text{weakly in } L^{p'}(\Omega). \end{aligned}$$

From these convergences we obtain that $\|u\|_{L^p(\Omega)} = 1$ (so that $u \neq 0$) and

$$\int_\Omega \eta \nabla v \, dx = \lambda \int_\Omega \rho(x) |u|^{p-1} u v \, dx \tag{4.4}$$

for every $v \in W_0^{1,p}(\Omega)$. So, the proof will be finished if we show that

$$\int_\Omega a(x, \nabla u) \nabla v \, dx = \int_\Omega \eta \nabla v \, dx \tag{4.5}$$

for every $v \in W_0^{1,p}(\Omega)$. For this purpose, we make use of the monotonicity inequality (H1) and the fact that u_j is an eigenfunction associated to λ_j. In fact, for every $w \in W_0^{1,p}(\Omega)$,

$$\begin{aligned} 0 &\leq \int_\Omega (a(x, \nabla u_j) - a(x, \nabla w))(\nabla u_j - \nabla w) \, dx \\ &= \int_\Omega \lambda_j \rho(x) |u_j|^{p-2} u_j (u_j - w) \, dx - \int_\Omega a(x, \nabla w)(\nabla u_j - \nabla w) \, dx. \end{aligned}$$

Taking the limit $j \to \infty$ in the former inequality, we get, using (4.4),

$$\begin{aligned} 0 &\leq \int_\Omega \lambda\rho(x)|u|^{p-2}u(u-w)\,dx - \int_\Omega a(x,\nabla w)(\nabla u - \nabla w)\,dx \\ &= \int_\Omega \eta(\nabla u - \nabla w)\,dx - \int_\Omega a(x,\nabla w)(\nabla u - \nabla w)\,dx. \end{aligned}$$

So, if we take $w = u - tv$ with $v \in W_0^{1,p}(\Omega)$ given and $t > 0$, we immediately get

$$0 \leq \int_\Omega (\eta - a(x, \nabla u - t\nabla v))\nabla v\,dx,$$

and taking $t \to 0+$ we arrive at

$$0 \leq \int_\Omega (\eta - a(x,\nabla u))\nabla v\,dx.$$

From this inequality it is easy to see that (4.5) holds and so the claim follows. □

The existence of a sequence of variational eigenvalues for (4.1) can be traced back to the papers of F. Browder, as we pointed out before. We state the result here for further reference.

Theorem 4.2. *Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the sequence defined by*

$$\lambda_k = \inf_{C\in\mathcal{C}_k} \sup_{v\in C} \frac{\int_\Omega \Phi(x,\nabla v)}{\int_\Omega \rho|v|^p},$$

where $\Phi(x,\xi)$ is the potential function given in Proposition 3.1,

$$\mathcal{C}_k = \{C \subset W_0^{1,p}(\Omega) : C \text{ closed},\ C = -C,\ \gamma(C) \geq k\}$$

and $\gamma(C)$ is the Kranoselskii genus.

Then $\{\lambda_k\}_{k\in\mathbb{N}} \subset \Sigma$ and $\lambda_k \to \infty$ as $k \to \infty$.

We call the sequence $\{\lambda_k\}_{k\geq 1}$ the *variational spectrum* of (4.1). We refer the reader to [40] for the definition and properties of γ. As for the asymptotic behavior of the sequence $\Sigma_{\text{var}} = \{\lambda_k\}_{k\in\mathbb{N}}$ this follows easily from the variational characterization given in Theorem 4.2, the coercivity inequality (3.1) and the asymptotic behaviors for the eigenvalues of the p-Laplacian found in [35] and refined in [31]. More precisely, we have

Theorem 4.3. *There exist $c, C > 0$ depending only on p, N such that*

$$c\frac{\alpha}{\rho^+}\left(\frac{k}{|\Omega|}\right)^{\frac{p}{N}} \leq \lambda_k \leq C\frac{\beta}{\rho^-}\left(\frac{k}{|\Omega|}\right)^{\frac{p}{N}},$$

where α, β are given in (3.1) *and ρ_-, ρ_+ are given in* (4.2).

Proof. From (3.1) and (4.2) it follows that, for every $v \in W_0^{1,p}(\Omega)$, we have

$$\frac{\alpha}{\rho^+}\frac{\int_\Omega |\nabla v|^p\,dx}{\int_\Omega |v|^p\,dx} \leq \frac{\int_\Omega \Phi(x,\nabla v)}{\int_\Omega \rho|v|^p} \leq \frac{\beta}{\rho^-}\frac{\int_\Omega |\nabla v|^p\,dx}{\int_\Omega |v|^p\,dx}.$$

From these inequalities and the variational characterization of Σ_{var} we obtain

$$\frac{\alpha}{\rho^+}\mu_k \leq \lambda_k \leq \frac{\beta}{\rho^-}\mu_k,$$

where $\{\mu_k\}_{k\in\mathbb{N}}$ are the variational eigenvalues of the p-Laplacian. Now, the conclusion of the Theorem follows from Weyl's asymptotic formula for $\{\mu_k\}_{k\in\mathbb{N}}$ proved in [31]. □

The following maximum principle for quasilinear operators was proved in [37] and it will be most useful in the sequel.

Theorem 4.4 ([37], Section 6.2)**.** *Assume that $u \in W^{1,p}_{\text{loc}}(\Omega)$ satisfies*

$$\int_\Omega a(x,\nabla u)\nabla\phi - \rho|u|^{p-2}u\phi \geq 0, \quad \forall \phi \in C_0^\infty(\Omega),\ \phi \geq 0.$$

Consider its zero set

$$\mathfrak{Z} := \{x \in \Omega : \tilde{u}(x) = 0\},$$

where $\tilde{u}$ is the p-quasi continuous representative of u.

Then, either $\text{Cap}_p(\mathfrak{Z}) = 0$ *or* $u = 0$.

For the properties of the p-capacity and the p-quasi continuous representative of a Sobolev function we refer to [27].

The following result gives the positivity of the first eigenfunction.

Theorem 4.5 ([37], Proposition 5.3)**.** *Let u be an eigenfunction corresponding to λ_1. Then exactly one of the following alternatives holds:*

$$u > 0 \qquad \text{or} \qquad u < 0,$$

and the set of zeroes of u satisfies

$$\text{Cap}_p(\{u = 0\}) = 0.$$

Proof. Assume that $u^+ \not\equiv 0$ and let us show then that $u^- \equiv 0$.

First observe that $a(x,\xi)\xi = \Phi(x,\xi)$. This fact follows from the homogeneity of Φ and Euler's differentiation formula for homogeneous mappings.

By using u^+ as test function in (1.1) we deduce that

$$\int_\Omega \Phi(x,\nabla u^+) = \lambda_1 \int_\Omega \rho|u^+|^p$$

and therefore u^+ is also an eigenfunction corresponding to λ_1. It satisfies hence (1.1) and we get

$$\begin{cases} -\operatorname{div}(a(x,\nabla u^+)) = \lambda_1\rho|u^+|^{p-1}, & \text{in } \Omega, \\ u^+ \geq 0, \quad u^+ \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By the maximum principle as stated in Theorem 4.4, we deduce that $u^- \equiv 0$ and $\text{Cap}_p(\{u = 0\}) = 0$. □

The following result gives the simplicity of the first eigenvalue. It follows by using a Picone type identity, see [2, 4, 36, 38]. Whenever the eigenfunctions associated to λ_1 are regular enough, the following Picone type identity holds.

Lemma 4.6. *Let $v > 0$, $u \geq 0$ be two continuous functions in Ω differentiable a.e. Let us denote*

$$L(u,v) = \Phi(x,\nabla u) + (p-1)\Big(\frac{u}{v}\Big)^p \Phi(x,\nabla v) - \Big(\frac{u}{v}\Big)^{p-1}\langle a(x,\nabla v),\nabla u\rangle,$$

$$R(u,v) = \langle a(x,\nabla u),\nabla u\rangle - \langle a(x,\nabla v),\nabla\Big(\frac{u^p}{v^{p-1}}\Big)\rangle.$$

*Then, **(i)** $L(u,v) = R(u,v)$, **(ii)** $L(u,v) \geq 0$, **(iii)** $L(u,v) = 0$ a.e. in Ω if and only if $u = cu$ for some $c \in \mathbb{R}$.*

For the p-Laplacian, the regularity of eigenfunctions is known and it is enough to use Picone's identity. For general operators the proof is the same assuming that regularity, and the full proof without this assumption can be found in [37].

Now, simplicity of the first eigenvalue can be proved with a standard argument by using Picone's identity given in Lemma 4.6.

Theorem 4.7. *Let u, v be two eigenfunctions corresponding to λ_1. Then there exists $c \in \mathbb{R}$ such that $u = cv$.*

Proof. Let u, v be two eigenfunctions associated to λ_1. We can assume that u and v are both positive in Ω. We apply Lemma 4.6 to the pair u, $v+\varepsilon$ and obtain

$$\begin{aligned} 0 \leq \int_\Omega L(u,v+\varepsilon)dx &= \int_\Omega R(u,v+\varepsilon)dx \\ &= \lambda_1 \int_\Omega \rho(x)|u|^p dx - \int_\Omega \langle a(x,\nabla v),\nabla\Big(\frac{u^p}{v^{p-1}}\Big)\rangle dx. \end{aligned}$$

Since the function $\frac{u^p}{(v+\varepsilon)^{p-1}} \in W^{1,p}(\Omega)$, it is admissible in the weak formulation of v. It follows that

$$0 \leq \int_\Omega L(u,v+\varepsilon)dx \leq \lambda_1 \int_\Omega \rho(x)|u|^p\big(1 - \frac{v^{p-1}}{(v+\varepsilon)^{p+1}}\big)dx.$$

Letting $\varepsilon \to 0$, we obtain

$$\int_\Omega L(u,v)dx = 0,$$

but then $L(u,v) = 0$ and by Lemma 4.6, there exists $c \in \mathbb{R}$ such that $u = cv$. □

The proof in the general case, when Lemma 4.6 is not true a.e. in Ω, is quite more complex and can be found in [37, Theorem 1.3].

Theorem 4.8 ([37], Section 6.2)**.** *Let u_1 be an eigenfunction corresponding to λ_1; then u_1 does not change sign on Ω. Also, the first eigenvalue is simple, that is, any other eigenfunction u associated to λ_1 is a multiple of u_1.*

Next, we show that the first eigenvalue λ_1 is isolated in Σ. The key step in the proof of the isolation is the following result:

Proposition 4.9. *Let $\lambda \in \Sigma$ and let w be an eigenfunction corresponding to $\lambda \neq \lambda_1$. Then w changes sign on Ω, that is, $u^+ \neq 0$ and $u^- \neq 0$. Moreover, there exists a positive constant C independent of w and λ such that*

$$|\Omega^+| \geq C\lambda^{-\gamma}, \quad |\Omega^-| \geq C\lambda^{-\gamma},$$

where $\Omega^\pm$ denotes the positivity and the negativity set of w respectively, γ is a positive parameter, and C depends on N p, ρ^+ and the coercivity constant α in (H2)*. Here, $\gamma = (N-p)/p$ if $p < N$, $\gamma = 1$ if $p = N$, and $\gamma = (p-N)/N$ if $p > N$.*

Proof. Let w be an eigenfunction corresponding to $\lambda \neq \lambda_1$ and let u be an eigenfunction corresponding to λ_1.

Assume that w does not change sign on Ω. We can assume that $w \geq 0$ and $u \geq 0$ in Ω. For each $k \in \mathbb{N}$, let us truncate u as follows:

$$u_k(x) := \min\{u(x), k\},$$

and for each $\varepsilon > 0$ we consider the function $u_k^p/(w+\varepsilon)^{p-1} \in W_0^{1,p}(\Omega)$. We get

$$\int_\Omega a(x,\nabla u)\nabla u - a(x,\nabla w)\nabla\Big(\frac{u_k^p}{(w+\varepsilon)^{p-1}}\Big) = \int_\Omega \lambda_1\rho u^p - \lambda\rho w^{p-1}\frac{u_k^p}{(w+\varepsilon)^{p-1}}. \quad (4.6)$$

We claim that the integral in the left hand side in (4.6) is non-negative. Indeed, let Φ be the potential function given by Proposition 3.1. Then, as Φ is p-homogeneous in the second variable we have (see [37, p. 19])

$$\begin{aligned} a(x,\nabla u)\nabla u - a(x,\nabla w)\nabla\Big(\frac{u_k^p}{(w+\varepsilon)^{p-1}}\Big) \\ = p\Big\{\Phi(x,\nabla u) + (p-1)\Phi(x,\frac{u_k}{w+\varepsilon}\nabla w) - a(x,\frac{u_k}{w+\varepsilon}\nabla w)\nabla u_k\Big\}. \quad (4.7) \end{aligned}$$

By using the property that $\xi \mapsto \Phi(x,\xi)$ is convex, we easily deduce that (4.7) is nonnegative. Therefore, coming back to (4.6) we get

$$\int_\Omega \lambda_1\rho u^p - \lambda\rho w^{p-1}\frac{u_k^p}{(w+\varepsilon)^{p-1}} \geq 0. \quad (4.8)$$

Since by the strong maximum principle for quasilinear operators (Theorem 4.4) the set $\{\tilde{w} = 0\}$, where $\tilde{w}$ is the p-quasi continuous representative of w, is of measure zero, then (4.8) is equivalent to

$$\int_{\{w>0\}} \lambda_1\rho u^p - \lambda\rho w^{p-1}\frac{u_k^p}{(w+\varepsilon)^{p-1}} \geq 0. \quad (4.9)$$

Now, letting $\varepsilon \to 0$ and $k \to \infty$ in (4.9), we get

$$(\lambda_1 - \lambda)\int_\Omega \rho|u|^p \geq 0,$$

which is a contradiction. Therefore w changes sign on Ω.

The second part of the proof follows almost exactly as in the p-Laplacian case. Let us suppose first that $p < N$. In fact, as w changes sign, we can use w^+ as a

test function in the equation satisfied by w to obtain

$$\begin{aligned}\int_\Omega a(x,\nabla w)\nabla w^+ &= \lambda \int_\Omega \rho |w|^{p-2} w w^+ \\ &= \lambda \int_{\Omega^+} \rho |w|^p \\ &\le \lambda \rho^+ \int_{\Omega^+} |w|^p \\ &\le \lambda \rho^+ \|w^+\|^p_{L^{p^*}(\Omega)} |\Omega^+|^{p/(N-p)} \\ &\le \lambda \rho^+ K_p |\Omega^+|^{p/(N-p)} \int_\Omega |\nabla w^+|^p,\end{aligned}$$

where K_p is the optimal constant in the Sobolev-Poincaré inequality.

Now, by (H2), it follows that

$$\int_\Omega a(x,\nabla w)\nabla w^+ \ge \alpha \int_\Omega |\nabla w^+|^p.$$

Combining these two inequalities, we obtain

$$|\Omega^+| \ge \Big(\frac{\alpha}{K_p \lambda \rho^+}\Big)^{(N-p)/p}.$$

The estimate for $|\Omega^-|$ follows in the same way.

The remaining cases are similar: $p = N$ follows by using the Sobolev inclusion $W_0^{1,N}(\Omega) \subset L^N(\Omega)$, and the case $p > N$ follows from Morrey's inequality. □

Now we are ready to prove the isolation of λ_1.

Theorem 4.10. *The first eigenvalue λ_1 is isolated. That is, there exists $\delta > 0$ such that $(\lambda_1, \lambda_1 + \delta) \cap \Sigma = \emptyset$.*

Proof. Assume by contradiction that there exists a sequence $\lambda_j \in \Sigma$ such that $\lambda_j \to \lambda_1$ as $j \to \infty$. Let u_j be the associated eigenfunctions normalized such that

$$\int_\Omega \rho |u_j|^p = 1.$$

By (H2) it follows that the sequence $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $W_0^{1,p}(\Omega)$, so, passing to a subsequence if necessary, there exists $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} u_j &\rightharpoonup u \qquad \text{weakly in } W_0^{1,p}(\Omega), \\ u_j &\to u \qquad \text{strongly in } L^p(\Omega). \end{aligned}$$

Now, as the functional

$$v \mapsto \int_\Omega \Phi(x,\nabla v)$$

is weakly sequentially lower semicontinuous (see [7]), it follows that u is an eigenfunction associated to λ_1.

Now, by Theorem 4.8, we can assume that $u \geq 0$ and by Proposition 4.9 we have $|\{u = 0\}| > 0$. But this is a contradiction to the strong maximum principle in [37], Theorem 4.4. □

As a consequence of Theorem 4.10 it makes sense to define the second eigenvalue Λ_2 as the infimum of the eigenvalues greater than λ_1. Next, we show that this second eigenvalue Λ_2 coincides with the second variational eigenvalue λ_2. This result is known to hold for the p-Laplacian (see [6]) and we extend it here for the general case (4.1).

Theorem 4.11. *Let λ_2 be the second variational eigenvalue, and let Λ_2 be defined as*

$$\Lambda_2 = \inf\{\lambda > \lambda_1 : \lambda \in \Sigma\}.$$

Then

$$\lambda_2 = \Lambda_2.$$

Proof. The proof of this Theorem follows closely the one in [30] where the analogous result for the Steklov problem for the p-Laplacian is analyzed.

Let us call

$$\mu = \inf\left\{\int_\Omega \Phi(x, \nabla u) : \|\rho u\|^p_{L^p(\Omega)} = 1 \text{ and } |\Omega^\pm| > c_{\lambda_2}\right\},$$

where $c_{\lambda_2} := C\lambda_2^{-\gamma}$ and C, γ are given by Proposition 4.9.

If we take u_2 an eigenfunction of (4.1) associated with Λ_2 such that $\|\rho u\|^p_{L^p(\Omega)} = 1$, by Theorem 4.9 we have that u_2 is admissible in the variational characterization of μ. It follows that $\mu \leq \Lambda_2$. The proof will follow if we show that $\mu \geq \lambda_2$. The inverse of μ can be written as

$$\frac{1}{\mu} = \sup\left\{\int_\Omega \rho|u|^p : \int_\Omega \Phi(x, \nabla u) = 1 \text{ and } |\Omega^\pm| > c_{\lambda_2}\right\}.$$

The supremum is attained by a function $w \in W_0^{1,p}(\Omega)$ such that $\int_\Omega \Phi(x, \nabla w) = 1$ and $|\Omega^\pm| > c_{\lambda_2}$. As w^+ and w^- are not identically zero, if we consider the set

$$C = \operatorname{span}\{w^+, w^-\} \cap \{u \in W_0^{1,p}(\Omega) : \|u\|_{W_0^{1,p}(\Omega)} = 1\},$$

then $\gamma(C) = 2$. Hence, we obtain

$$\frac{1}{\lambda_2} \geq \inf_{u \in C} \int_\Omega \rho|u|^p \tag{4.10}$$

but, as w^+ and w^- have disjoint supports, it follows that the infimum (4.10) can be computed by minimizing the two variable function

$$G(a, b) := |a|^p \int_\Omega \rho|w^+|^p + |b|^p \int_\Omega \rho|w^-|^p$$

with the restriction

$$H(a, b) := |a|^p \int_\Omega \Phi(x, \nabla w^+) + |b|^p \int_\Omega \Phi(x, \nabla w^-) = 1.$$

Now, an easy computation shows that

$$\frac{1}{\lambda_2} \geq \min\left\{ \frac{\int_\Omega \rho |w^+|^p}{\int_\Omega \Phi(x, \nabla w^+)}, \frac{\int_\Omega \rho |w^-|^p}{\int_\Omega \Phi(x, \nabla w^-)} \right\}.$$

We can assume that the minimum in the above inequality is realized with w^+. Then, for $t > -1$ the function $w + tw^+$ is admissible in the variational characterization of μ, hence if we denote

$$Q(t) := \frac{\int_\Omega \rho |w + tw^+|^p}{\int_\Omega \Phi(x, \nabla w + t\nabla w^+)},$$

we get

$$0 = Q'(0) = p \int_\Omega \rho |w|^{p-2} w w^+ - \frac{p}{\mu} \int_\Omega a(x, \nabla w) \nabla w^+;$$

therefore

$$\frac{\int_\Omega \rho |w^+|^p}{\int_\Omega \Phi(x, \nabla w^+)} = \frac{1}{\mu}$$

and the result follows. □

In dimensions $N > 1$ it is not known even in the p-Laplacian case whether λ_2 is isolated in Σ or not, or if Σ is countable or not. So we cannot expect to obtain much more information in the general case (4.1).

However, in the one dimensional problem $N = 1$ it is known since the work of Fučík and coauthors in [33] (see also the more recent works [28, 42]) that $\Sigma = \Sigma_{\text{var}}$. So now we generalize this fact to (4.1). That is, we study

$$\begin{cases} -(a(x)|u'|^{p-2}u')' = \lambda \rho(x)|u|^{p-2}u & \text{in } J := (0, \ell), \\ u(0) = u(\ell) = 0, \end{cases} \tag{4.11}$$

where $0 < \rho_- \leq \rho(x) \leq \rho_+$ and $0 < \alpha \leq a(x) \leq \beta$ for some constants ρ_-, ρ_+, α and β.

For the one dimensional p-Laplace operator in J with zero Dirichlet boundary conditions, that is (4.11) with $a(x) = \rho(x) = 1$, we denote by $\tilde{\Sigma} = \tilde{\Sigma}_{\text{var}} = \{\mu_k\}_{k \in \mathbb{N}}$ the spectrum given by

$$\mu_k = \inf_{C \in \mathcal{C}_k} \sup_{u \in C} \frac{\int_J |u'|^p \, dx}{\int_J |u|^p \, dx}. \tag{4.12}$$

Here, all the eigenvalues and eigenfunctions can be found explicitly:

Theorem 4.12 (Del Pino, Drabek and Manasevich, [23]). *The eigenvalues μ_k given by* (4.12) *and their corresponding eigenfunctions u_k on the interval J are given by*

$$\mu_k = \frac{\pi_p^p k^p}{\ell^p},$$
$$u_k(x) = \sin_p(\pi_p k x / \ell).$$

The function $\sin_p(x)$ is the solution of the initial value problem

$$\begin{cases} -(|u'|^{p-2}u')' = |u|^{p-2}u \\ u(0) = 0, \quad u'(0) = 1, \end{cases}$$

and is defined implicitly as

$$x = \int_0^{\sin_p(x)} \left(\frac{p-1}{1-t^p}\right)^{1/p} dt.$$

Moreover, its first zero is π_p, given by

$$\pi_p = 2\int_0^1 \left(\frac{p-1}{1-t^p}\right)^{1/p} dt.$$

In [5], problem (4.11) with $a \equiv 1$ is studied and, among other things, it is proved that any eigenfunction associated to λ_k has exactly k nodal domains. As a consequence of this fact, in [5] the simplicity of every variational eigenvalue is obtained.

The exact same proof of [5] works in our case, and so we obtain the following:

Theorem 4.13. *Every eigenfunction corresponding to the k-th eigenvalue λ_k has exactly $k-1$ zeroes. Moreover, for every k, λ_k is simple, and consequently the eigenvalues are ordered as $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k \nearrow +\infty$.*

Now, using the same ideas as in [28] is easy to prove that the spectrum of (4.11) coincides with the variational spectrum. In fact, we have:

Theorem 4.14. $\Sigma = \Sigma_{\mathrm{var}}$.

Proof. The proof of this theorem is completely analogous to that of Theorem 1.1 in [28]. □

5. Eigenvalue homogenization

In this section, as an application of the results in Section 3, we analyze the convergence of the spectrum Σ_ε of problem

$$\begin{cases} -\operatorname{div}(a_\varepsilon(x, \nabla u)) = \lambda^\varepsilon \rho_\varepsilon |u|^{p-2}u & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

to the spectrum Σ of the limit problem

$$\begin{cases} -\operatorname{div}(a(x, \nabla u)) = \lambda \rho |u|^{p-2}u & \text{on } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

under the assumption that $\mathcal{A}_\varepsilon$ G-converges to $\mathcal{A}$ and that $\rho_\varepsilon \overset{*}{\rightharpoonup} \rho$ in $L^\infty(\Omega)$. Moreover, we assume that $a_\varepsilon(x, \xi)$ satisfies (H0)–(H8) uniformly.

The results in this section are not original, since they were obtained in [19] (for $\rho_\varepsilon, \rho \equiv 1$ though). Nevertheless, the proofs that we provide are much simpler than those in [19].

In the linear case, it is well known (see [1]) that the G-convergence of the operators implies the convergence of their spectra in the sense that the k-th eigenvalue λ_k^ε converges to the k-th eigenvalue of the limit problem.

We want to study the convergence of the spectrum in the non-linear case. We begin with a general result for bounded sequences of eigenvalues. This result was already proved in [7] but we include here a simpler proof for the reader's convenience.

Along the proofs by normalized eigenfunctions we understand that $\|u\|_p = 1$.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^N$ be bounded. Let $\lambda^\varepsilon \in \Sigma_\varepsilon$ be a sequence of eigenvalues of problem* (5.1) *with $\{u^\varepsilon\}_{\varepsilon>0}$ associated normalized eigenfunctions. Assume that the sequence of eigenvalues is convergent*

$$\lim_{\varepsilon\to 0^+} \lambda^\varepsilon = \lambda.$$

Then, $\lambda \in \Sigma$ and there exists a sequence $\varepsilon_j \to 0^+$ such that

$$u^{\varepsilon_j} \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega),$$

with u a normalized eigenfunction associated to λ.

Remark 5.2. In most applications, we take the sequence λ^ε to be the sequence of the k-th variational eigenvalue of (5.1). In this case, it is not difficult to check that the sequence $\{\lambda_k^\varepsilon\}_{\varepsilon>0}$ is bounded and so, up to a subsequence, convergent.

In fact, by using the variational characterization of λ_k^ε, (3.1) and our assumptions on ρ we have that

$$\frac{\alpha}{\rho^+}\frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p} \le \frac{\int_\Omega \Phi_\varepsilon(x,\nabla v)}{\int_\Omega \rho_\varepsilon |v|^p} \le \frac{\beta}{\rho^-}\frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p};$$

therefore

$$\frac{\alpha}{\rho^+}\mu_k \le \lambda_k^\varepsilon \le \frac{\beta}{\rho^-}\mu_k,$$

where μ_k is the k-th variational eigenvalue of the p-Laplacian.

Proof. As λ_ε is bounded and u^ε is normalized, by (H2) it follows that the sequence $\{u^\varepsilon\}_{\varepsilon>0}$ is bounded in $W_0^{1,p}(\Omega)$.

Therefore, up to some sequence $\varepsilon_j \to 0$, we have that

$$\begin{aligned} u^{\varepsilon_j} &\rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega),\\ u^{\varepsilon_j} &\to u \quad \text{strongly in } L^p(\Omega), \end{aligned} \tag{5.3}$$

with u also normalized.

We define the sequence of functions $f_\varepsilon := \lambda^\varepsilon \rho_\varepsilon |u^\varepsilon|^{p-2}u^\varepsilon$. By using the fact that $\rho_\varepsilon \rightharpoonup \rho$ weakly $*$ in $L^\infty(\Omega)$ together with (5.3) it follows that

$$f_{\varepsilon_j} \rightharpoonup f := \lambda\rho|u|^{p-2}u \quad \text{weakly in } L^p(\Omega),$$

and therefore

$$f_{\varepsilon_j} \to f \quad \text{strongly in } W^{-1,p'}(\Omega).$$

By Proposition 3.6 we deduce that u^{ε_j} converges weakly in $W_0^{1,p}(\Omega)$ to the unique solution v of the homogenized problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla v)) = \lambda\rho|u|^{p-2}u & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

By uniqueness of the limit, $v = u$ is a normalized eigenfunction of the homogenized problem. □

Remark 5.3. In the case where the sequence λ^ε is the sequence of the k-th variational eigenvalues of (5.1) it would be desirable to prove that it converges to the k-th variational eigenvalue of the homogenized problem (5.2) (see Remark 5.2).

Unfortunately, our method only allows us to treat the first and second variational eigenvalues in the general setting. In the one dimensional case, one can be more precise and this fact holds true. See [19] for a general proof of this fact using the Γ-convergence method.

5.1. **Convergence of the first and second eigenvalues.** The first eigenvalue of (5.1) is the infimum of the Rayleigh quotient

$$\lambda_1^\varepsilon = \inf_{v\in W_0^{1,p}(\Omega)} \frac{\int_\Omega \Phi_\varepsilon(x,\nabla v)}{\int_\Omega \rho_\varepsilon |v|^p}.$$

In the following result we prove the convergence of λ_1^ε when ε tends to zero.

Theorem 5.4. *Let λ_1^ε be the first eigenvalue of* (5.1) *and λ_1 the first eigenvalue of the limit problem* (5.2)*; then*

$$\lim_{\varepsilon\to 0} \lambda_1^\varepsilon = \lambda_1.$$

Moreover, if u_1^ε and u_1 are the (normalized) nonnegative eigenfunctions of (5.1) *and* (5.2) *associated to λ_1^ε and λ_1 respectively, then*

$$u_1^\varepsilon \rightharpoonup u_1 \quad \textit{weakly in } W_0^{1,p}(\Omega).$$

Remark 5.5. In [7] using the theory of convergence of monotone operators the authors obtain the conclusions of Theorem 5.4. We propose here a simple proof of this result which exploits the fact that the first eigenfunction has constant sign.

Proof. Let u_1^ε be the nonnegative normalized eigenfunction associated to λ_1^ε; the uniqueness of u_1^ε follows from Theorem 4.8.

By Theorem 5.1, up to some sequence, u_1^ε converges weakly in $W_0^{1,p}(\Omega)$ to u, an eigenfunction of the homogenized eigenvalue problem associated to $\lambda = \lim_{\varepsilon\to 0}\lambda_1^\varepsilon$. But then, u is a nonnegative normalized eigenfunction of the homogenized problem (5.2) and so $u = u_1$. Therefore $\lambda = \lambda_1$ and the uniqueness implies that the whole sequences λ_1^ε and u_1^ε are convergent. □

Now we turn our attention to the second eigenvalue. For this purpose we use the fact that eigenfunctions associated to the second variational eigenvalue of problems (5.1) and (5.2) have, at least, two nodal domains (cf. Proposition 4.9).

Theorem 5.6. *Let λ_2^ε be the second eigenvalue of* (5.1) *and λ_2 be the second eigenvalue of the homogenized problem* (5.2)*. Then*

$$\lim_{\varepsilon\to 0}\lambda_2^\varepsilon=\lambda_2.$$

Proof. Let u_2 be a normalized eigenfunction associated to λ_2 and let $\Omega^\pm$ be the positivity and the negativity sets of u_2 respectively. By standard elliptic regularity theory, $\Omega^\pm$ are open sets. Now, the previous result about the positivity of the first eigenfunction implies that the restrictions of u_2 to $\Omega^\pm$ are the first eigenfunctions of the problem in those sets.

We denote by $u_\pm^\varepsilon$ the first eigenfunction of (5.1) in $\Omega^\pm$ respectively. Extending $u_\pm^\varepsilon$ to Ω by 0, these functions have disjoint supports and therefore they are linearly independent in $W_0^{1,p}(\Omega)$.

Let S be the unit sphere in $W_0^{1,p}(\Omega)$ and define the set C_2^ε as

$$C_2^\varepsilon:=\operatorname{span}\{u_+^\varepsilon,u_-^\varepsilon\}\cap S.$$

Clearly C_2^ε is compact, symmetric and $\gamma(C_2^\varepsilon)=2$. Hence,

$$\lambda_2^\varepsilon=\inf_{C\in\Gamma_2}\sup_{v\in C}\frac{\int_\Omega\Phi_\varepsilon(x,\nabla v)}{\int_\Omega\rho_\varepsilon|v|^p}\le\sup_{v\in C_2^\varepsilon}\frac{\int_\Omega\Phi_\varepsilon(x,\nabla v)}{\int_\Omega\rho_\varepsilon|v|^p}.$$

As C_2^ε is compact, the supremum is achieved for some $v^\varepsilon\in C_2^\varepsilon$ which can be written as

$$v^\varepsilon=a_\varepsilon u_+^\varepsilon+b_\varepsilon u_-^\varepsilon$$

with $a_\varepsilon,b_\varepsilon\in\mathbb{R}$ such that $|a_\varepsilon|^p+|b_\varepsilon|^p=1$. Since the functions u_+^ε and u_-^ε have disjoint supports, we obtain, using the p-homogeneity of Φ_ε (see Proposition 3.1),

$$\lambda_2^\varepsilon\le\frac{\int_\Omega\Phi_\varepsilon(x,\nabla v^\varepsilon)}{\int_\Omega\rho_\varepsilon|v^\varepsilon|^p}=\frac{|a_\varepsilon|^p\int_{\Omega^+}\Phi_\varepsilon(x,\nabla u_+^\varepsilon)+|b_\varepsilon|^p\int_{\Omega^-}\Phi_\varepsilon(x,\nabla u_-^\varepsilon)}{\int_\Omega\rho_\varepsilon|v^\varepsilon|^p}.$$

Using the definition of $u_\pm^\varepsilon$, the above inequality can be rewritten as

$$\lambda_2^\varepsilon\le\frac{|a_\varepsilon|^p\lambda_{1,+}^\varepsilon\int_{\Omega^+}\rho_\varepsilon|u_+^\varepsilon|^p+|b_\varepsilon|^p\lambda_{1,-}^\varepsilon\int_{\Omega^-}\rho_\varepsilon|u_-^\varepsilon|^p}{\int_\Omega\rho_\varepsilon|v^\varepsilon|^p}\le\max\{\lambda_{1,+}^\varepsilon,\lambda_{1,-}^\varepsilon\},\tag{5.4}$$

where $\lambda_{1,\pm}^\varepsilon$ is the first eigenvalue of (5.1) in the nodal domain $\Omega^\pm$ respectively.

Now, using Theorem 5.4, we have that $\lambda_{1,\pm}^\varepsilon\to\lambda_{1,\pm}$ respectively, where $\lambda_{1,\pm}$ are the first eigenvalues of (5.2) in the domains $\Omega^\pm$ respectively. Moreover, we observe that these eigenvalues $\lambda_{1,\pm}$ are both equal to the second eigenvalue λ_2 in Ω, therefore from (5.4) we get

$$\lambda_2^\varepsilon\le\lambda_2+\delta$$

for δ arbitrarily small and ε tending to zero. So,

$$\limsup_{\varepsilon\to 0}\lambda_2^\varepsilon\le\lambda_2.\tag{5.5}$$

On the other hand, suppose that $\lim_{\varepsilon\to 0}\lambda_2^\varepsilon=\lambda$, where $\lambda\in\Sigma_h$. We claim that $\lambda>\lambda_1$.

In fact, we have that $u_2^\varepsilon\rightharpoonup u$ in $W_0^{1,p}(\Omega)$, where u is a normalized eigenfunction associated to λ. As the measure of the positivity and negativity sets of u_2^ε are

bounded below uniformly in $\varepsilon > 0$ (see Proposition 4.9), we have that either u changes sign or $|\{u = 0\}| > 0$. In any case, this implies our claim.

Then, as $\lambda > \lambda_1$ it must be $\lambda \geq \lambda_2$. Then

$$\lambda_2 \leq \lambda = \lim_{\varepsilon \to 0} \lambda_2^\varepsilon. \tag{5.6}$$

Combining (5.5) and (5.6) we obtain the desired result. □

5.2. **Convergence of the full spectrum in the one dimensional case.** The goal of this subsection is to prove the following Theorem.

Theorem 5.7. *Let $N = 1$ and assume that $\mathcal{A}_\varepsilon$ G-converges to $\mathcal{A}$ and that $\rho_\varepsilon \rightharpoonup \rho$ weakly $*$ in $L^\infty(I)$. For each $k \geq 1$ let λ_k^ε be the k-th eigenvalue of* (5.1)*. Then we have that*

$$\lim_{\varepsilon \to 0} \lambda_k^\varepsilon = \lambda_k,$$

where λ_k is the k-th eigenvalue of (5.2)*.*

Moreover, up to a subsequence, an eigenfunction u_k^ε associated to λ_k^ε converges weakly in $W_0^{1,p}(I)$ to u_k, an eigenfunction associated to λ_k.

The main tool that allows us to prove that $\lambda = \lambda_k$ is Theorem 4.13, that says that any eigenfunction associated to the k-th eigenvalue of (5.1) has exactly k nodal domains.

Moreover, we need a refinement of this result, namely an estimate on the measure of each nodal domain independent on ε. This is the content of the next Lemma.

Lemma 5.8. *Let λ_k^ε be an eigenvalue of* (5.1) *with corresponding eigenfunction u_k^ε. Let $\mathcal{N} = \mathcal{N}(k, \varepsilon)$ be a nodal domain of u_k^ε. We have that*

$$|\mathcal{N}| > C,$$

where $C = C(k)$ is a positive constant independent of ε.

Proof. We can write λ_k^ε as

$$\lambda_k^\varepsilon(I) = \lambda_1^\varepsilon(\mathcal{N}) = \inf_{u \in W_0^{1,p}(\mathcal{N})} \frac{\int_{\mathcal{N}} a_\varepsilon(x)|u'|^p}{\int_{\mathcal{N}} \rho_\varepsilon(x)|u|^p};$$

by our assumptions (4.2) we get

$$\lambda_k^\varepsilon(I) \geq \frac{\alpha}{\rho_+} \mu_1(\mathcal{N}) = \frac{\alpha}{\rho_+} \frac{\pi_p^p}{|\mathcal{N}|^p},$$

where $\mu_1(\mathcal{N})$ is the first eigenvalue of the p-Laplacian on $\mathcal{N}$. Moreover,

$$\lambda_k^\varepsilon(I) \leq \frac{\beta}{\rho_-} \mu_k(I) = \frac{\beta}{\rho_-} \pi_p^p k^p.$$

Combining both inequalities we get

$$|\mathcal{N}|^p \geq \frac{\alpha}{\rho_+} \frac{\pi_p^p}{\lambda_k^\varepsilon(\Omega)} \geq \frac{\alpha}{\beta} \frac{\rho_-}{\rho_+} \frac{1}{k^p}$$

and the result follows. □

Now we are ready to establish the main result of this section:

Proof of Theorem 5.7. Let u_k be a normalized eigenfunction associated to λ_k and according to Theorem 4.13, let I_i, $i = 1, \dots, k$, be the nodal domains of u_k.

We denote by u_i^ε the first eigenfunction of (5.1) in I_i. Extending u_i^ε to I by 0, these functions have disjoint supports and therefore they are linearly independent in $W_0^{1,p}(I)$.

Let S be the unit sphere in $W_0^{1,p}(I)$ and we define the set C_k^ε as

$$C_k^\varepsilon := \operatorname{span}\{u_1^\varepsilon, \dots, u_k^\varepsilon\} \cap S.$$

Clearly C_k^ε is compact, symmetric and $\gamma(C_k^\varepsilon) = k$. Hence,

$$\lambda_k^\varepsilon = \inf_{C \in \Gamma_k} \sup_{v \in C} \frac{\int_I a_\varepsilon(x)|v'|^p}{\int_I \rho_\varepsilon |v|^p} \leq \sup_{v \in C_k^\varepsilon} \frac{\int_I a_\varepsilon(x)|v'|^p}{\int_I \rho_\varepsilon |v|^p}.$$

As C_k^ε is compact, the supremum is achieved for some $v^\varepsilon \in C_k^\varepsilon$ which can be written as

$$v^\varepsilon = \sum_{i=1}^k a_i^\varepsilon u_i^\varepsilon,$$

with $a_i^\varepsilon \in \mathbb{R}$ such that $\sum_{i=1}^k |a_i^\varepsilon|^p = 1$. Since the functions u_i^ε have non-overlapping supports, we obtain

$$\lambda_k^\varepsilon \leq \frac{\int_I a_\varepsilon(x)|v^{\varepsilon\prime}|^p}{\int_I \rho_\varepsilon |v^\varepsilon|^p} = \frac{\sum_{i=1}^k |a_i^\varepsilon|^p \int_{I_i} a_\varepsilon(x)|u_i^{\varepsilon\prime}|^p}{\int_I \rho_\varepsilon |v^\varepsilon|^p}.$$

Using the definition of u_i^ε, the above inequality can be rewritten as

$$\lambda_k^\varepsilon \leq \frac{\sum_{i=1}^k |a_i^\varepsilon|^p \lambda_{1,i}^\varepsilon \int_{I_i} \rho_\varepsilon |u_i^\varepsilon|^p}{\int_I \rho_\varepsilon |v^\varepsilon|^p} \leq \max_{1 \leq i \leq k} \{\lambda_{1,i}^\varepsilon\}, \tag{5.7}$$

where $\lambda_{1,i}^\varepsilon$ is the first eigenvalue of (5.1) in the nodal domain Ω_i.

Now, using that $\lambda_{1,i}^\varepsilon \to \lambda_{1,i}$, where $\lambda_{1,i}$ are the first eigenvalues of (5.2) in the domains I_i (see Theorem 4.4, [29]), and since these eigenvalues $\lambda_{1,i}$ are all equal to the k-th eigenvalue λ_k in I, from (5.7) we get

$$\lambda_k^\varepsilon \leq \lambda_k + \delta$$

for δ arbitrarily small and ε tending to zero. So

$$\limsup_{\varepsilon \to 0} \lambda_k^\varepsilon \leq \lambda_k. \tag{5.8}$$

On the other hand, suppose that $\lim_{\varepsilon \to 0} \lambda_k^\varepsilon = \lambda$. By Lemma 5.8 the k nodal domains of u_k^ε have positive measure independent of ε. Then it must be $\lambda \geq \lambda_k$. It follows that

$$\lambda_k \leq \lambda = \lim_{\varepsilon \to 0} \lambda_k^\varepsilon. \tag{5.9}$$

Combining (5.8) and (5.9) we obtain the desired result. □

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