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THREE DIMENSIONAL REAL LIE BIALGEBRAS

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Abstract. By different methods, we classify the real three dimensional Lie bialgebras and give their automorphism groups; in case of coboundary Lie bialgebras, the corresponding coboundaries $r \in \Lambda^2\mathfrak{g}$ are listed.

Introduction

Our goal is to classify the *real* three dimensional Lie bialgebras. Recall that a Lie bialgebra over a field $\mathbb{K}$ is a triple $(\mathfrak{g}, [-,-], \delta)$ where $(\mathfrak{g}, [-,-])$ is a Lie algebra over $\mathbb{K}$ and $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ is such that

- $\delta^* : \Lambda^2\mathfrak{g}^* \subseteq (\Lambda^2\mathfrak{g})^* \to \mathfrak{g}^*$ is a Lie algebra structure on $\mathfrak{g}^*$,
- $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ is a 1-cocycle in the Chevalley-Eilenberg complex of the Lie algebra $(\mathfrak{g}, [-,-])$ with coefficients in $\Lambda^2\mathfrak{g}$.

The Jacobi condition for δ^* is called co-Jacobi condition. We will usually denote a Lie bialgebra, with underlying Lie algebra $\mathfrak{g} = (\mathfrak{g}, [-,-])$, by $(\mathfrak{g}, \delta)$. A Lie bialgebra $(\mathfrak{g}, \delta)$ is said a coboundary Lie bialgebra, if there exists $r \in \Lambda^2\mathfrak{g}$ such that $\delta(x) = \mathrm{ad}_x(r)$ $\forall x \in \mathfrak{g}$; i.e. $\delta = \partial r$ is a 1-coboundary in the Chevalley-Eilenberg complex with coefficients in $\Lambda^2\mathfrak{g}$. Coboundary Lie bialgebras are denoted by $(\mathfrak{g}, r)$, although r is in general not unique. Recall that $r \in \mathfrak{g} \otimes \mathfrak{g}$ satisfies the *classical Yang-Baxter equation*, CYBE for short, if

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,$$

where the Lie bracket is taken in the repeated index; for example, if $r = \sum_i r_i \otimes r^i$ then $r^{12} := r \otimes 1$, $r^{13} := \sum_i r_i \otimes 1 \otimes r^i$, and $r^{23} := 1 \otimes r \in \mathfrak{U}(\mathfrak{g})^{\otimes 3}$, so $[r^{12}, r^{13}] = \sum_{i,j} [r_i, r_j] \otimes r^i \otimes r^j \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \hookrightarrow \mathfrak{U}(\mathfrak{g})^{\otimes 3}$, and so on for the other terms of CYBE. We denote the left hand-side of CYBE by $\mathrm{CYB}(r)$.

If $r \in \Lambda^2\mathfrak{g}$ then $\delta = \partial r$ satisfies the co-Jacobi identity if and only if $\mathrm{CYB}(r) \in \Lambda^3\mathfrak{g}$ is $\mathfrak{g}$-invariant. If $(\mathfrak{g}, r)$ is a coboundary Lie bialgebra and r satisfies CYBE, $(\mathfrak{g}, r)$ is said triangular. A Lie bialgebra is quasi-triangular if there exists $r \in \mathfrak{g} \otimes \mathfrak{g}$, not

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necessarily skewsymmetric, such that $\delta(x) = \mathrm{ad}_x(r)$ $\forall x \in \mathfrak{g}$ and r satisfies CYBE; if, moreover, the symmetric component of r induces a nondegenerate inner product on $\mathfrak{g}^*$, then $(\mathfrak{g}, \delta)$ is said factorizable. Any quasi-triangular Lie bialgebra $(\mathfrak{g}, r)$ is, in particular, coboundary, with coboundary given as the skewsymmetric component of r.

Classification of three dimensional Lie bialgebras was considered in [2] (for $k = \mathbb{R}, \mathbb{C}$), and more recently in [4] for k arbitrary, in geometric terms, involving bilinear forms and Lagrangian varieties. However, there are slight differences between them, and we also find differences between their classifications and ours. In the $\mathfrak{sl}(2,\mathbb{R})$ case, there are, apart from coabelian, two 1-parameter families which are factorizable, and the triangular case. In [2], both 1-parameter families appear as an arbitrary nonzero multiple of a given Lie cobracket, and in [4] they consider only arbitrary positive multiples. We found that in one family, $(\mathfrak{g}, \delta) \ncong (\mathfrak{g}, -\delta)$, while in the second $(\mathfrak{g}, \delta) \cong (\mathfrak{g}, -\delta)$, so in order to list isomorphism classes, one needs to consider arbitrary multiples in the first family and positive multiples in the second one. For the triangular case, both [2] and [4] exhibit only one representative $(\mathfrak{g}, \delta)$, while we found two nonisomorphic: $(\mathfrak{g}, \pm\delta)$. Here we give a table with structure constants and correspondences between these works and ours. Denote $u = \frac{h}{2}$, $v = \frac{x+y}{2}$, $w = \frac{(x-y)}{2}$.

Comparative results for $\mathfrak{sl}(2,\mathbb{R})$

$\mathfrak{sl}(2,\mathbb{R})$	[2]	[4]	Theorem 9.1
Structure constants	$[e_0,e_1] = -e_2$ $[e_0,e_2] = e_1$ $[e_1,e_2] = e_0$	$[e_1,e_2] = -e_3$ $[e_2,e_3] = e_1,$ $[e_3,e_1] = -e_2$	$[u,v] = w$ $[v,w] = -u,$ $[w,u] = -v$
Factorizable	$r = \lambda e_1 \wedge e_2,$ $\lambda \neq 0$	$(-a,0,0), R = -ae_2 \wedge e_1$ $a > 0$	$\delta_\beta,\ r = \beta v \wedge w$ $\beta > 0$
Almost factorizable	$r = \lambda e_0 \wedge (e_1 + e_2)$ $\lambda \neq 0$	$(0,0,a), R = ae_1 \wedge e_3$ $a > 0$	$\delta_\alpha,\ r = \alpha u \wedge v$ $\alpha \neq 0$
Triangular	$r = e_0 \wedge e_1$	$(0,1,1), R = -e_3 \wedge e_1 + e_1 \wedge e_2$	$\pm\delta_1,\ r = u \wedge v + v \wedge w$

The work [6] computes all the r-matrices for real 3-dimensional Lie algebras; but for 3-dimensional solvable Lie algebras, $H^1(\mathfrak{g}, \Lambda^2\mathfrak{g})$ is not trivial. We compute in all cases $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$ and all 1-cocycles, which imply the computation of $H^1(\mathfrak{g}, \Lambda^2\mathfrak{g})$, since the space of coboundaries is isomorphic to $\Lambda^2\mathfrak{g}/(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$. Besides, in our work, we distinguish the isomorphism classes of Lie bialgebras and show the r-matrices, in the cases of coboundary Lie bialgebras. The dimension of $\dim H^1(\mathfrak{g}, \Lambda^2\mathfrak{g})$ is given in the following table; see theorem 3.1 for notations. An Appendix containing comparative tables of the presentation of the Lie algebras is given at the end of the paper.

Dimension of $H^1(\mathfrak{g}, \Lambda^2\mathfrak{g})$ for the real 3-dimensional Lie algebras

$\mathfrak{g}$	$\mathfrak{h}_3$	$\mathfrak{r}_3$	$\mathfrak{r}_{3,\lambda}$ $\lambda\neq\pm1$	$\mathfrak{r}_{3,\lambda}$ $\lambda=-1$	$\mathfrak{r}_{3,\lambda}$ $\lambda=1$	$\mathfrak{r}'_{3,\lambda}$ $\lambda\neq0$	$\mathfrak{r}'_{3,\lambda}$ $\lambda=0$	$\mathfrak{su}(2)$	$\mathfrak{sl}(2,\mathbb{R})$
$\dim(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$	2	0	0	1	0	0	1	0	0
$\dim\big(\text{1-coboundaries}\big)$	1	3	3	2	3	3	2	3	3
$\dim\big(\text{1-cocycles}\big)$	6	4	4	4	6	4	4	3	3
$\dim(H^1(\mathfrak{g}, \Lambda^2\mathfrak{g}))$	5	1	1	2	3	1	2	0	0

Our method is different: we fix a Lie algebra structure, find all possible 1-cocycles, solve the co-Jacobi condition and let the Lie algebra automorphism group act on the set of solutions; in this way we find simultaneously the isoclasses of Lie bialgebras and its automorphism group as Lie bialgebras.

1. General results

The center. Given a Lie bialgebra $(\mathfrak{g}, [-,-], \delta)$, if one fixes the Lie algebra structure and varies δ, the 1-cocycle condition can be viewed as a set of linear equations in the matrix coefficients of δ. The following simplifies computations:

Proposition 1.1. *If $\mathfrak{g}$ is a Lie algebra and $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ a 1-cocycle, then $\delta(\mathcal{Z}\mathfrak{g}) \subseteq (\Lambda^2\mathfrak{g})^{\mathfrak{g}}$.*

Proof. Let $z \in \mathcal{Z}\mathfrak{g}$ and $x \in \mathfrak{g}$. The 1-cocycle condition reads $\delta[x,z] = [\delta x, z] + [x, \delta z]$. But $z \in \mathcal{Z}\mathfrak{g}$ implies $[\mathfrak{g}, z] = 0 = [z, \Lambda^2\mathfrak{g}] = 0$, and we conclude that $[\mathfrak{g}, \delta z] = 0$. □

The above proposition will be useful when $\mathcal{Z}\mathfrak{g}$ is "big" and $(\Lambda^2\mathfrak{g})^{\mathfrak{g}}$ "small". So, it will be useful to start computing the center and the invariant part or $\Lambda^2\mathfrak{g}$.

The derived ideal $[\mathfrak{g}, \mathfrak{g}]$. Recall that a coideal in a Lie bialgebra $\mathfrak{g}$ is a subspace $V \subseteq \mathfrak{g}$ such that $\delta V \subseteq V \wedge \mathfrak{g}$. Such a subspace occurs as kernel of a Lie coalgebra map. The 1-cocycle condition for δ implies the following:

Proposition 1.2. *If $(\mathfrak{g}, \delta)$ is a Lie bialgebra then $[\mathfrak{g}, \mathfrak{g}]$ is a coideal. In particular $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ admits a unique Lie bialgebra structure such that the canonical projection is a Lie bialgebra map. Moreover, if $(\mathfrak{g}, \delta_1) \cong (\mathfrak{g}, \delta_2)$ as Lie bialgebras, then $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], \overline{\delta}_1) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], \overline{\delta}_2)$.*

Notice that the Lie algebra structure on $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is trivial, so a Lie bialgebra structure on $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is equivalent to an usual Lie algebra structure on $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$.

Lemma 1.3. *Let $\mathfrak{g}$ be a Lie algebra and $\psi : \mathfrak{g} \to \mathfrak{g}$ a Lie algebra automorphism, then ψ induces Lie algebra morphisms $\psi|_{[\mathfrak{g},\mathfrak{g}]} : [\mathfrak{g}, \mathfrak{g}] \to [\mathfrak{g}, \mathfrak{g}]$ and $\overline{\psi} : \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. The applications* $\mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}([\mathfrak{g}, \mathfrak{g}])$ *and* $\mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])$ *defined respectively by $\psi \mapsto \psi|_{[\mathfrak{g},\mathfrak{g}]}$ and $\psi \mapsto \overline{\psi}$ are group homomorphisms.*

Remark 1.4. Proposition 1.2 says that by trivializing the bracket one gets a quotient Lie bialgebra. The dual statement of Proposition 1.2 is about a subobject of $\mathfrak{g}$ instead of a quotient: Ker δ is a Lie subalgebra (due to the 1-cocycle condition) and it is maximal with respect to the ones with trivial cobracket. If $\mathfrak{g}_1 \cong \mathfrak{g}_2$ are two isomorphic Lie bialgebras, then Ker $\delta_1 \cong$ Ker δ_2 as Lie algebras and also as bialgebras with trivial cobracket.

The characteristic bi-derivation. Let $(\mathfrak{g}, [-,-], \delta)$ be a Lie bialgebra. The characteristic endomorphism $\mathcal{D} : \mathfrak{g} \to \mathfrak{g}$ given by $\mathfrak{g} \xrightarrow{\delta} \Lambda^2\mathfrak{g} \xrightarrow{[-,-]} \mathfrak{g}$, $\mathcal{D} := [-,-]\circ\delta$, is clearly preserved by Lie bialgebra isomorphisms. Namely, if $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a Lie bialgebra isomorphism and $\mathcal{D}_{\mathfrak{g}}$ and $\mathcal{D}_{\mathfrak{g}'}$ denote the endomorphism associated to $\mathfrak{g}$ and $\mathfrak{g}'$ respectively, then $\mathcal{D}_{\mathfrak{g}'} = \phi\mathcal{D}_{\mathfrak{g}}\phi^{-1}$. As a consecuence, any function in $\mathcal{D}$ which is invariant under conjugation provides an invariant of the isomorphism class of the Lie bialgebra. For example, $\det(\mathcal{D})$ and $\mathrm{tr}(\mathcal{D})$ are (real) numerical invariants. The characteristic polynomial of $\mathcal{D}$ and its Jordan form are also invariants. Lie bialgebras $\mathfrak{g}$ such that $\mathcal{D}_{\mathfrak{g}} = 0$ are called **involutive**, but in many cases $\mathcal{D}$ is far from being zero. The following is standard.

Proposition 1.5. *If $\mathfrak{g}$ is a Lie bialgebra and $\mathcal{D} = [-,-]\circ\delta$ then $\mathcal{D}$ is a derivation with respect to the bracket and a coderivation with respect to the cobracket.*

Proof. We prove that $\mathcal{D}$ is a derivation, and the second claim follows by dualization. In Sweedler-type notation the 1-cocycle condition reads $\delta[x,y] = [x_1\wedge x_2, y]+[x, y_1\wedge y_2] = [x_1,y]\wedge x_2 + x_1\wedge[x_2,y] + [x,y_1]\wedge y_2 + y_1\wedge[x,y_2]$. It follows that $\mathcal{D}[x,y] = [[x_1,y],x_2] + [x_1,[x_2,y]] + [[x,y_1],y_2] + [y_1,[x,y_2]]$. Using the Jacobi identity and the definition of $\mathcal{D}$, we get $\mathcal{D}[x,y] = [[x_1,x_2],y]+[x,[y_1,y_2]] = [\mathcal{D}x,y]+[x,\mathcal{D}y]$. □

2. Two dimensional Lie bialgebras

In a similar way that one proves that there are only two isoclasses of Lie algebras of dimension 2, an easy manipulation of basis shows that the following list exhausts the isoclases of two dimensional Lie bialgebras. The structure is given in a basis $\{h, x\}$ of $\mathfrak{g}$. The first four lines are clearly non isomorphic among them, and non isomorphic to any of the last line. Finally, thanks to the invariant $\mathrm{tr}(\mathcal{D})$, one sees that they are not isomorphic to each other for different μ. Denote by $\mathfrak{aff}(\mathbb{K})$ the non-abelian 2-dimensional Lie algebra over $\mathbb{K}$. The same table is valid for any field $\mathbb{K}$, replacing $\mathfrak{aff}(\mathbb{R})$ by $\mathfrak{aff}(\mathbb{K})$. A similar table appears in [5], but without the parameter μ, which can not be eliminated, because $\mathrm{tr}(\mathcal{D})$ is an invariant of the Lie bialgebra.

2-dimensional Lie bialgebras isomorphism classes

$\mathfrak{g}$	$\mathfrak{g}^*$	$[-,-]$	δ	Invariants	Name	tr($\mathcal{D}$)
abelian	abelian	0	0			0
abelian	non abel	0	$\delta h = x \wedge h; \delta x = 0$			0
$\mathfrak{aff}(\mathbb{R})$	abelian	$[h, x] = x$	0			0
$\mathfrak{aff}(\mathbb{R})$	non abel	$[h, x] = x$	$\delta h = h \wedge x; \delta x = 0$	Ker $\delta = [\mathfrak{g}, \mathfrak{g}]$; $\delta = \partial r, r = h \wedge x$	$\mathfrak{aff}_{2,0}$	0
$\mathfrak{aff}(\mathbb{R})$	non abel	$[h, x] = x$	$\delta h = 0; \mu \neq 0$ $\delta x = \mu h \wedge x;$	Ker $\delta \neq [\mathfrak{g}, \mathfrak{g}]$	$\mathfrak{aff}_{2,\mu}$	μ

Simply by inspection, notice the following:

Proposition 2.1. *If* $(\mathfrak{g}, [-,-], \delta)$ *is a Lie bialgebra with* $\dim \mathfrak{g} = 2$*, then, whithin the non abelian and non coabelian cases,* tr($\mathcal{D}$) *is a total invariant;* $\mathfrak{aff}_{2,\mu} \cong \mathfrak{aff}_{2,\mu'}$ *if and only if* $\mu = \mu'$*.*

Corollary 2.2. *Let* $a, b, c, d \in \mathbb{K}$ *such that* $(a, b) \neq (0, 0)$ *and* $(c, d) \neq (0, 0)$*; consider* $\mathfrak{g}_{abcd}$ *the Lie bialgebra given by* $[h, x] = ah + bx$*,* $\delta h = ch \wedge x$*,* $\delta x = dh \wedge x$*; then* $\mathfrak{g}_{abcd} \cong \mathfrak{aff}_{2,\mu}(\mathbb{K})$*, with* $\mu = ac + bd$*.*

Proof. Since $(a, b) \neq (0, 0)$ and $(c, d) \neq (0, 0)$, we are not in the abelian or coabelian case. It suffices to compute the trace of $\mathcal{D}$, which is equal to $ac + db$. □

Automorphism groups in the non abelian and non co-abelian cases. Consider $\mathcal{D} = [-,-] \circ \delta$ as above and the ordered basis $\{h, x\}$ of $\mathfrak{g}$, then the Lie bialgebra automorphism groups in the non abelian and non co-abelian cases are as follows:

- Case $\mathfrak{g} = \mathfrak{aff}(\mathbb{R})$ with $[h, x] = x$ and $\delta h = h \wedge x$, $\delta x = 0$:

$$\mathrm{Aut}(\mathfrak{g}) = \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} : \ b \in \mathbb{R} \right\}.$$

In particular, any of these maps is the exponential of a multiple of $\mathcal{D} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

- Case $\mathfrak{g} = \mathfrak{aff}(\mathbb{R})$ with $[h, x] = x$ and $\delta_\mu h = 0$, $\delta_\mu x = \mu h \wedge x$:

$$\mathrm{Aut}(\mathfrak{g}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} : \ 0 \neq d \in \mathbb{R} \right\}.$$

Any of these maps with $d > 0$ is the exponential of a multiple of $\mathcal{D}_\mu = \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix}$. The fact that the exponential of (a multiple of) the endomorphism $\mathcal{D}$ gives an automorphism of the Lie bialgebra is not surprising, since we already knew that $\mathcal{D}$ is a derivation and a coderivation.

3. Three dimensional real Lie algebras

Theorem 3.1. [3] *Let $\{e_1, e_2, e_3\}$ be an ordered basis of $\mathfrak{g}$. The following list exhausts the 3-dimensional solvable real Lie algebras:*

$$\begin{aligned} \mathbb{R}^3 :&\quad \textit{the three dimensional abelian;}\\ \mathfrak{h}_3 :&\quad [e_1,e_2]=e_3,\ \textit{the three dimensional Heisenberg;}\\ \mathfrak{r}_3 :&\quad [e_1,e_2]=e_2,\ [e_1,e_3]=e_2+e_3;\\ \mathfrak{r}_{3,\lambda} :&\quad [e_1,e_2]=e_2,\ [e_1,e_3]=\lambda e_3,\ |\lambda|\le 1;\\ \mathfrak{r}'_{3,\lambda} :&\quad [e_1,e_2]=\lambda e_2-e_3,\ [e_1,e_3]=e_2+\lambda e_3,\ \lambda\ge 0. \end{aligned}$$

Denote $u = \frac{ih}{2}$, $v = \frac{x-y}{2}$, $w = \frac{i(x+y)}{2}$; the semisimple 3-dimensional real Lie algebras are

$$\mathfrak{sl}(2,\mathbb{R})\colon [h,x]=2x,\ [h,y]=-2y,\ [x,y]=h;$$

$$\mathfrak{su}(2)\colon [u,v]=w,\ [v,w]=u,\ [w,u]=v.$$

Three dimensional real Lie bialgebras: general strategy. In order to classify all real three dimensional Lie bialgebras we will proceed as follows:

(1) Given a Lie algebra $\mathfrak{g}$, we find the general 1-cocycle $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$.
(2) Determine when δ satisfies the co-Jacobi identity.
(3) Study the action of $\mathrm{Aut}(\mathfrak{g}, [-,-])$ on the set of cobrackets δ.
(4) Find a set of representatives, hence, the list of isomorphism classes of Lie bialgebras with underlying Lie algebra $\mathfrak{g}$.

To give a Lie bialgebra structure on the abelian Lie algebra $\mathbb{R}^3$ is the same as giving a Lie algebra structure on $(\mathbb{R}^3)^*$, so the list of all three dimensional Lie bialgebras with underlying Lie algebra $\mathbb{R}^3$ is in bijection with the list of three dimensional Lie algebras. Next, we proceed with the other cases: first $\mathfrak{h}_3$, the only 3-dimensional nilpotent, non abelian Lie algebra; secondly the solvable non nilpotent $\mathfrak{r}_3$, $\mathfrak{r}_{3,\lambda}$ and $\mathfrak{r}'_{3,\lambda}$, and finally the simple $\mathfrak{su}(2)$ and $\mathfrak{sl}(2,\mathbb{R})$.

The general co-Jacobi condition. If $\mathfrak{g}$ is any three dimensional Lie algebra, we write the structure in terms of basis $\{x, y, h\}$ of $\mathfrak{g}$ and $\{x\wedge y,\ y\wedge h,\ h\wedge x\}$ for $\Lambda^2(\mathfrak{g})$. Write, with a_i, b_i, $c_i \in \mathbb{R}$, $i = 1, 2, 3$, $\delta x = a_1 x\wedge y + a_2 y\wedge h + a_3 h\wedge x$; $\delta y = b_1 x\wedge y + b_2 y\wedge h + b_3 h\wedge x$; $\delta h = c_1 x\wedge y + c_2 y\wedge h + c_3 h\wedge x$.
For a linear map $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$, the co-Jacobi condition is equivalent to:

$$\begin{aligned} -a_1b_2 + a_2(b_1 - c_3) + a_3c_2 &= 0,\\ b_1a_3 - b_2c_3 + b_3(-a_1 + c_2) &= 0,\\ c_1(a_3 - b_2) + c_2b_1 - c_3a_1 &= 0. \end{aligned}$$

4. Lie bialgebra structures on $\mathfrak{h}_3$

Recall that the Lie algebra $\mathfrak{h}_3$ has a basis $\{x, y, h\}$, with the relations $[h, x] = 0$, $[h, y] = 0$, $[x, y] = h$. We list general properties of $\mathfrak{h}_3$:

- $[\mathfrak{h}_3, \mathfrak{h}_3] = \mathbb{R}h$, $[\mathfrak{h}_3, [\mathfrak{h}_3, \mathfrak{h}_3]] = 0$; $\mathfrak{h}_3$ is nilpontent.
- $\mathscr{Z}(\mathfrak{h}_3) = \mathbb{R}h$, $(\Lambda^2\mathfrak{h}_3)^{\mathfrak{h}_3} = \mathbb{R}y\wedge h \oplus \mathbb{R}h\wedge x$, $(\Lambda^3\mathfrak{h}_3)^{\mathfrak{h}_3} = \mathbb{R}x\wedge y\wedge h$.

- Since $\mathcal{Z}(\mathfrak{h}_3) = \mathbb{R}h$, if $\phi : \mathfrak{h}_3 \to \mathfrak{h}_3$ is an automorphism, it must satisfy $\phi(h) = \lambda h$ for some $0 \neq \lambda \in \mathbb{R}$, and if $\phi(x) = \mu x + \sigma y + ah$ and $\phi(y) = \nu x + \rho y + bh$, then

$$\lambda h = \phi(h) = [\phi(x), \phi(y)] = (\mu\nu - \rho\sigma)[x, y].$$

So, the automorphism group of $\mathfrak{h}_3$ is the following subgroup of $\mathrm{GL}(3, \mathbb{R})$:

$$\mathrm{Aut}(\mathfrak{h}_3) = \left\{ \phi_{\mu,\rho,\sigma,\nu,a,b} = \begin{pmatrix} \mu & \rho & 0 \\ \sigma & \nu & 0 \\ a & b & \lambda \end{pmatrix} : \mu\nu - \rho\sigma = \lambda \neq 0 \right\}$$

1-coboundaries: Let $r = ax \wedge y + by \wedge h + ch \wedge x$; then $\partial r = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ -a & 0 & 0 \end{pmatrix}$, $a, b, c \in \mathbb{R}$; $\mathrm{CYB}(r) = a^2 x \wedge y \wedge h$ is always $\mathfrak{h}_3$-invariant.

1-cocycle condition. Consider the basis $\{x \wedge y, y \wedge h, h \wedge x\}$ of $\Lambda^2(\mathfrak{h}_3)$ and write δ as in section 3. Proposition 1.1 implies $\delta h = c_2 y \wedge h + c_3 h \wedge x$, namely $c_1 = 0$. The 1-cocycle condition for $[h, x]$ and $[h, y]$ is the content of the proof of this proposition, so it gives no further information in this case. Besides, the 1-cocycle for $[x, y] = h$ reads $\delta h = \delta[x, y] = [\delta x, y] + [x, \delta y]$, then

$$\begin{aligned} \delta h &= c_2 y \wedge h + c_3 h \wedge x \\ &= [a_1 x \wedge y + a_2 y \wedge h + a_3 h \wedge x, y] + [x, b_1 x \wedge y + b_2 y \wedge h + b_3 h \wedge x] \\ &= a_1 h \wedge y + b_1 x \wedge h, \end{aligned}$$

so $c_2 = -a_1$ and $c_3 = -b_1$. Hence, a general 1-cocycle is $\delta = \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & -a_1 \\ a_3 & b_3 & -b_1 \end{pmatrix}$.

The co-Jacobi condition (see section 3), restricted to a 1-cocycle in $\mathfrak{h}_3$, reduces to

$$\begin{aligned} 2a_2 b_1 - a_1(a_3 + b_2) &= 0, \\ b_1 a_3 + b_1 b_2 - 2a_1 b_3 &= 0. \end{aligned}$$

These equations are not so easy to solve, so we use a dimensional reduction procedure, thanks to the results of section 1.

Lemma 4.1. *For $\mathfrak{g} = \mathfrak{h}_3$, the natural application of Lemma 1.3,*

$$\mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]), \quad \psi \mapsto \overline{\psi},$$

is a split epimorphism.

Proof. Consider the basis $\{x, y, h\}$. The splitting may be defined as

$$\mathrm{Aut}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \to \mathrm{Aut}(\mathfrak{g})$$

$$\phi = \begin{pmatrix} \mu & \rho \\ \sigma & \nu \end{pmatrix} \mapsto \widehat{\phi} = \left(\begin{array}{cc|c} \mu & \rho & 0 \\ \sigma & \nu & 0 \\ \hline 0 & 0 & \mu\nu - \rho\sigma \end{array}\right). \qquad \square$$

We know that $\dim(\mathfrak{h}_3/[\mathfrak{h}_3,\mathfrak{h}_3]) = 2$ and $(\mathfrak{h}_3/[\mathfrak{h}_3,\mathfrak{h}_3])$ is abelian. According to section 2, there are only two classes of isomorphisms of 2-dimensional Lie bialgebras with abelian bracket: the co-abelian one and the one with $\overline{\delta}(\overline{x}) = 0$ and $\overline{\delta}(\overline{y}) = \overline{x}\wedge\overline{y}$. Observe that, in virtue of the form of the Lie algebra automorphims, there is no loss of generality in assuming that the basis $\{\overline{x},\overline{y}\}$ is the one which allows us to write $\overline{\delta}$ in this form, since any automorphism of $(\mathfrak{h}_3/[\mathfrak{h}_3,\mathfrak{h}_3])$ may be lifted to an automorphism of $\mathfrak{h}_3$. Explicitly, we may assume $a_1 = 0$, then there are two possibilities for b_1: $b_1 = 0$ or $b_1 = 1$. We get

$$\delta_{b_1=0} = \begin{pmatrix} 0 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{pmatrix} \quad \text{and} \quad \delta_{b_1=1} = \begin{pmatrix} 0 & 1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & -1 \end{pmatrix}.$$

With the assumption $a_1 = 0$, the co-Jacobi condition is automatically satisfied in the case $b_1 = 0$, and it reduces to $b_2 + a_3 = 0 = 2a_2$ if $b_1 = 1$.

Case $b_1 = 1$. In this case, $\delta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a_3 & 0 \\ a_3 & b_3 & -1 \end{pmatrix}$. Conjugating by $\phi = \phi_{\mu,\nu,\rho,\sigma,a,b}$ one gets

$$\begin{aligned}(\phi^{-1}\wedge\phi^{-1})\delta\phi &=: \delta' \\ &= \frac{1}{\mu\nu-\rho\sigma}\begin{pmatrix} \sigma & \nu & 0 \\ \frac{b_3\sigma^2}{\mu\nu-\rho\sigma} & \frac{a\nu-a_3\mu\nu-b\sigma+b_3\nu\sigma+a_3\rho\sigma}{\mu\nu-\rho\sigma} & -\sigma \\ \frac{-a\nu+a_3\mu\nu+b\sigma+b_3\nu\sigma-a_3\rho\sigma}{\mu\nu-\rho\sigma} & \frac{b_3\nu^2}{\mu\nu-\rho\sigma} & -\nu \end{pmatrix}.\end{aligned}$$

If one wants to preserve $a_1' = 0$ one needs $\sigma = 0$, so

$$\delta' = \begin{pmatrix} 0 & \frac{1}{\mu} & 0 \\ 0 & \frac{a-a_3\mu}{\mu^2\nu} & 0 \\ \frac{-a+a_3\mu}{\mu^2\nu} & \frac{b_3}{\mu^2} & -\frac{1}{\mu} \end{pmatrix}.$$

The condition $b_1' = 1$ forces $\mu = 1$, so, with $\sigma = 0$ and $\mu = 1$,

$$\delta' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{a-a_3}{\nu} & 0 \\ \frac{-(a-a_3)}{\nu} & b_3 & -1 \end{pmatrix}.$$

Taking an automorphism with $a = a_3$ we get $a_3' = 0$, namely δ changes into

$$\delta' = \delta_{b_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & b_3 & -1 \end{pmatrix}.$$

An automorphim preserving also $a_3' = 0$ must have $a = 0$, and in this case $\delta' = \delta$. We conclude that the list of isoclasses of Lie bialgebras with $a_1 = 0$ and $b_1 = 1$ is given by the cobrackets $\{\delta_{b_3} : b_3 \in \mathbb{R}\}$ given above. Also, for each of these, the automorphism group of Lie bialgebras is

$$G = \left\{ \phi_{\rho,\nu,b} = \begin{pmatrix} 1 & \rho & 0 \\ 0 & \nu & 0 \\ 0 & b & \nu \end{pmatrix} : \nu \neq 0, b,\ \rho \in \mathbb{R} \right\}.$$

Case $b_1 = 0$: In this situation, co-Jacobi is automatically satisfied. Let $\delta = \begin{pmatrix} 0 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{pmatrix}$ and let us compute $\delta' := (\phi\wedge\phi)^{-1}\delta\phi$ with $\phi = \phi_{\mu,\nu,\rho,\sigma,a,b}$; then

$$\delta' = \frac{1}{(\mu\nu-\rho\sigma)^2} \begin{pmatrix} 0 & 0 & 0 \\ a_2\mu^2 + \sigma\mu(a_3+b_2) + b_3\sigma^2 & b_2\mu\nu + a_2\mu\rho + b_3\nu\sigma + a_3\rho\sigma & 0 \\ a_2\mu\rho + b_2\rho\sigma + b_3\nu\sigma + a_3\nu\mu & b_3\nu^2 + (a_3+b_2)\rho\nu + a_2\rho^2 & 0 \end{pmatrix}.$$

Although the matrix $\begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$ does not correspond to a symmetric bilinear form, it changes according to the following rule:

$$\begin{pmatrix} a_2' & b_2' \\ a_3' & b_3' \end{pmatrix} = \frac{1}{(\mu\nu-\rho\sigma)^2} \begin{pmatrix} \mu & \sigma \\ \rho & \nu \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} \mu & \rho \\ \sigma & \nu \end{pmatrix}.$$

Namely, it changes as a bilinear form, divided by the square of the determinant $\mu\nu - \rho\sigma$. We conclude that this block may be written as the sum of its symmetric plus its antisymmetric parts, and the group of automorphisms of $\mathfrak{h}_3$ preserves this decomposition. We know that its symmetric part can be diagonalized; hence, we may assume that the symmetric part is diagonal, so, up to isomorphism, one may assume that $b_2 = -a_3$. One possibility is when $a_2 = b_3 = 0$, namely $\delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix}$. Under general automorphism, δ transforms as $\delta \mapsto \delta' = \frac{\mu\nu-\rho\sigma}{\delta}$, so up to isomorphism there are two cases, $a_3 = 0$ (namely the coabelian case), and $a_3 = 1$; in this latter case a representative of the isoclass is $\delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and the automorphism group is the Lie algebra isomorphisms satisfying $\mu\nu - \rho\sigma = 1$.

For general a_2 and a_3, this block transforms as

$$\begin{pmatrix} a_2 & -a_3 \\ a_3 & b_3 \end{pmatrix} \mapsto$$
$$\frac{1}{(\mu\nu-\rho\sigma)^2} \begin{pmatrix} a_2\mu^2 + b_3\sigma^2 & \nu\sigma b_3 + a_2\mu\rho + (-\mu\nu+\rho\sigma)a_3 \\ a_2\mu\rho + (-\rho\sigma+\nu\mu)a_3 + b_3\nu\sigma & b_3\nu^2 + a_2\rho^2 \end{pmatrix}$$

So, the fact of being a diagonal matrix plus an antisymmetric one is not preserved, unless

$$a_2\mu\rho + b_3\nu\sigma = 0.$$

We notice that if $a_2 = b_3 \neq 0$, this equation means that the vector (μ,ν) is orthogonal to (ρ,σ), while if $a_2 = -b_3 \neq 0$ then the vector (μ,ν) is orthogonal to (ρ,σ) with respect to the bilinear form with 1 and -1 on the diagonal.

Considering the particular case when $\rho=\sigma=0$, this block changes respectively into

$$\begin{pmatrix} a_2 & -a_3 \\ a_3 & b_3 \end{pmatrix} \mapsto \begin{pmatrix} \frac{a_2}{\nu^2} & -\frac{a_3}{\mu\nu} \\ \frac{a_3}{\mu\nu} & \frac{b_3}{\mu^2} \end{pmatrix},$$

so we see that a_2 and b_3 may be chosen up to positive scalar. We distinguish then the cases $(\pm1,\pm1)$, $(0,\pm1)$ and $(\pm1,0)$. But also, considering the automorphism with $\mu=\nu=0$, $\rho=1$ and $\sigma=-1$, then $a_2'=b_3$,$b_3'=a_2$, and $a_3'=a_3$, so the choices $(a_2,b_3)=(1,-1)$ or $(-1,1)$ are isomorphic; ditto $(0,1)$ with $(1,0)$ and $(0,-1)$ with $(-1,0)$. We conclude that the symmetric part may be chosen equal to

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

and since they all can be distinguished by the rank and signature, they correspond to non-isomorphic cases.

Now in each of these cases, we have to see how much freedom we have for a_3.

We begin with the case $(a_2,b_3)=(1,0)$:

In this case, the block changes with the rule

$$\begin{pmatrix} 1 & -a_3 \\ a_3 & 0 \end{pmatrix} \mapsto \frac{1}{(\mu\nu-\rho\sigma)^2}\begin{pmatrix} \mu^2 & \mu\rho+(-\mu\nu+\rho\sigma)a_3 \\ \mu\rho+(-\rho\sigma+\nu\mu)a_3 & \rho^2 \end{pmatrix},$$

so if we want to preserve the condition $b_3'=0$ we need $\rho=0$, so

$$\begin{pmatrix} 1 & -a_3 \\ a_3 & 0 \end{pmatrix} \mapsto \frac{1}{(\mu\nu)^2}\begin{pmatrix} \mu^2 & -\mu\nu a_3 \\ \nu\mu a_3 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\nu^2} & -\frac{a_3}{\mu\nu} \\ \frac{a_3}{\mu\nu} & 0 \end{pmatrix}.$$

If in addition we force to preserve $a_2'=1$, then $\nu=\pm1$; but we see that choosing properly μ we can change a_3 up to an arbitrary scalar, so the cases are covered by $a_3=1$ or $a_3=0$.

We conclude that the group of Lie bialgebra isomorphisms is, in the $a_3=1$ and $a_3=0$ cases, respectively

$$\left\{\begin{pmatrix} 1 & 0 & 0 \\ \sigma & \nu & 0 \\ a & b & \nu \end{pmatrix} : \sigma,a,b\in\mathbb{R},\ \nu=\pm1\right\}$$

and

$$\left\{\begin{pmatrix} \mu & 0 & 0 \\ \sigma & \nu & 0 \\ a & b & \nu \end{pmatrix} : \sigma,a,b\in\mathbb{R},\ \nu=\pm1,\ \mu\neq0\right\}.$$

The case $(a_2,b_3)=(-1,0)$ is completely analogous. Let us see the case $(a_2,b_3)=(1,1)$. In this case, this 2×2 block changes with the rule

$$\begin{pmatrix} 1 & -a_3 \\ a_3 & 1 \end{pmatrix} \mapsto \frac{1}{(\mu\nu-\rho\sigma)^2} \times \begin{pmatrix} \mu^2+\sigma^2 & \nu\sigma+\mu\rho+(-\mu\nu+\rho\sigma)a_3 \\ \mu\rho+\nu\sigma+(-\rho\sigma+\nu\mu)a_3 & \nu^2+\rho^2 \end{pmatrix},$$

so this 2×2 block is unchanged if and only if the matrix $\begin{pmatrix} \mu & \rho \\ \sigma & \nu \end{pmatrix}$ is orthogonal, and if $a_3 \neq 0$, then the determinant must be equal to one, because otherwise $a_3 \mapsto a_3' = (\mu\nu - \rho\sigma)$; so, in particular, one can choose the sign of a_3. We conclude that in this case we have a 1-parameter family, with $a_3 \in R_{\geq 0}$; the Lie bialgebra automorphisms consist of matrices

$$\left\{ \begin{pmatrix} \mu & \rho & 0 \\ \sigma & \nu & 0 \\ a & b & \Delta \end{pmatrix} : a, b \in \mathbb{R},\ \Delta = \mu\nu - \rho\sigma \right\}$$

where, in addition, the matrix $\begin{pmatrix} \mu & \rho \\ \sigma & \nu \end{pmatrix}$ belongs to $O(2,\mathbb{R})$ in the case $a_3 = 0$, and to $SO(2,\mathbb{R})$ if $a_3 > 0$.

The case $(a_2, b_3) = (-1, -1)$ is completely analogous to this one. It remains to consider the case $(a_2, b_3) = (1, -1)$; in this case the commutations are almost the same, the difference being that in this case the equations for $\begin{pmatrix} \mu & \rho \\ \sigma & \nu \end{pmatrix}$ correspond to the group $O(1,1)$ when $a_3 = 0$, and to $SO(1,1)$ if $a_3 > 0$.

We conclude that the list of isomorphism classes is given by δ_{a_2,b_3,a_3} obtained by choosing the parameters $(a_2, b_3) = (0, 0)$ and $a_3 = 0$ or 1, and $(a_2, b_3) = (1, 1)$, $(-1, -1)$, $(1, -1)$, $(1, 0)$, $(-1, 0)$ and $a_3 \geq 0$. This completes the proof of the next theorem.

Theorem 4.2. *For the Lie algebra $\mathfrak{h}_3$, the exhaustive list of the isomorphism classes of Lie bialgebra structures is given by the following set of cobrackets:*

$$\delta_{b_3} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & b_3 & -1 \end{pmatrix} : b_3 \in \mathbb{R}, \quad \text{and} \quad \delta_{a_2,b_3,a_3} = \begin{pmatrix} 0 & 0 & 0 \\ a_2 & -a_3 & 0 \\ a_3 & b_3 & 0 \end{pmatrix},$$

with $(a_2, b_3) = \pm(1, 1)$, $(\pm 1, 0)$, $(1, -1)$, and $a_3 \geq 0$, or $(a_2, b_3) = (0, 0)$ and $a_3 = 0$ or 1.

Among them, the only isoclass of non trivial coboundary Lie bialgebras is the one with $(a_2, b_3) = (0, 0)$ and $a_3 = 1$; in particular, for $\delta = \partial r$, $r = -x \wedge y \in \Lambda^2\mathfrak{h}_3$, $CYB(r) = x \wedge y \wedge h \neq 0$, so it is not triangular. The automorphism group of the Lie bialgebra with cobracket $\delta_{a_2,b_3,a_3=0}$ is the following:

$$G_{a_2,b_3} = \left\{ \phi_{\rho,\nu,b} = \begin{pmatrix} \mu & \rho & 0 \\ \sigma & \nu & 0 \\ a & b & \mu\nu - \rho\sigma \end{pmatrix} : a, b \in \mathbb{R} \text{ and } \begin{pmatrix} \mu & \rho \\ \sigma & \nu \end{pmatrix} \in \mathrm{O}_{a_2,b_3} \right\},$$

where O_{a_2,b_3} denotes the orthogonal group $\mathrm{O}(2,\mathbb{R})$ for $(a_2, b_3) = \pm(1, 1)$, $\mathrm{O}(1, 1)$ for $(a_2, b_3) = (1, -1)$, $\mathrm{GL}(2,\mathbb{R})$ for $(a_2, b_3) = (0, 0)$ and upper triangular with $\nu = \pm 1$ for $(a_2, b_3) = (\pm 1, 0)$. The automorphism group for $\delta_{a_2,b_3,a_3 \neq 0}$ is SG_{a_2,b_3}, namely $G_{a_2,b_3} \cap \mathrm{SL}(3,\mathbb{R})$. The automorphism group for δ_{b_3} is

$$\left\{ \begin{pmatrix} 1 & \rho & 0 \\ 0 & \nu & 0 \\ 0 & b & \nu \end{pmatrix} : \rho, b \in \mathbb{R},\ \nu \neq 0 \right\}.$$

5. Lie bialgebra structures on $\mathfrak{r}_3$

Recall $\mathfrak{r}_3$ is the real Lie algebra with brackets $[h,x]=x,\ [h,y]=x+y,\ [x,y]=0$. It is straightforward to check that $\mathcal{Z}(\mathfrak{r}_3)=0$, $(\Lambda^2\mathfrak{r}_3)^{\mathfrak{r}_3}=0$, and $(\Lambda^3\mathfrak{r}_3)^{\mathfrak{r}_3}=0$.

Proposition 5.1. *1-cocycles on $\mathfrak{r}_3$ are of the form*

$$\delta=\begin{pmatrix} a_1 & b_1 & c_1\\ 0 & 0 & a_1\\ 0 & b_3 & b_1-2a_1\end{pmatrix},\quad a_1,b_1,b_3,c_1\in\mathbb{R};$$

among them, those which verify the co-Jacobi identity are of the form

$$\delta=\begin{pmatrix} 0 & b_1 & c_1\\ 0 & 0 & 0\\ 0 & b_3 & b_1\end{pmatrix}.$$

Besides, the 1-coboundaries are

$$\partial r=\begin{pmatrix} b & b+c & 2a\\ 0 & 0 & b\\ 0 & 0 & c-b\end{pmatrix},\quad a,b,c\in\mathbb{R},$$

with $r=ax\wedge y+by\wedge h+ch\wedge x$, and $CYB(r)=-2b^2x\wedge y\wedge h$ is $\mathfrak{r}_3$-invariant if and only if $b=0$.

Proof. Let us write, as in section 3, $\delta(x)=a_1x\wedge y+a_2y\wedge h+a_3h\wedge x$, $\delta(y)=b_1x\wedge y+b_2y\wedge h+b_3h\wedge x$, $\delta(h)=c_1x\wedge y+c_2y\wedge h+c_3h\wedge x$, with a_i, b_i, $c_i\in\mathbb{R}$. The 1-cocycle condition for $[x,y]=0$ and $\delta[x,y]=[\delta x,y]+[x,\delta y]$ gives

$$\begin{aligned}0&=[a_1x\wedge y+a_2y\wedge h+a_3h\wedge x,y]+[x,b_1x\wedge y+b_2y\wedge h+b_3h\wedge x]\\&=a_2y\wedge x+a_3y\wedge x-b_2y\wedge x,\end{aligned}$$

so $a_2+a_3-b_2=0$. Now, $[h,x]=x$ and $\delta[h,x]=[\delta h,x]+[h,\delta x]$ imply

$$\begin{aligned}&a_1x\wedge y+a_2y\wedge h+a_3h\wedge x\\&\quad=[c_1x\wedge y+c_2y\wedge h+c_3h\wedge x,x]+[h,a_1x\wedge y+a_2y\wedge h+a_3h\wedge x]\\&\quad=c_2y\wedge x+2a_1x\wedge y+a_2(x+y)\wedge h+a_3h\wedge x,\end{aligned}$$

so $a_1=-c_2+2a_1$ and $a_3=-a_2+a_3$. This is equivalent to $a_1=c_2$ and $a_2=0$. Finally, $[h,y]=x+y$ and $\delta[h,y]=[\delta h,y]+[h,\delta y]$ imply

$$\begin{aligned}&(a_1+b_1)x\wedge y+(a_2+b_2)y\wedge h+(a_3+b_3)h\wedge x\\&\quad=[c_1x\wedge y+c_2y\wedge h+c_3h\wedge x,y]+[h,b_1x\wedge y+b_2y\wedge h+b_3h\wedge x]\\&\quad=c_2y\wedge x+c_3y\wedge x+2b_1x\wedge y+b_2(x+y)\wedge h+b_3h\wedge x.\end{aligned}$$

So, $a_1+b_1=-c_2-c_3+2b_1$, $a_2+b_2=b_2$, $a_3+b_3=-b_2+b_3$. Solving all the linear equations, we obtain $a_2=a_3=b_2=0$, $c_2=a_1$, $c_3=b_1-2a_1$. Hence, a general 1-cocycle δ is given by

$$\begin{aligned}\delta(x)&=a_1x\wedge y,\\\delta(y)&=b_1x\wedge y+b_3h\wedge x,\\\delta(h)&=c_1x\wedge y+a_1y\wedge h+(b_1-2a_1)h\wedge x.\end{aligned}$$

The co-Jacobi condition for a general 1-cocycle is simply $2a_1^2 = 0$. Hence, a 1-cocycle satisfying also co-Jacobi is of the same form but with $a_1 = 0$. □

The Lie algebras automorphism group of $\mathfrak{r}_3$ is the following subgroup of $\mathrm{GL}(3,\mathbb{R})$:

$$\left\{ \phi_{\mu,\rho,a,b} = \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & b \\ 0 & 0 & 1 \end{pmatrix} : \mu,\rho,a,b \in \mathbb{R},\ \mu \neq 0 \right\}.$$

Proof. Since $[\mathfrak{r}_3,\mathfrak{r}_3] = \mathbb{R}x \oplus \mathbb{R}y$ and any automorphism preserves the subspace $[\mathfrak{r}_3,\mathfrak{r}_3]$ we have that a general automorphism ϕ must be of the form

$$\begin{aligned} \phi(x) &= \mu x + \sigma y \\ \phi(y) &= \rho x + \nu y \\ \phi(h) &= ax + by + ch, \end{aligned}$$

with $c \neq 0$. And from the equation $[\phi h, \phi x] = \phi x$ we get

$$\begin{aligned} \mu x + \sigma y = \phi(x) &= [\phi h, \phi x] \\ &= [ax + by + ch, \mu x + \sigma y] \\ &= c(\mu + \sigma)x + c\sigma y. \end{aligned}$$

So, $\sigma = 0$ and $c = 1$. Now, from the equation $[\phi h, \phi y] = \phi x + \phi y$ we get

$$\begin{aligned} \mu x + \rho x + \nu y = \phi(x) + \phi y &= [\phi h, \phi y] \\ &= [ax + by + h, \rho x + \nu y] \\ &= \rho x + \nu(x + y), \end{aligned}$$

so $\mu + \rho = \rho + \sigma$ and hence $\mu = \nu$. □

Under the action of the automorphism group, a 1-cocycle δ maps into

$$\delta' = \begin{pmatrix} 0 & \frac{b_1 + bb_3}{\mu} & \frac{2bb_1 + b^2 b_3 + c_1}{\mu^2} \\ 0 & 0 & 0 \\ 0 & b_3 & \frac{b_1 + bb_3}{\mu} \end{pmatrix}.$$

Notice that b_3 is an invariant.

Case $b_3 \neq 0$. Taking $b = -b_1/b_3$ we get $b_1' = 0$, so we may assume $b_1 = 0$. The condition $b_1' = 0$ is preserved only if $b = 0$, and in this case δ changes into $\delta' = \begin{pmatrix} 0 & 0 & \frac{c_1}{\mu^2} \\ 0 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix}$. So, c_1 can be chosen up to positive scalar and we may take the numbers $c = 0, \pm 1$ as representatives. Hence, the isomorphism classes of Lie bialgebras consists of three 1-parameter families with cobrackets

$$\delta_{b_3,c_1} = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix} : c_1 = 0, 1, -1,\ b_3 \neq 0.$$

For $c_1 = 0$, the automorphism group is $\left\{ \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mu, a \in \mathbb{R},\ \mu \neq 0 \right\}$, and for $c_1 = \pm 1$, we have $c_1' = c_3/\mu^2$, then μ^2 must be equal to 1 and the automorphism group is $\left\{ \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \in \mathbb{R},\ \mu = \pm 1 \right\}$.

Case $b_3 = 0$. We have then

$$\delta = \begin{pmatrix} 0 & b_1 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & b_1 \end{pmatrix} \mapsto \delta' = \begin{pmatrix} 0 & b_1/\mu & (2bb_1 + c_1)/\mu^2 \\ 0 & 0 & 0 \\ 0 & 0 & b_1/\mu \end{pmatrix};$$

hence, b_1 is determined up to a multiple; we consider the cases $b_1 \neq 0$ and $b_1 = 0$.

Case $b_1 \neq 0$. We may assume $b_1 = 1$ then $\delta' = \begin{pmatrix} 0 & 1/\mu & (2b + c_1)/\mu^2 \\ 0 & 0 & 0 \\ 0 & 0 & 1/\mu \end{pmatrix}$. If we whish to preserve $b_1 = 1$, we need to impose $\mu = 1$; we obtain $\delta' = \begin{pmatrix} 0 & 1 & 2b + c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. We may choose $b = -\frac{c_1}{2}$, so the new $c_1' = 0$ and take as a representative of the class of isomorphism $\delta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
The automorphism group of this Lie bialgebra is

$$G_{b_1 \neq 0} = \left\{ \begin{pmatrix} 1 & \rho & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : \rho, a \in \mathbb{R} \right\}.$$

Case $b_1 = 0$. We have then

$$\delta = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \delta' = \begin{pmatrix} 0 & 0 & c_1/\mu^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, we obtain that $c_1 = 0, \pm 1$ are all the possibilities for c_1. The automorphism group of such Lie bialgebra with $c_1 = 0$ is

$$G_{b_1 = c_1 = 0} = \left\{ \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & b \\ 0 & 0 & 1 \end{pmatrix} : \mu, \rho, a, b \in \mathbb{R},\ \mu \neq 0 \right\}.$$

On the other hand, the automorphism group of the Lie bialgebras classes with $c_1 = \pm 1$ consists of the Lie algebra maps satisfying $\mu^2 = 1$; hence

$$G_{b_1 = 0, c_1 \neq 0} = \left\{ \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & b \\ 0 & 0 & 1 \end{pmatrix} : \mu, \rho, a, b \in \mathbb{R},\ \mu = \pm 1 \right\}.$$

This discussion leads to the following result.

Theorem 5.2. *The isomorphism classes of Lie bialgebra structures on* $\mathfrak{r}_3$ *are given by the following list of cobrackets:*

$$\delta_{b_1\neq 0, b_3=0} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \textit{and} \quad \delta_{c_1,b_3} = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & b_3 & 0 \end{pmatrix} : b_3 \in \mathbb{R},\ c_1 = 0, \pm 1.$$

The first cobracket corresponds to a coboundary structure with $r = h \wedge x$, *as well as the second one in the case* $b_3 = 0$, *with* $r = \frac{c_1}{2} x \wedge y$; *both are triangular. The corresponding Lie bialgebras automorphisms are given in the previous paragraph.*

6. Lie bialgebra structures on $\mathfrak{r}_{3,\lambda}$, with $|\lambda| \leq 1$

Recall that $\mathfrak{r}_{3,\lambda}$ is the Lie algebra with bases $\{x, y, h\}$, and brackets $[h, x] = x$, $[h, y] = \lambda y$, $[x, y] = 0$. We list general properties for $\mathfrak{r}_{3,\lambda}$:

- If $\lambda \neq 0$ then $\mathcal{Z}(\mathfrak{g}) = 0$; if $\lambda = 0$ then $\mathcal{Z}(\mathfrak{g}) = \langle y \rangle$.
- If $\lambda \neq -1$ then $\Lambda^2(\mathfrak{g})^{\mathfrak{g}} = 0$; if $\lambda = -1$ then $\Lambda^2(\mathfrak{g})^{\mathfrak{g}} = \langle x \wedge y \rangle$.
- If $\lambda \neq -1$ then $(\Lambda^3\mathfrak{g})^{\mathfrak{g}} = 0$; if $\lambda = -1$ then $(\Lambda^3\mathfrak{g})^{\mathfrak{g}} = \Lambda^3\mathfrak{g} = \langle x \wedge y \wedge h \rangle$.

Proposition 6.1. *All the 1-cocycles on the Lie algebra* $\mathfrak{r}_{3,\lambda}$ *with* $|\lambda| \leq 1$ *are*

$$\delta = \begin{pmatrix} a_1 & \lambda c_3 & c_1 \\ 0 & \lambda a_3 & \lambda a_1 \\ a_3 & 0 & c_3 \end{pmatrix} \quad \textit{if } \lambda \neq 1, \qquad \delta = \begin{pmatrix} a_1 & c_3 & c_1 \\ a_2 & a_3 & a_1 \\ a_3 & b_3 & c_3 \end{pmatrix} \quad \textit{if } \lambda = 1,$$

in the basis $\{x \wedge y,\ y \wedge h,\ h \wedge x\}$ *of* $\Lambda^2(\mathfrak{g})$ *and notations as in section 3. On the other hand, all the coboundaries are, with* $r = a x \wedge y + b y \wedge h + c h \wedge x$,

$$\partial r = \begin{pmatrix} b & c\lambda & a(1+\lambda) \\ 0 & 0 & b\lambda \\ 0 & 0 & c \end{pmatrix} : a, b, c \in \mathbb{R}, \textit{with } CYB(r) = bc(1-\lambda) x \wedge y \wedge h.$$

Proof. Let $\delta : \mathfrak{g} \to \Lambda^2(\mathfrak{g})$ be a 1-cocyle, then $\delta[h, x] = [\delta h, x] + [h, \delta x]$ and $[h, x] = x$ imply

$$\begin{aligned} & a_1 x \wedge y + a_2 y \wedge h + a_3 h \wedge x \\ &\quad = [c_1 x \wedge y + c_2 y \wedge h + c_3 h \wedge x, x] + [h, a_1 x \wedge y + a_2 y \wedge h + a_3 h \wedge x] \\ &\quad = c_2 y \wedge x + a_1(1+\lambda) x \wedge y + \lambda a_2 y \wedge h + a_3 h \wedge x. \end{aligned}$$

We conclude that $\lambda a_1 = c_2$, $a_2 = \lambda a_2$, so $a_2 = 0$ if $\lambda \neq 1$, and no condition in a_2 for $\lambda = 1$. In an analogous way, using the cocycle condition $\delta[h, y] = [\delta h, y] + [h, \delta y]$ for $[h, y] = \lambda y$, we get

$$\begin{aligned} & \lambda(b_1 x \wedge y + b_2 y \wedge h + b_3 h \wedge x) \\ &\quad = [c_1 x \wedge y + c_2 y \wedge h + c_3 h \wedge x, y] + [h, b_1 x \wedge y + b_2 y \wedge h + b_3 h \wedge x] \\ &\quad = \lambda c_3 y \wedge x + (1+\lambda) b_1 x \wedge y + \lambda b_2 y \wedge h + b_3 h \wedge x, \end{aligned}$$

so $b_1 = \lambda c_3$ and $b_3 = \lambda b_3$, so again $b_3 = 0$ if $\lambda \neq 1$ and no restriction on b_3 for $\lambda = 1$. The third condition is $\delta[x,y] = [\delta x, y] + [x, \delta y]$; since $[x,y] = 0$, we get

$$0 = [a_1 x\wedge y + a_2 y\wedge h + a_3 h\wedge x, y] + [x, b_1 x\wedge y + b_2 y\wedge h + b_3 h\wedge x] = \lambda a_3 y\wedge x - b_2 y\wedge x,$$

so $b_2 = \lambda a_3$. As a consequence, the general form of a 1-cocycle is, for $\lambda \neq 1$:

$$\begin{aligned}
\delta(x) &= a_1\, x\wedge y + a_3 h\wedge x\\
\delta(y) &= \lambda c_3\, x\wedge y + \lambda a_3 y\wedge h\\
\delta(h) &= c_1\, x\wedge y + \lambda a_1 y\wedge h + c_3 h\wedge x,
\end{aligned}$$

and for $\lambda = 1$:

$$\begin{aligned}
\delta(x) &= a_1 x\wedge y + a_2 y\wedge h + a_3 h\wedge x\\
\delta(y) &= c_3 x\wedge y + a_3 y\wedge h + b_3 h\wedge x\\
\delta(h) &= c_1 x\wedge y + a_1 y\wedge h + c_3 h\wedge x.
\end{aligned}$$

The statement for the coboundaries is shown similarly. □

6.1. **Lie bialgebra structures on $\mathfrak{r}_{3,\lambda}$, with $\lambda \neq \pm 1$.** Since the cases $\lambda = \pm 1$ are different, we will consider them later, and concentrate on the generic case $\lambda \in (-1, 1)$.

Proposition 6.2. *The automorphism group of the Lie algebra $\mathfrak{r}_{3,\lambda}$ with $\lambda \neq \pm 1$ is the following subgroup of $\mathrm{GL}(3,\mathbb{R})$:*

$$\mathrm{Aut}(\mathfrak{g}) = \left\{ \begin{pmatrix} \mu & 0 & a\\ 0 & \nu & b\\ 0 & 0 & 1 \end{pmatrix} : \ \mu\nu \neq 0 \right\}.$$

Proof. Let us see that if $\phi : \mathfrak{g} \to \mathfrak{g}$ is an automorphism of Lie algebra, then

$$\phi(x) = \mu x, \quad \phi(y) = \nu y, \quad \phi(h) = h + ax + by,$$

for some $\mu, \nu, a, b \in \mathbb{R}$, $\mu, \nu \neq 0$. Actually, the elements x and y may be characterized, up to scalar multiple, as the generators of $[\mathfrak{g},\mathfrak{g}]$ and eigenvectors of ad_z, for all $z \in \mathfrak{g} \setminus [\mathfrak{g},\mathfrak{g}]$. Moreover, given such z, the element x distinguishes from y as being the eigenvector corresponding to the eigenvalue with smaller absolute value. Explicitly, if $z \notin [\mathfrak{g},\mathfrak{g}]$, $z = ch + ax + by$ with $c \neq 0$,

$$\mathrm{ad}_z(x) = [ch + ax + by, x] = cx, \quad \mathrm{ad}_z(y) = [ch + ax + by, y] = c\lambda y$$

(recall $|\lambda| < 1$). This implies $\phi(x) = \mu x$ and $\phi(y) = \nu y$ for some μ, $\nu \neq 0$. If $\phi(x) = \tilde{x} = \mu x$, $\phi(y) = \tilde{y} = \nu y$, and $\phi(h) = \tilde{h} = ch + ax + by$, then the equation $[\tilde{h}, \tilde{x}] = \tilde{x}$ implies $c = 1$. □

Consider $\lambda \neq \pm 1$, let ϕ be an automorphism as above, and $\delta' := (\phi\wedge\phi)^{-1}\delta\phi$; explicitly

$$\delta' = \begin{pmatrix} \frac{a_1 + a_3 b}{\nu} & \lambda\frac{c_3 + a a_3}{\mu} & \frac{c_1 + a(a_1 + a_3 b)(1+\lambda) + b c_3(1+\lambda)}{\mu\nu}\\ 0 & \lambda a_3 & \lambda\frac{a_1 + a_3 b}{\nu}\\ a_3 & 0 & \frac{c_3 + a a_3}{\mu} \end{pmatrix}.$$

Co-Jacobi for δ' is $(1-\lambda)(a_3c_1-a_1c_3(1+\lambda))=0$; equivalently, $a_3c_1-a_1c_3(1+\lambda)=0$. If $a_3=0$, the condition reduces to $a_1c_3=0$. Note that, up to isomorphism, we may independently change a_1 and c_3 by $a_1'=a_1/\nu$ and $c_3'=c_3/\mu$, respectively. Moreover, since one of them is zero, we get the possibilities:

- $(a_1,c_3)=(0,0)$, $c_1=0$ or 1 because c_1 is determined (up to isomorphism) up to scalar multiple.
- $(a_1,c_3)=(1,0)$, and c_1 changes into $c_1'=(c_1+a(1+\lambda))/\mu$ (we need $\nu=1$ in order to preserve $a_1'=1$); we see that we can choose a such that $c_1'=0$.
- $(a_1,c_3)=(0,1)$, and c_1 changes into $c_1'=(c_1+b(1+\lambda))/\nu$, so we can choose $c_1=0$.

Notice that, if $a_3\neq 0$, by means of an automorphism with $b=-a_1/a_3$ and $a=-c_3/a_3$, we get δ' with $a_1'=0=c_3'$; explicitly

$$\delta'=\begin{pmatrix} 0 & 0 & \frac{a_3c_1-a_1c_3(1+\lambda)}{\mu\nu} \\ 0 & \lambda a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix}=\begin{pmatrix} 0 & 0 & c_1' \\ 0 & \lambda a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix}.$$

So, if $a_3\neq 0$, we may assume $a_1=0=c_3$, then co-Jacobi implies $c_1=0$. Hence, every cocycle with $a_3\neq 0$ satisfying co-Jacobi is equivalent to

$$\left\{\delta_{a_3}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & a_3\lambda & 0 \\ a_3 & 0 & 0 \end{pmatrix} : 0\neq a_3\in\mathbb{R}\right\}.$$

The parameter a_3 can not be modified using a Lie algebra automorphism, it is an invariant; we get a 1-parameter family of isoclasses, parametrized by a_3.

The bialgebra automorphisms are of the form $\phi=\begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 1 \end{pmatrix}:\mu,\ \nu\neq 0$. This discussion leads to the following result.

Theorem 6.3. *The set of representatives of all isomorphisms classes of Lie bialgebras on $\mathfrak{r}_{3,\lambda}:\lambda\neq\pm 1$ is given by the following cobrackets:*

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \lambda \\ 0 & 0 & 0 \end{pmatrix};\ \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ \delta_{a_3}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix}: a_3\in\mathbb{R}.$$

The first three cobrackets are coboundaries with $r=\frac{1}{1+\lambda}x\wedge y$, $r=y\wedge h$ and $r=h\wedge x$ respectively, which result in triangular Lie bialgebras.

6.2. ***Case* $\mathfrak{g}=\mathfrak{r}_{3,\lambda}$ *with* $\lambda=-1$.** Recall that $\mathfrak{r}_{3,\lambda=-1}$ is the Lie algebra with bracket $[h,x]=x$, $[h,y]=-y$, $[x,y]=0$. The automorphism group of $\mathfrak{r}_{3,\lambda=-1}$, in the ordered basis $\{x,y,h\}$, identifies with the following subgroup of $\mathrm{GL}(3,\mathbb{R})$:

$$\mathrm{Aut}(\mathfrak{r}_{3,\lambda=-1})=\left\langle \phi_0=\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},\ \phi_{\mu,\nu,a,b}=\begin{pmatrix} \mu & 0 & a \\ 0 & \nu & b \\ 0 & 0 & 1 \end{pmatrix}:\ \mu\nu\neq 0\right\rangle.$$

$\phi_{\mu,\nu,a,b}$ is an automorphism by the same reasons as in generic λ, but in this case, the absolute value of the eigenvalues of x and y are the same. Actually,

$$x \mapsto y, \quad y \mapsto x, \quad h \mapsto -h$$

is an automorphism, which we denote by ϕ_0. Moreover, any automorphism is obtained by compositions of the above ones. The set of 1-cocycles was computed for any λ but for convenience in the case $\lambda = -1$ we write them by $\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & -a_3 & -a_1 \\ a_3 & 0 & -b_1 \end{pmatrix}$ instead of $\delta = \begin{pmatrix} a_1 & \lambda c_3 & c_1 \\ 0 & \lambda a_3 & \lambda a_1 \\ a_3 & 0 & c_3 \end{pmatrix}$. For these cocycles, co-Jacobi reads $2a_3c_1 = 0$. The action of the automorphism group is

$$\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & -a_3 & -a_1 \\ a_3 & 0 & -b_1 \end{pmatrix} \xmapsto{\phi_{\mu,\nu,a,b}} \delta' = \begin{pmatrix} \frac{a_1+a_3b}{\nu} & \frac{b_1-aa_3}{\mu} & \frac{c_1}{\mu\nu} \\ 0 & -a_3 & \frac{-a_1-a_3b}{\nu} \\ a_3 & 0 & \frac{-b_1+aa_3}{\mu} \end{pmatrix},$$

$$\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & -a_3 & -a_1 \\ a_3 & 0 & -b_1 \end{pmatrix} \xmapsto{\phi_0} \delta' = \begin{pmatrix} -b_1 & -a_1 & c_1 \\ 0 & a_3 & b_1 \\ -a_3 & 0 & a_1 \end{pmatrix}.$$

Case $a_3 \neq 0$: Co-Jacobi implies $c_1 = 0$. But also, taking an automorphism ϕ with parameters $a = b_1/a_3$ and $b = -a_1/a_3$, we get δ' with $a_1' = b_1' = 0$. Besides, using ϕ_0, $a_3 \mapsto a_3' = -a_3$, so we may choose $a_3 > 0$. We conclude that inside this isomorphism class, we have the representative

$$\delta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a_3 & 0 \\ a_3 & 0 & 0 \end{pmatrix} : a_3 > 0.$$

Case $a_3 = 0$: In this case, co-Jacobi condition is automatically satisfied. For each 3-uple (a_1, b_1, c_1) we have $\delta_{a_1,b_1,c_1} \cong \delta_{\frac{a_1}{\mu},\frac{b_1}{\nu},\frac{c_1}{\mu\nu}}$, i.e.

$$\delta_{a_1,b_1,c_1} = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & -a_1 \\ 0 & 0 & -b_1 \end{pmatrix} \cong \delta_{\frac{a_1}{\mu},\frac{b_1}{\nu},\frac{c_1}{\mu\nu}} = \begin{pmatrix} \frac{a_1}{\nu} & \frac{b_1}{\mu} & \frac{c_1}{\mu\nu} \\ 0 & 0 & \frac{-a_1}{\nu} \\ 0 & 0 & \frac{-b_1}{\mu} \end{pmatrix}.$$

By means of the isomorphism ϕ_0 we obtain aditionally $\delta_{a_1,b_1,c_1} \cong \delta_{-b_1,-a_1,c_1}$. Choosing conveniently μ and ν, we arrive at the following result.

Theorem 6.4. *The set of representatives of all isomorphisms classes of Lie bialgebras on $\mathfrak{r}_{3,\lambda} : \lambda = -1$ is given by the following cobrackets: $\delta_{0,0,0} = 0$, and*

$$\delta_{0,0,1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \delta_{1,0,0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\delta_{1,0,1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \delta_{1,1,c_1} = \begin{pmatrix} 1 & 1 & c_1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix} : c_1 \in \mathbb{R},$$

where $\delta_{1,0,0} \cong \delta_{0,1,0}$*,* $\delta_{1,0,1} \cong \delta_{0,1,1}$*. Among them, the coboundary ones are those with* $\delta_{1,0,0}$ *(with* $r = y \wedge h$*), which is triangular, and* $\delta_{1,1,c_1=0}$ *(with* $r = y \wedge h - h \wedge x$*), which is not triangular.*

6.3. **Lie bialgebra structures on $\mathfrak{r}_{3,\lambda}$ with $\lambda = 1$.** Recall that $\mathfrak{r}_{3,1}$ is the Lie algebra bracket defined by $[h,x] = x$, $[h,y] = y$, $[x,y] = 0$. It can be easily verified that the automorphism group is the subgroup of $\mathrm{GL}(3,\mathbb{R})$ expressed as matrices as:

$$\mathrm{Aut}(\mathfrak{r}_{3,1}) = \left\{ \phi^{a,b}_{\mu,\nu,\rho,\sigma} = \begin{pmatrix} \mu & \rho & a \\ \sigma & \nu & b \\ 0 & 0 & 1 \end{pmatrix} : \mu\nu - \rho\sigma \neq 0 \right\}.$$

Recall that 1-cocycles are given by $\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & a_3 & a_1 \\ a_3 & b_3 & b_1 \end{pmatrix}$, $a_1, a_2, a_3, b_1, b_3, c_1 \in \mathbb{R}$; the co-Jacobi identity is always satisfied.

Theorem 6.5. *The exhaustive list of representatives of the isomorphism classes of Lie bialgebras with underlying Lie algebra* $\mathfrak{g} = \mathfrak{r}_{3,\lambda=1}$ *is given by the following Lie cobrackets:*

$$\delta = 0, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\delta^{0,c_1=0,\pm 1}_{a_2=0} = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \delta^{c_1=0,\pm 1}_{a_2>0} = \begin{pmatrix} 0 & 0 & c_1 \\ a_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\delta^{c_1=0,1}_{a_2<0} = \begin{pmatrix} 0 & 0 & c_1 \\ a_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Among them, the two first cobrackets give coboundary Lie bialgebras with $r = y \wedge h$ *and* $r = \frac{1}{2} x \wedge y$*, respectively, which are, moreover, triangular.*

We dedicate the rest of the section to the proof of this result, which proceeds in the cases ordered as in the statement of the theorem. We begin with the general formula for the action of the automorphism group.

Action of the automorphism group. If $\delta' = (\phi \wedge \phi)^{-1} \delta \phi$, $\phi = \phi^{a,b}_{\mu,\nu,\rho,\sigma}$, then δ' equals

$$\begin{pmatrix} \frac{\mu(a_1+aa_2+a_3b)+\sigma(aa_3+b_1+bb_3)}{\mu\nu-\rho\sigma} & \frac{\nu(aa_3+b_1+bb_3)+\rho(a_1+aa_2+a_3b)}{\mu\nu-\rho\sigma} & \frac{a(aa_2+2a_1+2a_3b)+2bb_1+b^2b_3+c_1}{\mu\nu-\rho\sigma} \\ \frac{a_2\mu^2+\sigma(2a_3\mu+b_3\sigma)}{\mu\nu-\rho\sigma} & \frac{a_3\mu\nu+a_2\mu\rho+b_3\nu\sigma+a_3\rho\sigma}{\mu\nu-\rho\sigma} & \frac{\mu(a_1+aa_2+a_3b)+\sigma(aa_3+b_1+bb_3)}{\mu\nu-\rho\sigma} \\ \frac{a_3\mu\nu+a_2\mu\rho+b_3\nu\sigma+a_3\rho\sigma}{\mu\nu-\rho\sigma} & \frac{b_3\nu^2+\rho(2a_3\nu+a_2\rho)}{\mu\nu-\rho\sigma} & \frac{\nu(aa_3+b_1+bb_3)+\rho(a_1+aa_2+a_3b)}{\mu\nu-\rho\sigma} \end{pmatrix}.$$

By inspection, one can see that the submatrix $\begin{pmatrix} a_2 & a_3 \\ a_3 & b_3 \end{pmatrix}$ transforms as

$$\begin{pmatrix} a_2 & a_3 \\ a_3 & b_3 \end{pmatrix} \mapsto \begin{pmatrix} a_2' & a_3' \\ a_3' & b_3' \end{pmatrix} = \frac{1}{\mu\nu - \rho\sigma} \begin{pmatrix} \mu & \sigma \\ \rho & \nu \end{pmatrix} \begin{pmatrix} a_2 & a_3 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} \mu & \sigma \\ \rho & \nu \end{pmatrix}^t.$$

So, up to a scalar factor, it transforms as the matrix of a (symmetric) bilinear form. As a consequence, it can be diagonalized; in other words, we may assume that, up to isomorphism, $a_3 = 0$. So we may take δ with $a_3 = 0$ from the beginning. Let us consider separately $(a_2, b_3) = (0, 0)$ or $(a_2, b_3) \neq (0, 0)$.

Case $(a_2, b_3) = (0, 0)$, namely $\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ 0 & 0 & a_1 \\ 0 & 0 & b_1 \end{pmatrix}$. After applying a general isomorphism it is mapped into

$$\delta' = \frac{1}{\mu\nu - \rho\sigma} \begin{pmatrix} a_1\mu + b_1\sigma & a_1\rho + b_1\nu & 2aa_1 + 2bb_1 + c_1 \\ 0 & 0 & a_1\mu + b_1\sigma \\ 0 & 0 & a_1\rho + b_1\nu \end{pmatrix}.$$

If the pair $(a_1, b_1) \neq (0, 0)$ then there exists a linear transformation $\begin{pmatrix} \mu & \sigma \\ \rho & \nu \end{pmatrix}$ with determinant 1, such that $(a_1\mu + b_1\sigma, a_1\rho + b_1\nu) = (1, 0)$. This says that the cobracket δ belongs to the same isoclass that one with $a_1 = 1$ and $b_1 = 0$. If we make such a choice, namely $\delta = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, then it transforms under a general automorphism into $\delta' = \frac{1}{\mu\nu - \rho\sigma} \begin{pmatrix} \mu & \rho & 2a + c_1 \\ 0 & 0 & \mu \\ 0 & 0 & \rho \end{pmatrix}$. If $b_1' = 0$ we need $\rho = 0$, and so δ' equals $\delta' = \begin{pmatrix} \frac{1}{\nu} & 0 & \frac{2a+c_1}{\mu\nu} \\ 0 & 0 & \frac{1}{\nu} \\ 0 & 0 & 0 \end{pmatrix}$. In order to preserve also $a_1' = 1$ we need $\nu = 1$; then $\delta' = \begin{pmatrix} 1 & 0 & \frac{2a+c_1}{\mu} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ and we may choose $c_1 = 0$. Hence, in this case we get only one representative given by $\delta_{a_2=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Its automorphism group consists of $\left\{ \begin{pmatrix} \mu & 0 & a \\ \sigma & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : \mu, \sigma, a, b \in \mathbb{R}, \mu \neq 0 \right\}$.

If the pair $(a_1, b_1) = (0, 0)$ then

$$\delta = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longmapsto \delta' = \begin{pmatrix} 0 & 0 & \frac{c_1}{\mu\nu - \rho\sigma} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We conclude that there are two representatives for this case, namely

$$\delta^{0,c_1}_{a_2=0} = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c_1 = 0 \text{ or } c_1 = 1.$$

If $c_1 = 0$, the automorphism group is the same as for the Lie algebra. If $c_1 \neq 0$, the automorphism group consists of $\left\{ \begin{pmatrix} \mu & 0 & a \\ \sigma & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : \mu\nu - \rho\sigma = 1 \right\}$.

Case $(a_2, b_3) \neq (0,0)$. Under an automorphism ϕ with $a = b = 0$, $\mu = \nu = 0$, $\sigma = 1$ and $\rho = -1$, the cobracket changes following the rule

$$\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & 0 & a_1 \\ 0 & b_3 & b_1 \end{pmatrix} \mapsto \delta' = \begin{pmatrix} b_1 & -a_1 & c_1 \\ b_3 & 0 & b_1 \\ 0 & a_2 & -a_1 \end{pmatrix}.$$

So we may assume $b_3 \neq 0$. Under an automorphism ϕ with $a = b = 0$, $\nu = 1$, $\sigma = 0 = \rho$, the cobracket changes following the rule

$$\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & 0 & a_1 \\ 0 & b_3 & b_1 \end{pmatrix} \mapsto \delta' = \begin{pmatrix} a_1 & \frac{b_1}{\mu} & \frac{c_1}{\mu} \\ a_2\mu & 0 & a_1 \\ 0 & \frac{b_3}{\mu} & \frac{b_1}{\mu} \end{pmatrix}.$$

So we can set $b_3 = 1$ and start with $\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & 0 & a_1 \\ 0 & 1 & b_1 \end{pmatrix}$. By means of an automorphism $\sigma = \rho = 0$ and $\mu = \nu = 1$,

$$\delta = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & 0 & a_1 \\ 0 & 1 & b_1 \end{pmatrix} \mapsto \delta' = \begin{pmatrix} a_1 + aa_2 & b_1 + b & c_1 + a(2a_1 + aa_2) + b(b + 2b_1) \\ a_2 & 0 & a_1 + aa_2 \\ 0 & 1 & b_1 + b \end{pmatrix}.$$

Then, taking $b = -b_1$, we get $b_1' = 0$, so we may assume $\delta = \begin{pmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & a_1 \\ 0 & 1 & b_1 \end{pmatrix}$.

Case $a_2 = 0$. Let δ be as above but with $a_2 = 0$, under a general automorphism

$$\delta \longmapsto \delta' = \frac{1}{\mu\nu - \rho\sigma} \begin{pmatrix} a_1\mu + b\sigma & b\nu + a_1\rho & 2aa_1 + b^2 + c_1 \\ \sigma^2 & \nu\sigma & a_1\mu + b\sigma \\ \nu\sigma & \nu^2 & b\nu + a_1\rho \end{pmatrix}.$$

In order to get $a_2' = 0$ we need $\sigma = 0$; so

$$\delta' = \frac{1}{\mu\nu} \begin{pmatrix} a_1\mu & b\nu + a_1\rho & 2aa_1 + b^2 + c_1 \\ 0 & 0 & a_1\mu \\ 0 & \nu^2 & b\mu + a_1\rho \end{pmatrix}.$$

To preserve $b_3 = 1$ and $b_1 = 0$, we need, respectively, $\mu = \nu$ and $b = -\frac{a_1\rho}{\nu}$; then $\delta' = \begin{pmatrix} \frac{a_1}{\mu} & 0 & \frac{2aa_1 + \left(\frac{a_1\rho}{\mu}\right)^2 + c_1}{\mu^2} \\ 0 & 0 & \frac{a_1}{\mu} \\ 0 & 1 & 0 \end{pmatrix}$.

We distinguish cases $a_1 \neq 0$ and $a_1 = 0$. If $a_1 \neq 0$, set $\mu = a_1$ to get $a_1' = 1$; then

$$\delta' = \begin{pmatrix} 1 & 0 & (2a + \rho^2 + c_1) \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

then a may be chosen such that $c_1' = 0$, so the isoclass has only one representative $\delta_{a_2=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. The automorphism group consists of

$$\left\{ \begin{pmatrix} 1 & \rho & \frac{-\rho^2}{2} \\ 0 & 1 & -\rho \\ 0 & 0 & 1 \end{pmatrix} : \rho \in \mathbb{R} \right\}.$$

In case $a_1 = 0$ we have $\delta' = \begin{pmatrix} 0 & 0 & c_1/\mu^2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, so c_1 may be chosen up to positive scalar. The automorphism group in case $c_1 = 0$ consists of

$$\left\{ \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mu \neq 0 \right\};$$

while if $c_1 \neq 0$, the automorphism group consists of

$$\left\{ \begin{pmatrix} \mu & \rho & a \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mu = \pm 1 \right\}.$$

Hence, if $a_2 = 0$, $b_3 \neq 0$, the set of isoclasses is

$$\left\{ \delta_{a_2=0} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \ \delta_{a_2=0}^{0,c_1} = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : c_1 = 0, \pm 1 \right\}$$

Case $a_2 \neq 0$. Recall $a_3 = 0$, $b_3 = 1$ and $b_1 = 0$, i.e. $\delta = \begin{pmatrix} a_1 & 0 & c_1 \\ a_2 & 0 & a_1 \\ 0 & 1 & 0 \end{pmatrix}$. Using the action of an automorphism with $\sigma = \rho = 0$, $\mu = \nu = 1$ and $b = 0$:

$$\delta \mapsto \delta' = \begin{pmatrix} a_1 + aa_2 & 0 & 2aa_1 + a^2 a_2 + c_1 \\ a_2 & 0 & a_1 + aa_2 \\ 0 & 1 & 0 \end{pmatrix},$$

so with $a = -a_1/a_2$ we get $a'_1 = 0$, so in this case, since $a_2 \neq 0$, we may assume that $a_1 = 0$, namely $\delta = \begin{pmatrix} 0 & 0 & c_1 \\ a_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Under a general automorphism,

$$\delta \longmapsto \delta' = \frac{1}{\mu\nu - \rho\sigma} \begin{pmatrix} aa_2\mu + b\sigma & b\nu + aa_2\rho & c_1 + a^2 a_2 + b^2 \\ a_2\mu^2 + \sigma^2 & a_2\mu\rho + \nu\sigma & aa_2\mu + b\sigma \\ a_2\mu\rho + \nu\sigma & \nu^2 + \rho^2 a_2 & b\nu + aa_2\rho \end{pmatrix}.$$

Now we look for automorphisms preserving the conditions $a_1 = 0 = b_1 = a_3$ and $b_3 = 1$. From the entries of the matrix we see that we need to solve

$$0 = \mu(aa_2) + \sigma b, \tag{1}$$

$$0 = \rho(aa_2) + \nu b, \tag{2}$$

$$0 = a_2\mu\rho + \nu\sigma, \tag{3}$$

$$1 = \frac{\nu^2 + a_2\rho^2}{\mu\nu - \rho\sigma}. \tag{4}$$

Since the 2×2 matrix with entries μ, σ, ρ, ν is invertible, the first two equations have general solution $aa_2 = b = 0$, so $a = 0 = b$. In order to solve (3) and (4) we consider first the special case $\sigma = 0$. Remember that $\mu\nu - \rho\sigma \neq 0$, so $\sigma = 0$ implies $\mu \neq 0$, and together with (3) we get $\rho = 0$. Now, (4) gives $\nu/\mu = 1$, so $\mu = \nu$.

Going back to the action of the automorphism group, if the parameters are $a = b = 0 = \sigma = \rho$, $\mu = \nu$, we get $\delta' = \begin{pmatrix} 0 & 0 & \frac{c_1}{\mu^2} \\ a_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and, in particular, we see that c_1 may be taken up to positive scalar.

If we try to solve equations (3) and (4) with $\sigma \neq 0$, from (3) we can solve $\nu = -a_2\mu\rho/\sigma$, and (4) gives $-a_2\rho/\sigma = 1$, so $\sigma = -a_2\rho$; hence $\nu = \mu$. Using an automorphism with $\mu = \nu$ and $\sigma = -a_2\rho$, we get

$$\delta' = \begin{pmatrix} 0 & 0 & \frac{c_1}{\mu^2 + a_2\rho^2} \\ a_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Notice that under the assumption $\mu = \nu$ and $\sigma = -a_2\rho$, we have $\mu\nu - \rho\sigma = \mu^2 + a_2\rho^2$, so μ and ρ must be such that this quantity is not zero.

If $a_2 > 0$, c_1 is multiplied by a positive number and we obtain the same possibilities for c_1 as before. If $a_2 < 0$, we can choose μ and ρ such that $\mu^2 + a_2\rho^2$ is any nonzero real number, so c_1 can be chosen up to any nonzero multiple. Hence, the Lie bialgebra isomorphism classes when $a_2 \neq 0$ is

$$\delta^{c_1}_{a_2 \neq 0} = \begin{pmatrix} 0 & 0 & c_1 \\ a_2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : a_2 > 0; c_1 = 0, \pm 1; \text{ or } a_2 < 0; c_1 = 0, 1$$

The automorphism group of Lie bialgebra, when $c_1 = 0$, regardless of the sign of a_2, is given by $\left\{ \phi^{a_2}_{\mu,\rho} = \begin{pmatrix} \mu & \rho & 0 \\ -a_2\rho & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mu^2 + a_2\rho^2 \neq 0 \right\}$; while if $c_1 \neq 0$, it consists of $\left\{ \phi^{a_2}_{\mu,\rho} = \begin{pmatrix} \mu & \rho & 0 \\ -a_2\rho & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mu^2 + a_2\rho^2 = 1 \right\}$.

7. Lie bialgebra structures on $\mathfrak{r}'_{3,\lambda}$

In basis $\{h, x, y\}$, the Lie algebra $\mathfrak{r}'_{3,\lambda}$ has the following brackets: $[h, x] = \lambda x - y$, $[h, y] = x + \lambda y$, $[x, y] = 0$. Remark that, in the basis $\{x, y\}$, the linear transformation ad_h has matrix $\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$. Notice the similarity with the matrix associated to multiplication by the complex number $\lambda - i$. A straightforward computation shows for $\mathfrak{g} = \mathfrak{r}'_{3,\lambda}$:

- Case $\lambda \neq 0$: $(\Lambda^2\mathfrak{g})^{\mathfrak{g}} = 0$ and $(\Lambda^3\mathfrak{g})^{\mathfrak{g}} = 0$.
- Case $\lambda = 0$: $(\Lambda^2\mathfrak{g})^{\mathfrak{g}} = \mathbb{R}x \wedge y$ and $(\Lambda^3\mathfrak{g})^{\mathfrak{g}} = \mathbb{R}x \wedge y \wedge h$.

1-coboundaries: For $r = ax\wedge y + by\wedge h + ch\wedge x$, $\partial r = \begin{pmatrix} \lambda b - c & b + \lambda c & 2\lambda a \\ 0 & 0 & \lambda b + c \\ 0 & 0 & \lambda c - b \end{pmatrix}$, $a, b, c \in \mathbb{R}$, $\mathrm{CYB}(r) = -4(b^2 + c^2)x \wedge y \wedge h$ is $\mathfrak{g}$-invariant if and only if

- $\lambda \neq 0$, $b = c = 0$, so $r = ax \wedge y$; or
- $\lambda = 0$, $r = ax \wedge y + by \wedge h + ch \wedge x$, for any $a, b, c \in \mathbb{R}$; if $b = c = 0$, $\partial r = 0$.

1-cocycle condition. Let $\delta : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ be a 1-cocycle, from $0 = \delta[x, y] = [\delta x, y] + [x, \delta y]$ we get

$$\begin{aligned} 0 &= [a_1 x \wedge y + a_2 y \wedge h + a_3 h \wedge x, y] + [x, b_1 x \wedge y + b_2 y \wedge h + b_3 h \wedge x] \\ &= (a_2 + a_3\lambda - b_2\lambda + b_3) y \wedge x; \end{aligned}$$

hence, $a_2 + a_3\lambda - \lambda b_2 + b_3 = 0$. The 1-cocycle condition for $[h, x] = \lambda x - y$ gives

$$\begin{aligned} \lambda\delta x - \delta y &= [\delta h, x] + [h, \delta x] \\ &= [c_1 x \wedge y + c_2 y \wedge h + c_3 h \wedge x, x] + [h, a_1 x \wedge y + a_2 y \wedge h + a_3 h \wedge x] \\ &= (-\lambda c_2 + c_3 + 2\lambda a_1) x \wedge y + (a_2\lambda + a_3) y \wedge h + (-a_2 + \lambda a_3) h \wedge x). \end{aligned}$$

So, $\lambda a_1 - b_1 = -\lambda c_2 + c_3 + 2\lambda a_1,\ \lambda a_2 - b_2 = a_2\lambda + a_3,\ \lambda a_3 - b_3 = -a_2 + \lambda a_3$; then $\lambda a_1 + b_1 = \lambda c_2 - c_3$, $-b_2 = a_3$, $b_3 = a_2$. Similarly, $[h, y] = x + \lambda y$ gives

$$\begin{aligned} \delta x + \lambda\delta y &= [\delta h, y] + [h, \delta y] \\ &= (c_2 + c_3\lambda) y \wedge x + 2\lambda b_1 x \wedge y + (b_2\lambda + b_3) y \wedge h + (-b_2 + \lambda b_3) h \wedge x, \end{aligned}$$

then $a_1 + \lambda b_1 = -c_2 - \lambda c_3 + 2\lambda b_1$, $a_2 + \lambda b_2 = \lambda b_2 + b_3$, $a_3 + \lambda b_3 = -b_2 + \lambda b_3$. Summarizing, we have

$$a_2 = b_3, \quad a_3 = -b_2,$$
$$\lambda b_2 - b_3 = \lambda a_3 + a_2, \quad \lambda b_1 - a_1 = c_2 + \lambda c_3, \quad b_1 + \lambda a_1 = \lambda c_2 - c_3.$$

The last two equations are equivalent to $c_2 = \frac{a_1(\lambda^2-1)+2\lambda b_1}{1+\lambda^2}$, $c_3 = \frac{b_1(\lambda^2-1)-2\lambda a_1}{1+\lambda^2}$, while the first three ones are equivalent to $a_2 = -\lambda a_3$, $b_2 = -a_3$, $b_3 = -\lambda a_3$. Hence, the general 1-cocycle, in basis $\{x, y, h\}$, $\{x \wedge y, y \wedge h, h \wedge x\}$, is given by

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ -\lambda a_3 & -a_3 & \frac{a_1(\lambda^2-1)+2\lambda b_1}{1+\lambda^2} \\ a_3 & -\lambda a_3 & \frac{b_1(\lambda^2-1)-2\lambda a_1}{1+\lambda^2} \end{pmatrix}.$$

For a 1-cocycle, the co-Jacobi condition is $2\frac{(a_1^2+b_1^2)\lambda+a_3c_1(1+\lambda^2)}{1+\lambda^2} = 0$.

Proposition 7.1. *The automorphism group of the Lie algebra* $\mathfrak{r}'_{3,\lambda}$*, expressed as matrices in basis* $\{x, y, h\}$*, is the following subgroup of* $\mathrm{GL}(3, \mathbb{R})$*:*

$$\left\{ \begin{pmatrix} \mu & -\sigma & a \\ \sigma & \mu & b \\ 0 & 0 & 1 \end{pmatrix} : \mu, \sigma, a, b \in \mathbb{R},\ \mu^2 + \sigma^2 \neq 0 \right\}.$$

Proof. Using that $[\mathfrak{g}, \mathfrak{g}]$ is generated by x and y and it is invariant under automorphism, we conclude that any automorphism ϕ restricted to $[\mathfrak{g}, \mathfrak{g}]$ must be of the form $\phi(x) = \mu x + \rho y$ and $\phi y = \sigma x + \nu y$, with $\mu\nu - \rho\sigma \neq 0$. Also, writing $\phi(h) = ax + by + ch$, since ϕ is an automorphism of the Lie algebra, we have

$$c[h, \phi x] = [\phi h, \phi x] = \lambda\phi x - \phi y, \quad c[h, \phi y] = [\phi h, \phi y] = \phi x + \lambda\phi y;$$

in matrix notation, $c\begin{pmatrix} \mu & \sigma \\ \rho & \nu \end{pmatrix}\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}\begin{pmatrix} \mu & \sigma \\ \rho & \nu \end{pmatrix}$. Taking determinant we get $c = 1$, and if $\begin{pmatrix} \mu & \sigma \\ \rho & \nu \end{pmatrix}$ commutes with $\begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix}$ then it must be of the form $\begin{pmatrix} \mu & -\sigma \\ \sigma & \mu \end{pmatrix}$. □

Action of the automorphism group on 1-cocycles. The effect of an arbitrary automorphism on a general 1-cocycle is the following

$$\delta' = \begin{pmatrix} \frac{a_1\mu+b_1\sigma-a_3(\mu(a\lambda-b)+\sigma(a+b\lambda))}{\mu^2+\sigma^2} & \frac{b_1\mu-a_1\sigma-a_3(b(\lambda\mu+\sigma)+a(\mu-\lambda\sigma))}{\mu^2+\sigma^2} & c'_1 \\ -a_3\lambda & -a_3 & * \\ a_3 & -a_3\lambda & * \end{pmatrix},$$

with

$$c'_1 = \frac{c_1(1+\lambda^2) - \lambda(-2a(b_1+a_1\lambda) + a^2a_3(1+\lambda^2) + b(2a_1 - 2b_1\lambda + a_3b(1+\lambda^2)))}{(\mu^2+\sigma^2)(1+\lambda^2)}.$$

Recall that the co-Jacobi condition reads $2\frac{(a_1^2+b_1^2)\lambda+a_3c_1(1+\lambda^2)}{1+\lambda^2} = 0$, or, equivalently, $(a_1^2+b_1^2)\lambda + a_3c_1(1+\lambda^2) = 0$. We will make simplifications using the automorphism group.

Case $a_3 \neq 0$. If we choose an automorphism with $\mu = 1$, $\sigma = 0$, then δ' has $a'_1 = a_1 + a_3(b - \lambda a)$ and $b'_1 = b_1 - a_3(\lambda b + a)$, so we can choose a and b such that

$a_1' = 0 = b_1'$; hence, we may suppose from the beginning that $a_1 = 0$ and $b_1 = 0$. But now $a_1 = b_1 = 0$ together with co-Jacobi imply $a_3c_1(1+\lambda^2) = 0$, so $a_3c_1 = 0$; since $a_3 \neq 0$, $c_1 = 0$, hence

$$\delta_{a_3,\lambda} = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda a_3 & -a_3 & 0 \\ a_3 & -\lambda a_3 & 0 \end{pmatrix}.$$

To compute the automorphism group, notice that $\phi_{\mu,\nu,a,b}$ transforms $\delta_{a_1=b_1=c_1=0}$ into δ' with $c_1' = -\lambda a_3 \frac{a^2+b^2}{\mu^2+\nu^2}$, so in case $\lambda \neq 0$ the only possibility to preserve $c_1 = 0$ is $a = b = 0$. Hence, the automorphism group in case $a_3 \neq 0$, $\lambda \neq 0$ is

$$\left\{ \begin{pmatrix} \mu & -\sigma & 0 \\ \sigma & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} : \mu, \sigma \in \mathbb{R},\ \mu^2 + \sigma^2 \neq 0 \right\}.$$

But if $\lambda = 0$, $\delta_{a_1=b_1=c_1=0}$ transforms by $\phi_{\mu,\nu,a,b}$ into

$$\delta' = \begin{pmatrix} a_3 \frac{b\mu - a\sigma}{\mu^2+\sigma^2} & -a_3 \frac{b\sigma + a\mu}{\mu^2+\sigma^2} & 0 \\ 0 & -a_3 & -a_3 \frac{b\mu - a\sigma}{\mu^2+\sigma^2} \\ a_3 & 0 & a_3 \frac{b\sigma + a\mu}{\mu^2+\sigma^2} \end{pmatrix}.$$

Since $\begin{pmatrix} \mu & -\sigma \\ \sigma & \mu \end{pmatrix}$ is invertible, the only way to preserve $a_1 = b_1 = 0$ is with $a = b = 0$. Hence, the automorphism group in case $a_3 \neq 0$, $\lambda = 0$ is the same as in case $\lambda \neq 0$.

Case $a_3 = 0$. If $\lambda \neq 0$, co-Jacobi implies $a_1 = b_1 = 0$; conjugation by $\phi_{\mu,\nu,a,b}$ gives

$$\delta = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \delta' = \begin{pmatrix} 0 & 0 & \frac{c_1}{\mu^2+\sigma^2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

so c_1 can be chosen up to positive scalar. We may take $0, \pm 1$ as representatives. In case $a_3 = 0$ but $\lambda = 0$, co-Jacobi identity gives no further information. We study the action of the automorphism group in this case. $\delta_{a_3=0,\lambda=0}$ transforms by $\phi_{\mu,\nu,a,b}$ into

$$\delta' = \begin{pmatrix} \frac{a_1\mu + b_1\sigma}{\mu^2+\sigma^2} & \frac{-a_1\sigma + b_1\mu}{\mu^2+\sigma^2} & \frac{c_1}{\mu^2+\sigma^2} \\ 0 & 0 & -\frac{a_1\mu + b_1\sigma}{\mu^2+\sigma^2} \\ 0 & 0 & \frac{a_1\sigma - b_1\mu}{\mu^2+\sigma^2} \end{pmatrix}.$$

The pair (a_1, b_1) transforms as $a_1 + ib_1 \mapsto \frac{a_1 + ib_1}{\mu + i\sigma}$ in the complex plane. We know that there are two orbits: $(a_1, b_1) = (0, 0)$, which has trivial action and gives the same cobrackets as for $\lambda \neq 0$, and $\{(a_1, b_1) \neq (0, 0)\}$, which has free $\mathbb{C}^*$-action. For the second case, one can take $(a_1, b_1) = (1, 0)$ as a representative. Hence, a set of representatives of Lie cobrackets in case $a_3 = 0$, $\lambda = 0$ follows:

$$\delta = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c_1 = 0, \pm 1, \quad \delta_{c_1}^{(1,0)} = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} : c_1 \in \mathbb{R}.$$

The Lie bialgebra automorphism group with $\delta_{c_1}^{(1,0)}$ is $\left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b \in \mathbb{R} \right\}$.

This discussion leads to the following result.

Theorem 7.2. *The set of isomorphism classes of Lie bialgebra with underlying Lie algebra $\mathfrak{r}'_{3,\lambda}$ in the case $\lambda \neq 0$ is given by the following list of cobrackets:*

$$\delta_{a_3,\lambda} = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda a_3 & -a_3 & 0 \\ a_3 & -\lambda a_3 & 0 \end{pmatrix} : a_3 \in \mathbb{R} \quad \text{and} \quad \delta_{c_1}^{(0,0)} = \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : c_1 = \pm 1.$$

The Lie bialgebra with $\delta_{c_1}^{(0,0)}$ is a coboundary with $r = \frac{c_1}{2\lambda} x \wedge y$, and it is triangular. In case $\lambda = 0$, we have the previous set specialized in $\lambda = 0$, together with the following 1-parameter family $\delta_{c_1}^{(1,0)} = \begin{pmatrix} 1 & 0 & c_1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} : c_1 \in \mathbb{R}$. In this case, only $\delta_{c_1=0}^{(1,0)}$ is a coboundary, with $r = -h \wedge x$, but it is not a triangular structure.

8. Lie bialgebra structures on $\mathfrak{su}(2)$

1-cocycles. Consider $\mathfrak{su}(2)$ as the $\mathbb{R}$-span of the following matrices:

$$u = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad v = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad w = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then the Lie brackets verify $[u, v] = w$, $[v, w] = u$, $[w, u] = v$. This is a simple Lie algebra, then every 1-cocycle is a 1-coboundary. If $r = \alpha u \wedge v + \beta v \wedge w + \gamma w \wedge u \in \Lambda^2 \mathfrak{su}(2)$, with α, β, $\gamma \in \mathbb{R}$, the 1-cocycle associated to it is $\delta(x) = \mathrm{ad}_x(r) = [x, r]$ for any $x \in \mathfrak{su}(2)$. The co-Jacobi condition for δ is equivalent to $[r, r] \in (\Lambda^3 \mathfrak{g})^{\mathfrak{g}}$ with $[r, r] = 2(\alpha^2 + \beta^2 + \gamma^2) u \wedge w \wedge v$, so it is satisfied for any r since $(\Lambda^3 \mathfrak{g})^{\mathfrak{g}} = \Lambda^3 \mathfrak{g}$ for $\mathfrak{g} = \mathfrak{su}(2)$. We get

$$\begin{aligned} \delta(u) &= \gamma u \wedge v - \alpha w \wedge u, \\ \delta(v) &= -\beta u \wedge v + \alpha v \wedge w, \\ \delta(w) &= -\gamma v \wedge w + \beta w \wedge u, \end{aligned}$$

or, in matrix notation, $\delta = \begin{pmatrix} \gamma & -\beta & 0 \\ 0 & \alpha & -\gamma \\ -\alpha & 0 & \beta \end{pmatrix}$.

Automorphisms and isomorphism classes. If $U \in \mathrm{SU}(2)$, then conjugation by U gives an automorphism of the Lie algebra $\mathfrak{su}(2)$. If we parametrize such a matrix by $U = \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$, with $a,b,c,d \in \mathbb{R}$, $a^2+b^2+c^2+d^2=1$, we have the automorphism $\phi_U(M) = UMU^{-1}$, where $M \in \mathfrak{su}(2)$. Straightforward computation shows the following:

$$\begin{aligned} \phi_U(u) &= (a^2+b^2-c^2-d^2)u + 2(-ac+bd)v - 2(bc+ad)w, \\ \phi_U(v) &= 2(ac+bd)u + (a^2-b^2-c^2+d^2)v + 2(ab-cd)w, \\ \phi_U(w) &= -2(bc-ad)u - 2(ab+cd)v + (a^2-b^2+c^2-d^2)w, \end{aligned}$$

and $\delta' := (\phi\wedge\phi)^{-1}\delta\phi$ is given by $\frac{1}{2}\delta' =$

$$\left(\begin{array}{c|c|c} \alpha(ab-cd)+\beta(ac+bd) & \alpha(ad+bc)+\gamma(ac-bd) & 0 \\ +\frac{1}{2}\gamma(a^2-b^2-c^2+d^2) & +\frac{1}{2}\beta(-a^2-b^2+c^2+d^2) & \\ \hline 0 & \frac{1}{2}\alpha(a^2-b^2+c^2-d^2) & \alpha(cd-ab)-\beta(bd+ac) \\ & \beta(ad-bc)-\gamma(ab+cd) & +\frac{1}{2}\gamma(-a^2+b^2+c^2-d^2) \\ \hline \frac{1}{2}\alpha(-a^2+b^2-c^2+d^2) & 0 & -\alpha(ad+bc)+\gamma(bd-ac) \\ +\beta(bc-ad)+\gamma(ab+cd) & & +\frac{1}{2}\beta(a^2+b^2-c^2-d^2) \end{array}\right).$$

Suppose $\gamma \neq 0$, and take $b=d=0$; then

$$\delta' = \begin{pmatrix} \gamma(a^2-c^2)+2\beta ac & 2\gamma ac+\beta(c^2-a^2) & 0 \\ 0 & \alpha(a^2+c^2) & -2\beta ac+\gamma(c^2-a^2) \\ -\alpha(a^2+c^2) & 0 & -2\gamma ac+\beta(a^2-c^2) \end{pmatrix}.$$

So, if $c = a\left(\frac{\beta+\sqrt{\beta^2+\gamma^2}}{\gamma}\right)$ then $\gamma'=0$. Assuming $\gamma=0$ and letting $a=d=0$, we get

$$\delta' = \begin{pmatrix} 0 & 2\alpha bc+\beta(-b^2+c^2) & 0 \\ 0 & -\alpha(b^2-c^2)-2\beta bc & 0 \\ \alpha(b^2-c^2)+2\beta bc & 0 & -2\alpha bc+\beta(b^2-c^2) \end{pmatrix}.$$

So if $\beta \neq 0$, we can use ϕ_U with $a=d=0$, $c = b\left(\frac{-\alpha+\sqrt{\alpha^2+\beta^2}}{\beta}\right)$, then δ' has $\gamma'=0$ and $\beta'=0$. So we may assume from the beginning $\gamma=\beta=0$. We get

$$\delta' = \alpha\begin{pmatrix} 2(ab-cd) & 2(ad+bc) & 0 \\ 0 & (a^2-b^2+c^2-d^2) & 2(cd-ab) \\ (-a^2+b^2-c^2+d^2) & 0 & -2(ad+bc) \end{pmatrix}.$$

In case $\alpha = 0$, we have the trivial cobracket. If $\alpha \neq 0$, the condition on the automorphism preserving the condition $\beta = \gamma = 0$ is given by the equations:

$$ab = cd, \quad ad = -bc.$$

These equations imply $a^2 b = -bc^2$, so if $b \neq 0$, then $a = c = 0$ and δ' is given by

$$\delta' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha(b^2+d^2) & 0 \\ \alpha(b^2+d^2) & 0 & 0 \end{pmatrix},$$

but $b^2 + d^2 = 1$, so $\delta_\alpha \cong \delta_{-\alpha}$. Next we consider the case $b = 0$. We have $cd = 0 = ad$. If $d = 0$,

$$\delta' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha(a^2+c^2) & 0 \\ -\alpha(a^2+c^2) & 0 & 0 \end{pmatrix} = \delta.$$

If $d \neq 0$ then $a = c = 0$, $d = \pm 1$ and $\delta' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\alpha d^2 & 0 \\ \alpha d^2 & 0 & 0 \end{pmatrix} = -\delta.$

This discussion leads to the following result.

Theorem 8.1. *The set of isomorphism classes of Lie bialgebras with underlying Lie algebra $\mathfrak{su}(2)$ is given by the 1-parameter family*

$$\delta_\alpha = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ -\alpha & 0 & 0 \end{pmatrix} : \alpha \geq 0.$$

For each $\alpha \neq 0$, the automorphism group is

$$\left\{ \phi_U = \begin{pmatrix} a^2-c^2 & -2ac & 0 \\ 2ac & a^2-c^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a, c \in \mathbb{R},\ a^2+c^2 = 1 \right\} \cong S^1.$$

Remark 8.2. For any $r = \alpha u \wedge v + \beta v \wedge w + \gamma w \wedge u \in \Lambda^2 \mathfrak{su}(2)$,

$$[r, r] = 2(\alpha^2 + \beta^2 + \gamma^2) u \wedge w \wedge v \in (\Lambda^3 \mathfrak{su}(2))^{\mathfrak{su}(2)}.$$

We have the following two possibilities:

- $[r, r] = 0$, this happens if and only if $r = 0$, and so if and only if $\delta = 0$. Hence, the only triangular structure is the trivial one.
- $0 \neq [r, r]$, and so $(\mathfrak{su}(2), \delta)$, with $\delta(-) = \mathrm{ad}_{(-)}(r)$, is almost factorizable (i.e. it is not factorizable, but its complexification is).

It was known (see [1]) that $\mathfrak{su}(2)$ admites a unique almost factorizable structure up to scalar multiple. This is in perfect agreement with the results of this section.

9. Lie bialgebra structures on $\mathfrak{sl}(2,\mathbb{R})$

1-Cocycles in $\mathfrak{sl}(2,\mathbb{R})$. The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is usually presented as generated by $\{x,y,h\}$ with brackets $[h,x]=2x$, $[h,y]=-2y$, $[x,y]=h$. But it is convenient to consider instead the ordered basis $\{u,v,w\}$ of $\mathfrak{sl}(2,\mathbb{R})$ and $\{u\wedge v, v\wedge w, w\wedge u\}$ of $\Lambda^2\mathfrak{sl}(2,\mathbb{R})$, where $u=h/2$, $v=(x+y)/2$ and $w=(x-y)/2$:

$$u=\frac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad v=\frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad w=\frac{1}{2}\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

In this basis, the brackets are given by $[u,v]=w$, $[v,w]=-u$, $[w,u]=-v$. As in the case $\mathfrak{su}(2)$, the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is simple, then every 1-cocycle is a 1-coboundary and the general considerations made for that case hold here. If $r=\alpha u\wedge v+\beta v\wedge w+\gamma w\wedge u$, the 1-cocycle associated to it is $\delta(z)=\mathrm{ad}_z(r)=[z,r]$ for any $z\in\mathfrak{sl}(2,\mathbb{R})$. Hence

$$\delta(u)=-\gamma u\wedge v-\alpha w\wedge u,\quad \delta(v)=\beta u\wedge v+\alpha v\wedge w,\quad \delta(w)=\gamma v\wedge w-\beta w\wedge u;$$

in matrix form, $\delta=\begin{pmatrix}-\gamma&\beta&0\\0&\alpha&\gamma\\-\alpha&0&-\beta\end{pmatrix}$. Co-Jacobi is automatically satisfied.

Automorphisms. In a similar way to the $\mathfrak{su}(2)$ case, the automorphisms of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ consist of conjugations by elements $S\in\mathrm{SL}(2,\mathbb{R})$. If $S=\begin{pmatrix}a&b\\c&d\end{pmatrix}$, with $a,b,c,d\in\mathbb{R}$, $ad-bc=1$, then the automorphism ϕ_S given by $\phi_S(M)=SMS^{-1}$ for any $M\in\mathfrak{sl}(2)$; ϕ_S maps u,v,w to

$$\begin{aligned}\phi_S(u)&=(ad+bc)u+(cd-ab)v-(cd+ab)w\\ \phi_S(v)&=(bd-ac)u+\frac{a^2-b^2-c^2+d^2}{2}v+\frac{a^2-b^2+c^2-d^2}{2}w\\ \phi_S(w)&=-(bd+ac)u+\frac{a^2+b^2-c^2-d^2}{2}v+\frac{a^2+b^2+c^2+d^2}{2}w.\end{aligned}$$

In matrix notation,

$$\phi_S=\begin{pmatrix}ad+bc&bd-ac&-(bd+ac)\\ cd-ab&\frac{a^2-b^2-c^2+d^2}{2}&\frac{a^2+b^2-c^2-d^2}{2}\\ -(cd+ab)&\frac{a^2-b^2+c^2-d^2}{2}&\frac{a^2+b^2+c^2+d^2}{2}\end{pmatrix}$$

Let us denote by $\kappa=a^2+b^2+c^2+d^2$, $\kappa_{1,3}=-a^2+b^2-c^2+d^2$, $\kappa_{3,4}=a^2+b^2-c^2-d^2$, $\kappa_{1,4}=-a^2+b^2+c^2-d^2$, etc.; i.e., the subindices point out the places of the negative signs. If $\delta':=(\phi\wedge\phi)^{-1}\delta\phi$ then it is given by

$$\frac{1}{2}\begin{pmatrix}\alpha\kappa_{1,3}+2\beta(ac-bd)+\gamma\kappa_{1,4}&2(-\alpha(ab+cd)+\beta(bc+ad)+\gamma(-ab+cd))&0\\ 0&\alpha\kappa-2\beta(ac+bd)+\gamma\kappa_{3,4}&-\alpha\kappa_{1,3}-2\beta(ac-bd)-\gamma\kappa_{1,4}\\ -\alpha\kappa+2\beta(ac+bd)-\gamma\kappa_{3,4}&0&2(\alpha(ab+cd)-\beta(ad+bc)+\gamma(ab-cd))\end{pmatrix}$$

If $a = d$ and $c = -b$, with $ad - bc = a^2 + b^2 = 1$, we get

$$\delta' = \begin{pmatrix} -2\beta ab - \gamma(a^2 - b^2) & \beta(a^2 - b^2) - 2\gamma ab & 0 \\ 0 & \alpha & 2\beta ab + \gamma(a^2 - b^2) \\ -\alpha & 0 & -\beta(a^2 - b^2) + 2\gamma ab \end{pmatrix}.$$

Since $a^2 + b^2 = 1$, we can write $a = \cos(\theta)$ and $b = \sin(\theta)$ for some $\theta \in \mathbb{R}$, so

$$\beta' = \beta(a^2 - b^2) - 2\gamma ab = \beta\cos(2\theta) - \gamma\sin(2\theta)$$
$$\gamma' = 2\beta ab + \gamma(a^2 - b^2) = \beta\sin(2\theta) + \gamma\cos(2\theta).$$

Namely, the pair (β, γ) transform as a rotation, so we can change it, for example, into $(\sqrt{\beta^2 + \gamma^2}, 0)$. In other words, we can assume that $\gamma = 0$ and $\beta \geq 0$.

Now if $\delta = \begin{pmatrix} 0 & \beta & 0 \\ 0 & \alpha & 0 \\ -\alpha & 0 & -\beta \end{pmatrix}$ with $\beta \geq 0$, we can take an automorphism $\phi_{a,b,c,d}$ with $d = a$, $b = c$ and $ad - bc = a^2 - b^2 = 1$. Such a, b may be written in the form $a = \cosh\theta$, $b = \sinh\theta$, for some $\theta \in \mathbb{R}$. Under such $\phi_{a,b,c,d}$, δ changes into

$$\delta' = \begin{pmatrix} 0 & \beta(a^2 + b^2) - 2\alpha ab & 0 \\ 0 & -2\beta ab + \alpha(a^2 + b^2) & 0 \\ 2\beta ab - \alpha(a^2 + b^2) & 0 & -\beta(a^2 + b^2) + 2\alpha ab \end{pmatrix}.$$

This says, for the coefficients of δ', that

$$\gamma' = 0$$
$$\beta' = \beta(a^2 + b^2) - 2\alpha ab = \beta\cosh(2\theta) - \alpha\sinh(2\theta)$$
$$\alpha' = -2\beta ab + \alpha(a^2 + b^2) = -\beta\sinh(2\theta) + \alpha\cosh(2\theta).$$

There are three possibilities:

(1) $\beta^2 - \alpha^2 > 0$. In this case, we can choose θ such that $\alpha' = 0$.
(2) $\beta^2 - \alpha^2 < 0$. In this case, we can choose θ such that $\beta' = 0$.
(3) $\beta^2 - \alpha^2 = 0$. In this case $\alpha = \pm\beta$. But the automorphism with $a = d = 0$, $b = 1 = -c$ changes $\alpha' = \alpha$ and $\beta' = -\beta$. So, we can assume $\beta = \alpha$.

Case $\alpha = 0$, $\beta \neq 0$. In this situation, under a general automorphism, δ changes into

$$\delta' = \begin{pmatrix} \beta(ac - bd) & \beta(ad + bc) & 0 \\ 0 & -\beta(ac + bd) & -\beta(ac - bd) \\ \beta(ac + bd) & 0 & -\beta(ad + bc) \end{pmatrix}.$$

An automorphism preserving $\alpha' = 0 = \gamma'$ must satisfy $ac = 0 = bd$. If $a \neq 0$ then $c = 0$; but $ad - bc = 1$, so $d \neq 0$, $b = 0$ and $d = a^{-1}$; in this case $\delta' = \delta$.

If $a = 0$, then $b \neq 0$ since $ad - bc = 1$; so $d = 0$ and $c = -b^{-1}$. In this case $\delta' = -\delta$. We conclude that β may be chosen up to a sign, so a list of representatives of the isomorphism classes is given by

$$\left\{\delta_\beta=\begin{pmatrix}0&\beta&0\\0&0&0\\0&0&-\beta\end{pmatrix}:\beta>0\right\}.$$

For each of these, the automorphism group is $\left\{\phi_S : S=\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix};0\neq a\in\mathbb{R}\right\}$. Notice that $\phi_S=\phi_{-S}$, so this group is connected, and it is isomorphic to $(\mathbb{R},+)$.

Case $\alpha\neq 0,\ \beta=0$. Under an automorphism ϕ_S, $\delta=\begin{pmatrix}0&0&0\\0&\alpha&0\\-\alpha&0&0\end{pmatrix}$ changes into

$$\delta'=\frac{\alpha}{2}\begin{pmatrix}-a^2+b^2-c^2+d^2&-2(ab+cd)&0\\0&a^2+b^2+c^2+d^2&a^2-b^2+c^2-d^2)\\-(a^2+b^2-c^2+d^2)&0&2(ab+cd)\end{pmatrix}.$$

If $\beta'=\gamma'=0$ then

$$\begin{aligned}-a^2+b^2-c^2+d^2&=0\\ab+cd&=0\\ad-bc&=1.\end{aligned}\tag{5}$$

If $c=0$, then $-a^2+b^2+d^2=0$, $ab=0$, $ad=1$ then $a\neq 0$, $b=0$, $d=a^{-1}$, so $a^4=1$, hence $a=\pm1$. In both cases, the automorphism acts trivially.

If $c\neq 0$ we solve $d=-ab/c$ from (5) and the other equations transform into

$$\frac{(a^2+c^2)(c^2-b^2)}{c^2}=0,\quad -b\frac{a^2+c^2}{c}=1.$$

Since $c\neq 0$, $c^2=b^2$ then $b=\pm c$. But if $b=c$, the last equation gives $-(a^2+c^2)=1$, which is absurd. Hence $b=-c$, $a^2+c^2=1$, $d=a$ and $\delta'=\delta$. The set of isomorphism classes of this type is parametrized by

$$\left\{\delta_\alpha=\alpha\begin{pmatrix}0&0&0\\0&1&0\\-1&0&0\end{pmatrix}:\alpha\in\mathbb{R},\ \alpha\neq 0\right\}.$$

For each of these classes, the automorphism group consists of

$$\left\{\phi_S : S=\begin{pmatrix}a&-c\\c&a\end{pmatrix},a^2+c^2=1\right\}=\left\{\begin{pmatrix}a^2-c^2&-2ac&0\\2ac&a^2-c^2&0\\0&0&1\end{pmatrix}:a^2+c^2=1\right\}.$$

Case $\alpha=\beta$: If $\delta=\begin{pmatrix}0&\alpha&0\\0&\alpha&0\\-\alpha&0&-\alpha\end{pmatrix}$ then under an isomorphism ϕ_S we get

$$\delta'=\alpha\begin{pmatrix}\frac{-a^2+b^2+2ac-c^2-2bd+d^2}{2}&-2(a-c)(b-d)&0\\0&\frac{a^2+b^2-2ac+c^2-2bd+d^2}{2}&\frac{a^2-b^2-2ac+c^2+2bd-d^2}{2}\\\frac{-a^2-b^2+2ac-c^2+2bd-d^2}{2}&0&2(a-c)(b-d)\end{pmatrix}.$$

In order to preserve $\alpha'=\beta'$, $\gamma'=0$ we have two possibilities: $\alpha=0$, or

$$-a^2+b^2+2ac-c^2-2bd+d^2=0,$$
$$a^2+b^2-2ac+c^2-2bd+d^2=-2(a-c)(b-d),$$

which are equivalent to $(b-d)^2=-(a-c)(b-d)$ and $(a-c)^2=(c-a)(b-d)$, then $c-a=b-d$, so, setting $d=a-c+b$, we get $\delta'=\alpha(a-c)^2\begin{pmatrix}0&1&0\\0&1&0\\-1&0&-1\end{pmatrix}$.

The condition on a,b,c,d is $ad-bc=1$, then

$$1=ad-bc \qquad =a(a-c+b)-bc$$
$$=a(a+b)-c(a+b)=(a+b)(a-c).$$

So, there is no restriction on $a-c$, except being different from zero. In its isomorphism class, α is determined up to positive scalar and it is enough to take $\alpha=\pm1$. Hence, the isoclasses of triangular Lie bialgebras are

$$\delta_0=0,\quad \delta_1=\begin{pmatrix}0&1&0\\0&1&0\\-1&0&-1\end{pmatrix},\quad \delta_{-1}=\begin{pmatrix}0&-1&0\\0&-1&0\\1&0&1\end{pmatrix}.$$

This discussion leads to the following result.

Theorem 9.1. *The set of isomorphism classes of Lie bialgebras on $\mathfrak{sl}(2,\mathbb{R})$ is given by the following representatives:*

$$\textit{Factorizable: } \delta_\beta=\begin{pmatrix}0&\beta&0\\0&0&0\\0&0&-\beta\end{pmatrix},\ \beta>0;$$

$$\textit{Almost factorizable: } \delta_\alpha=\begin{pmatrix}0&0&0\\0&\alpha&0\\-\alpha&0&0\end{pmatrix},\ \alpha\neq0;$$

$$\textit{Triangular: } \delta=0,\quad \delta_1=\begin{pmatrix}0&1&0\\0&1&0\\-1&0&-1\end{pmatrix},\quad \delta_{-1}=\begin{pmatrix}0&-1&0\\0&-1&0\\1&0&1\end{pmatrix}.$$

Besides, the Lie bialgebra corresponding to δ_β verifies $\mathfrak{g}\cong\mathfrak{g}^{\rm cop}$. On the other hand, although $\delta_{-1}=-\delta_1$, the corresponding Lie bialgebras are non isomorphic.

Remark 9.2. For any $r = \alpha u \wedge v + \beta v \wedge w + \gamma w \wedge u \in \Lambda^2\mathfrak{sl}(2,\mathbb{R})$,

$$[r,r] = 2(\alpha^2 - \beta^2 - \gamma^2) u \wedge v \wedge w \in \Lambda^3\mathfrak{sl}(2,\mathbb{R}) = (\Lambda^3\mathfrak{sl}(2,\mathbb{R}))^{\mathfrak{sl}(2,\mathbb{R})}.$$

We have the following possibilities:

- $[r,r] = 0$ if and only if $\alpha^2 - \beta^2 - \gamma^2 = 0$. So, unlike the $\mathfrak{su}(2)$ case, there are non trivial triangular structures, explicitly given by $\delta_{\pm 1}$ with $\alpha = \beta = \pm 1$.
- $0 \neq [r,r]$. The case $(\mathfrak{sl}(2,\mathbb{R}), \delta_\beta)$ is factorizable, with $\delta_\beta(-) = \mathrm{ad}_{(-)}(r)$, $r = \beta v \wedge w$, since $r \pm \beta\Omega$ satisfies CYBE. In fact, consider $\Omega = u \otimes u + v \otimes v - w \otimes w$ the Casimir element of the adjoint representation, then $\mathrm{CYB}(\Omega) = u \wedge v \wedge w$ is a generator of the 1-dimensional space $\left(\Lambda^3_{\mathbb{R}}\mathfrak{sl}(2,\mathbb{R})\right)^{\mathfrak{sl}(2,\mathbb{R})}$ and

  $$\mathrm{CYB}(r \pm \beta\Omega) = \mathrm{CYB}(r) + \beta^2\mathrm{CYB}(\Omega) = (-\beta^2 + \beta^2) u \wedge v \wedge w = 0.$$

  Notice that $\partial r = \partial(r \pm \beta\Omega)$ since Ω is invariant. Analogously, the case $(\mathfrak{sl}(2,\mathbb{R}), \delta_\alpha)$ is only *almost factorizable*, with $\delta_\alpha(-) = \mathrm{ad}_{(-)}(r)$, $r = \alpha u \wedge v$, since $r \pm i\alpha\Omega$ satisfies CYBE, and this element does not belong to $\mathfrak{sl}(2,\mathbb{R}) \otimes \mathfrak{sl}(2,\mathbb{R})$ but only to its complexification. In fact,

  $$\mathrm{CYB}(r \pm i\alpha\Omega) = \mathrm{CYB}(r) + i^2\alpha^2\mathrm{CYB}(\Omega) = (\alpha^2 - \alpha^2) u \wedge v \wedge w = 0.$$

The factorizable and almost factorizable structures for $\mathfrak{sl}(2,\mathbb{R})$, up to a scalar multiple, are well known (see [1]). This is in perfect agreement with the results of this section.

Appendix: Tables comparing structure constants

Structure constants of the Bianchi type Lie algebras: It is well known that in terms of the Bianchi classification the brackets of any three dimensional real Lie bialgebra can be described in the form

$$[e_1, e_2] = -\alpha e_2 + n^3 e_3, \quad [e_2, e_3] = n^1 e_1, \quad [e_3, e_1] = n^2 e_2 + \alpha e_3,$$

where the values of the parameters n^i, $i = 1, 2, 3$, and α are listed in the following table, together with the corresponding name used in this paper:

Name	Bianchi	α	n^1	n^2	n^3	A	κ^t
abelian	I	0	0	0	0	0	0
$\mathfrak{h}_3$ Heisenberg	II	0	1	0	0	$\mathrm{diag}(1,0,0)$	0
$\mathfrak{r}'_{3,\lambda=0}$	VII	0	1	1	0	$\mathrm{diag}(1,1,0)$	0
$\mathfrak{r}_{3,\lambda=-1}$	VI	0	1	-1	0	$\mathrm{diag}(1,-1,0)$	0
$\mathfrak{su}(2)$	IX	0	1	1	1	$\mathrm{diag}(1,1,1)$	0
$\mathfrak{sl}(2,\mathbb{R})$	$VIII$	0	1	1	-1	$\mathrm{diag}(1,1,-1)$	0
$\mathfrak{r}_{3,\lambda=1}$	V	1	0	0	0	0	$(0,0,1)$
$\mathfrak{r}_3$	IV	1	0	0	1	$\mathrm{diag}(1,0,0)$	$(0,0,1)$
$\mathfrak{r}'_{3,\lambda=a}$	II_a	a	0	1	1	$\mathrm{diag}(a',a',0) : a' > 0$	$(0,0,1)$
$\mathfrak{r}_{3,\lambda=\frac{a+1}{a-1}}$	VI_a	a	0	1	-1	$\mathrm{diag}(a',-a',0), a' > 0$	$(0,0,1)$
$\mathfrak{r}_{3,\lambda=0} = \mathfrak{aff}_2(\mathbb{R}) \times \mathbb{R}$	III	1	0	1	-1	$\mathrm{diag}(1,-1,0)$	$(0,0,1)$

Comments: $VI_a \cong VI_{-a}$, $VI_{a=1} = III$. For $a \neq \pm 1$, $\mathfrak{r}_{3,\lambda=\frac{a+1}{a-1}} \cong \mathfrak{r}_{3,\frac{1}{\lambda}=\frac{a-1}{a+1}}$, then one can always choose $|\lambda| \leq 1$. In II_a and VI_a one must take $a' = 1/a$.

Structure constants used in [4]: The authors decompose the bracket of a three dimensional Lie algebra $\mathfrak{g}$ as a sum $[x,y] = [x,y]_\kappa + [x,y]_A$, where $\kappa \in \mathfrak{g}^*$, $A : \mathfrak{g}^* \to \mathfrak{g}$ verifies $A = A^*$, $[x,y]_\kappa = i_\kappa(x \wedge y) = \kappa(x)y - \kappa(y)x$ and $[x,y]_A = A(i_{x\wedge y}(e^1 \wedge e^2 \wedge e^3))$. They show that under change of basis, these data change very likely as bilinear forms, so they can be diagonalized. In basis $\{e_1, e_2, e_3\}$ with $\kappa^t = (0,0,\nu)$ and $A = \mathrm{diag}(a^1, a^2, a^3)$, one gets $[e_1,e_2]_\kappa = 0$, $[e_2,e_3]_\kappa = -\nu e_2$, $[e_3,e_1]_\kappa = \nu e_1$, $[e_1,e_2]_A = A(e_3) = a^3 e_3$, $[e_2,e_3]_A = A(e_1) = a^1 e_1$, $[e_3,e_1]_A = A(e_2) = a^2 e_2$. So we see that, choosing $(a^1,a^2,a^3) = (n^1,n^2,n^3)$ and $\nu = \alpha$, we get very similar structure constants as in the Bianchi classification, but in order to get exactly the same, one must perform the cyclic permutation $1 \to 2 \to 3$. Nevertheless, in [4] the authors choose $\nu = 0, 1$ and leave the continuous parameter to (a^1,a^2,a^3). The table above also compares the structure constants given in the first item of Theorem 2.5 of [4] and the others. Remark that, for $a > 0$, $\lambda = \frac{1-a}{1+a}$ runs over all possible values in $(-1,1)$; the limit $a \to 0$ corresponds to $\lambda = 1$ and the limit $a \to +\infty$ corresponds to $\lambda = -1$, in agreement with the table.

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