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LOCAL BOUNDS, HARNACK'S INEQUALITY AND HÖLDER CONTINUITY FOR DIVERGENCE TYPE ELLIPTIC EQUATIONS WITH NON-STANDARD GROWTH

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ABSTRACT. We obtain a Harnack type inequality for solutions to elliptic equations in divergence form with non-standard $p(x)$-type growth. A model equation is the inhomogeneous $p(x)$-Laplacian. Namely,

$$\Delta_{p(x)}u := \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = f(x) \quad \text{in } \Omega,$$

for which we prove Harnack's inequality when $f \in L^{q_0}(\Omega)$ if $\max\{1, \frac{N}{p_1}\} < q_0 \leq \infty$. The constant in Harnack's inequality depends on u only through $\||u|^{p(x)}\|_{L^1(\Omega)}^{p_2-p_1}$. Dependence of the constant on u is known to be necessary in the case of variable $p(x)$. As in previous papers, log-Hölder continuity on the exponent $p(x)$ is assumed. We also prove that weak solutions are locally bounded and Hölder continuous when $f \in L^{q_0(x)}(\Omega)$ with $q_0 \in C(\Omega)$ and $\max\{1, \frac{N}{p(x)}\} < q_0(x)$ in Ω. These results are then generalized to elliptic equations

$$\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u)$$

with $p(x)$-type growth.

1. Introduction

The $p(x)$-Laplacian, defined as

$$\Delta_{p(x)}u := \operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u),$$

extends the Laplacian, where $p(x) \equiv 2$, and the p-Laplacian, where $p(x) \equiv p$ with $1 < p < \infty$. This operator has been used in the modelling of electrorheological fluids ([15]) and in image processing ([3, 4]), for instance.

Up to these days, a great deal of results have been obtained for solutions to equations related to this operator. We will only state in this introduction those results that are related to the ones we address in this paper.

One of the first issues that come into mind is the regularity of solutions to equations involving the $p(x)$-Laplacian or more general elliptic equations with $p(x)$-type growth. Another result —that among other things implies Hölder continuity

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of solutions— is Harnack's inequality. These two issues have been addressed in several papers and we will describe in this introduction those results we are aware of.

Let us state, for the record, that our main concern when starting our research was to obtain Harnack's inequality for nonnegative weak solutions of the inhomogeneous equation

$$\Delta_{p(x)} u = f(x) \quad \text{in } \Omega \tag{1.1}$$

that, strangely enough, had not been addressed previously.

By a weak solution we mean a function in $W^{1,p(x)}(\Omega)$ that satisfies (1.1) in the weak sense. (See the definition and some properties of these spaces below).

When dealing with equations of $p(x)$-type growth it is always assumed that $1 < p_1 \leq p(x) \leq p_2 < \infty$ in Ω. Also, some kind of continuity is assumed since most results on L^p spaces cease to hold without any continuity assumption. In particular, in order to get Harnack's inequality, log-Hölder continuity is always assumed and we will do so in this paper. (See the definition of log-Hölder continuity below).

Harnack's inequality for solutions of (1.1) with $f \equiv 0$ states that, for any nonnegative bounded weak solution u, there exists a constant C —that depends on u— such that, for balls $B_R(x_0)$ such that $B_{4R}(x_0) \subset \Omega$,

$$\sup_{B_R(x_0)} u \leq C\big[\inf_{B_R(x_0)} u + R\big].$$

The dependence of C on u cannot be removed as observed with an example in [11]. In [11] the authors get this inequality for quasiminimizers of the functional

$$J(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\, dx.$$

Solutions to (1.1) with $f \equiv 0$ are minimizers, and therefore, quasiminimizers.

In [11] the authors improve the dependence of C on u. In fact, in [18] Harnack's inequality had been obtained with C depending on the L^∞ norm of u. In [11] instead, the dependence was improved to the L^t norm of u for arbitrarily small $t > 1$ if R is small enough depending only on p and t. In particular, by taking $t = p_1 = \inf_\Omega p(x)$ they get a dependence on $\|u^{p(x)}\|_{L^1(B_{4R}(x_0))}$ that is finite by the definition of a weak solution. In particular, no a priori L^∞ bound is involved in Harnack's inequality.

Later on, the same inequality with a similar dependence on u was obtained for solutions of an obstacle problem related to the functional $J(u)$ in [10].

We would like to comment that [18] dealt with a more general equation. Namely,

$$\Delta_{p(x)} u = (\lambda b(x) - a(x))|u|^{p(x)-2}u \quad \text{in } \Omega$$

with a and b nonnegative and bounded and λ a positive constant.

Also, Harnack's inequality was proved for an operator called by the authors the strong $p(x)$-Laplacian in [1].

As is well known, Hölder continuity is deduced form Harnack's inequality. Anyway, there are methods that give Hölder continuity for weak solutions without going through Harnack's inequality. A result of this kind that applies to more general equations —possibly inhomogeneous— can be found in [7] where the authors prove that bounded weak solutions to

$$\operatorname{div} A(x,u,\nabla u) = B(x,u,\nabla u) \quad \text{in } \Omega \tag{1.2}$$

are locally Hölder continuous if $A(x,s,\xi), B(x,s,\xi)$ satisfy the structure conditions: For any $M_0>0$ there exist positive constants α, C_1, C_2, b such that, for $x\in\Omega$, $|s|\le M_0$, $\xi\in\mathbb{R}^N$,

(a) $A(x,s,\xi)\cdot\xi \ge \alpha|\xi|^{p(x)} - b$.
(b) $\big|A(x,s,\xi)\big| \le C_1|\xi|^{p(x)-1} + b$.
(c) $\big|B(x,s,\xi)\big| \le C_2|\xi|^{p(x)} + b$.

The condition that u is bounded is essential when the growth of B in the gradient variable is the one in (c). Boundedness is proved in [7] under the condition that $B(x,s,\xi)$ grows as $(|s|+|\xi|)^{p(x)-1}$, for instance.

Finally, let us comment that, under additional regularity assumptions on A and B and some different structure conditions (in particular, under the necessary assumption that $p(x)$ be Hölder continuous), Hölder continuity of the derivatives was obtained in [6]. (See also [2] for this result in the case of minimizers of the functional $J(u)$).

In the present paper we are mainly concerned with Harnack's inequality. Our main goal is to obtain this inequality in the case of an inhomogeneous equation with minimal integrability conditions on the right hand side —that in the case of p constant stand for $f\in L^q(\Omega)$ with $\max\{1,N/p\}<q\le\infty$— (see the classical paper [16]).

On the other hand, in several applications we found ourselves dealing with families of bounded nonnegative weak solutions —that are not uniformly bounded, not even in $L^{p(x)}$-norm— and in need of using Harnack's inequality with the same constant C for all the functions in the family. As stated above, we could not use any of the known results (not even for solutions of (1.1) with $f\equiv 0$).

In the present paper, a careful follow up of the constants involved in the proofs allows us to see that the dependence of C on u is actually through $\|u^{p(x)}\|_{L^1(B_{4R})}^{p_+^{4R}-p_-^{4R}}$ where $p_+^{4R}=\sup_{B_{4R}} p$ and $p_-^{4R}=\inf_{B_{4R}} p$. This makes all the difference in many applications. Anyway, this was also the case in the previous papers on the homogeneous equation. Unfortunately, the results were not stated in this way so that they could not be used in many situations.

We start our paper with the case of (1.1) in order to show the ideas and techniques in the simplest possible inhomogeneous case. Then, in Section 3 we consider weak solutions to (1.2) under the structure assumption: For any $M_0>0$ there exist a constant α and nonnegative functions $g_0, C_0\in L^{q_0}(\Omega)$, $g_1, C_1\in L^{q_1}(\Omega)$, $f, C_2\in L^{q_2}(\Omega)$, $K_1\in L^\infty(\Omega)$, $K_2^{p(x)}\in L^{t_2}(\Omega)$ with $\max\{1,\frac{N}{p_1}\}<q_2,t_2\le\infty$ ($p_1=\inf_\Omega p$), $\max\{1,\frac{N}{p_1-1}\}<q_0,q_1\le\infty$ such that, for every $x\in\Omega$, $|s|\le M_0$, $\xi\in\mathbb{R}^N$,

(1) $A(x,s,\xi)\cdot\xi\ge\alpha|\xi|^{p(x)}-C_0(x)|s|^{p(x)}-g_0(x)$,

(2) $\big|A(x,s,\xi)\big| \le g_1(x) + C_1(x)|s|^{p(x)-1} + K_1(x)|\xi|^{p(x)-1}$,
(3) $\big|B(x,s,\xi)\big| \le f(x) + C_2(x)|s|^{p(x)-1} + K_2(x)|\xi|^{p(x)-1}$,

and we prove

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, open set and let p be log-Hölder continuous in Ω. Let $A(x,s,\xi)$, $B(x,s,\xi)$ satisfy the structure conditions (1), (2) and (3). Let $u \ge 0$ be a bounded weak solution to* (1.2) *and let M_0 be such that $u \le M_0$ in Ω. Let $\Omega' \subset\subset \Omega$. Then, there exist C and $0 < R_0 \le \min\{1, \frac14 \operatorname{dist}(\Omega', \partial\Omega)\}$ such that, for every $x_0 \in \Omega'$, $0 < R \le R_0$,*

$$\sup_{B_R(x_0)} u \le C\big[\inf_{B_R(x_0)} u + R + \mu R\big],$$

where

$$\mu = \Big[R^{1-\frac{N}{q_2}} \|f\|_{L^{q_2}(B_{4R}(x_0))}\Big]^{\frac{1}{p_-^{4R}-1}} + \Big[R^{-\frac{N}{q_0}} \|g_0\|_{L^{q_0}(B_{4R}(x_0))}\Big]^{\frac{1}{p_-^{4R}-1}} + \Big[R^{-\frac{N}{q_1}} \|g_1\|_{L^{q_1}(B_{4R}(x_0))}\Big]^{\frac{1}{p_-^{4R}-1}}.$$

The constant C depends only on α, q_i, the log-Hölder modulus of continuity of p in Ω, $\mu^{p_+^{4R}-p_-^{4R}}$, $M^{p_+^{4R}-p_-^{4R}}$, $\|C_i\|_{L^{q_i}(B_{4R}(x_0))}$, $\|K_1^{p(x)}\|_{L^\infty(B_{4R}(x_0))}$, and $\|K_2^{p(x)}\|_{L^{t_2}(B_{4R}(x_0))}$, where $p_+^{4R} = \sup_{B_{4R}(x_0)} p$, $p_-^{4R} = \inf_{B_{4R}(x_0)} p$ and $M = \|u\|_{L^{p_-^{4R}}(\Omega)}$. (Theorem 3.1).

Observe that $\mu^{p_+^{4R}-p_-^{4R}}$ is bounded independently of R.

Observe that, when the functions in the structure conditions are independent of M_0, neither C nor μ depend on the L^∞ norm of u. Moreover, in this case any weak solution is locally bounded (see Remark 3.2).

As usual, from Harnack's inequality we get Hölder continuity of bounded weak solutions (Corollary 3.1).

Let us remark that in this paper we prove that solutions to (1.1) with $f \in L^{q_0(x)}(\Omega)$ with $q_0 \in C(\Omega)$ and $\max\{1, \frac{N}{p(x)}\} < q_0(x)$ in Ω are locally bounded (Proposition 2.1). In the case of equation (1.2), if the functions in the structure conditions are independent of M_0, the local boundedness of weak solutions also holds (see Remark 3.2).

For solutions of (1.1) with $f \in L^{q_0(x)}(\Omega)$, with q_0 as above, we also get local Hölder continuity with constant and exponent depending only on the compact subset, $p(x), q_0(x), \||f|^{q_0(x)}\|_{L^1(\Omega)}$ and $\||u|^{p(x)}\|_{L^1(\Omega)}^{p_2-p_1}$ (Corollary 2.3).

With the same ideas, a similar result can be obtained for solutions to (1.2) although we do not state this result.

On the other hand, if we replace the structure condition (3) by

(3') $\big|B(x,s,\xi)\big| \le f(x) + C_2(x)|s|^{p(x)-1} + K_2(x)|\xi|^{p(x)-1} + b|\xi|^{p(x)}$

with $b \in \mathbb{R}_{>0}$, we obtain Harnack's inequality for bounded weak solutions (Theorem 3.2). In this case, the constant in Harnack's inequality depends also on bM_0 where M_0 is a bound of u.

Again under the structure condition (3'), we deduce that if u is a bounded weak solution, then u is locally Hölder continuous (Corollary 3.2).

Finally, let us observe that even for the simplest homogeneous equation (1.1) with $f \equiv 0$, Harnack's inequality does not imply the strong maximum principle which, in the case of p constant, states that a nonnegative weak solution that vanishes at a point of a connected set must be identically zero. Therefore, a proof of this principle that does not make use of Harnack's inequality is needed. For the case of p constant, an alternative proof was produced in [17]. We adapt this proof for the variable exponent case in Section 4. We also prove a boundary Hopf lemma. For the sake of simplicity, we restrict ourselves to the $p(x)$-Laplacian.

Notation and assumptions

Throughout the paper N will denote the spatial dimension and Ω will be an open subset of $\mathbb{R}^N$.

Assumptions on $p(x)$. We will assume that the function $p(x)$ verifies

$$1 < p_1 \le p(x) \le p_2 < \infty, \qquad x \in \Omega.$$

When we are restricted to a ball B_r we use $p_-^r = p_-(B_r)$ and $p_+^r = p_+(B_r)$ to denote the infimum and the supremum of $p(x)$ over B_r.

We also assume that $p(x)$ is continuous up to the boundary and that it has a modulus of continuity $\omega_R : \mathbb{R} \to \mathbb{R}$, i.e. $|p(x) - p(y)| \le \omega_R(|x-y|)$ if $x, y \in B_R(x_0) \subset \Omega$. We will assume that

$$\omega_R(r) = \frac{C_R}{|\log r|} \quad \text{for } 0 < r \le 1/2,$$

and will refer to such a ω_R as a log-Hölder modulus of continuity of p in $B_R(x_0)$.

Observe that p log-Hölder continuous implies that

$$r^{-(p_+^r - p_-^r)} \le K_R \quad \text{for } 0 < r \le R$$

for a constant K_R related to C_R. This fact will be used throughout the paper.

We will say that p is log-Hölder continuous in Ω if ω_R is independent of the ball $B_R(x_0) \subset \Omega$.

Definition of weak solution. Let $1 < p_1 \le p(x) \le p_2 < \infty$ in Ω.

The space $L^{p(x)}(\Omega)$ stands for the set of measurable functions u such that $|u(x)|^{p(x)} \in L^1(\Omega)$. This is a Banach space with norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\Big\{\lambda > 0 : \int_\Omega \Big(\frac{|u(x)|}{\lambda}\Big)^{p(x)} dx \le 1\Big\}.$$

The dual space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for $x \in \Omega$ and duality pairing $\int_\Omega fg\,dx$.

Then, we let $W^{1,p(\cdot)}(\Omega)$ denote the space of measurable functions u such that u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}$ a Banach space.

We call $W_0^{1,p(\cdot)}(\Omega)$ the closure in the norm of $W^{1,p(\cdot)}$ of the set of those functions in $W^{1,p(\cdot)}(\Omega$ that have compact support in Ω. When p is log-Hölder continuous, it coincides with the closure of $C_0^\infty(\Omega)$.

Observe that $u \in W^{1,p(\cdot)}$ implies that $|\nabla u|^{p(x)-2}\nabla u \in \left(L^{p'(x)}\right)^N$.

For more definitions and results on these spaces we refer to [5] and [13].

Definition 1.1. We say that u is a weak solution to (1.2) if $u \in W^{1,p(x)}(\Omega)$ and, for every $\phi \in W_0^{1,p(x)}(\Omega)$, there holds that

$$\int A(x,u(x),\nabla u(x)) \cdot \nabla \phi(x)\,dx = \int B(x,u(x),\nabla u(x))\phi(x)\,dx.$$

2. Harnack's inequality for solutions to $\Delta_{p(x)}u = f$

In this section we will prove the following result.

Theorem 2.1. *Assume that p is locally log-Hölder continuous in Ω. Let $x_0 \in \Omega$ and $0 < R \le 1$ is such that $\overline{B_{4R}(x_0)} \subset \Omega$. There exists C such that, if u is a nonnegative weak solution of the problem*

$$\Delta_{p(x)}u = f \quad \text{in } \Omega,$$

with $f \in L^{q_0}(\Omega)$ for some $\max\{1, \frac{N}{p_-^{4R}}\} < q_0 \le \infty$, then

$$\sup_{B_R} u \le C\Big[\inf_{B_R} u + R + R\mu\Big], \tag{2.1}$$

where

$$\mu = \big[R^{1-\frac{N}{q_0}}\|f\|_{L^{q_0}(B_{4R}(x_0))}\big]^{\frac{1}{p_-^{4R}-1}}.$$

The constant C depends only on N, p_-^{4R}, p_+^{4R}, s, q_0, ω_{4R}, $\mu^{p_+^{4R}-p_-^{4R}}$, $\|u\|_{L^{sq'}(B_{4R}(x_0))}^{p_+^{4R}-p_-^{4R}}$, $\|u\|_{L^{sr_0}(B_{4R}(x_0))}^{p_+^{4R}-p_-^{4R}}$ (for certain $q' = \frac{q}{q-1}$, with r_0, $q \in (1,\infty)$ and $\frac{1}{q_0} + \frac{1}{q} + \frac{1}{r_0} = 1$ depending on N, q_0 and p_-^{4R}). Here $s \ge p_+^{4R} - p_-^{4R}$ is arbitrary and ω_{4R} is the modulus of log-Hölder continuity of p in $B_{4R}(x_0)$.

The proof will be a consequence of three lemmas.

Lemma 2.1 (Caccioppoli type estimate)**.** *Let $u \ge 1$ and bounded such that $\Delta_{p(x)}u \ge -H(x)u^{p(x)-1}$ in a ball B and $\gamma > 0$, or $\Delta_{p(x)}u \le H(x)u^{p(x)-1}$ in B and $\gamma < 0$. Here $H \ge 0$ is a measurable function. Then, for $\eta \in C_0^\infty(B)$ there holds that*

$$\begin{aligned}\int_B u^{\gamma-1}|\nabla u|^{p_-}\eta^{p_+} \le \int_B u^{\gamma-1}\eta^{p_+} + C|\gamma|^{-p_+}\int_B u^{\gamma+p(x)-1}\eta^{p_+-p(x)}|\nabla\eta|^{p(x)} \\ + C|\gamma|^{-1}\int_B H(x)u^{\gamma+p(x)-1}\eta^{p_+},\end{aligned} \tag{2.2}$$

with $C = C(p_+, p_-)$. Here $p_+ = \max_{\overline{B}} p$, $p_- = \min_{\overline{B}} p$.

Proof. As is usual in the proof of these type of estimates we take as a test function $u^\gamma \eta^{p_+} \in W_0^{1,p(x)}(\Omega)$, since $u \in W^{1,p(x)}(\Omega)$ and we are assuming that $1 \le u \in L^\infty(\Omega)$.

Assume first that $\Delta_{p(x)} u \ge -H(x) u^{p(x)-1}$ and $\gamma > 0$. We get

$$\begin{aligned}\gamma \int u^{\gamma-1}\eta^{p_+}|\nabla u|^{p(x)} &\le -p_+ \int u^\gamma \eta^{p_+-1}|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\eta + \int H(x)u^{\gamma+p(x)-1}\eta^{p_+}\\ &\le \varepsilon p_+ \int \frac{1}{p'(x)}|\nabla u|^{p(x)} u^{\gamma-1}\eta^{p_+}\\ &\quad + \int \frac{p_+}{\varepsilon^{p(x)-1}p(x)} u^{\gamma+p(x)-1}\eta^{p_+-p(x)}|\nabla \eta|^{p(x)}\\ &\quad + \int H(x)u^{\gamma+p(x)-1}\eta^{p_+},\end{aligned}$$

where $0 < \varepsilon \le 1$ is to be chosen, and $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Now, we choose $\varepsilon = \min\{1, \frac{\gamma}{2(p_+-1)}\}$ so that

$$\frac{\varepsilon p_+}{p'(x)} \le \frac{\gamma}{2}, \qquad \frac{p_+}{\varepsilon^{p(x)-1}p(x)} \le C(p_+,p_-)\gamma^{-p_++1},$$

and, in order to get (2.2), we bound

$$\int u^{\gamma-1}\eta^{p_+}|\nabla u|^{p_-} \le \int u^{\gamma-1}\eta^{p_+} + \int u^{\gamma-1}\eta^{p_+}|\nabla u|^{p(x)}.$$

Now, if $\Delta_{p(x)} u \le H(x)u^{p(x)-1}$ and $\gamma < 0$, since $u \ge 1$ we can proceed as before and we get

$$\gamma \int u^{\gamma-1}\eta^{p_+}|\nabla u|^{p(x)} \ge -p_+ \int u^\gamma\eta^{p_+-1}|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\eta - \int H(x)u^{\gamma+p(x)-1}\eta^{p_+}.$$

Dividing by γ we get

$$\begin{aligned}\int u^{\gamma-1}\eta^{p_+}|\nabla u|^{p(x)} &\le Cp_+|\gamma|^{-p_+}\int u^\gamma\eta^{p_+-1}|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\eta\\ &\quad + C|\gamma|^{-1}\int H(x)u^{\gamma+p(x)-1}\eta^{p_+}.\end{aligned}$$

Now the proof continues as before and we obtain (2.2). □

Lemma 2.2. *Let p be log-Hölder continuous in B_4. Let $u \ge 1$ be bounded and such that $\Delta_{p(x)}u \ge -H(x)u^{p(x)-1}$ in B_4, where $0 \le H \in L^{q_0}(B_4)$, with $\max\{1, \frac{N}{p_-^4}\} < q_0 \le \infty$. Let $t > 0$. Then, for every $0 < \rho_1 < \rho_2 \le 4$ there holds that*

$$\sup_{B_{\rho_1}} u \le C\Big(\frac{\rho_2}{\rho_2-\rho_1}\Big)^C \Big(\fint_{B_{\rho_2}} u^t\Big)^{1/t}. \tag{2.3}$$

The constant C depends only on s, p_+^4, p_-^4, $M^{p_+^4-p_-^4}$, ω_4, $\|H(x)\|_{L^{q_0}(B_4)}$, q_0, q and t. Here $M = \big(\fint_{B_4} u^{sq'}\big)^{sq'} + \big(\fint_{B_4} u^{sr_0}\big)^{sr_0}$, with $r_0, q' \in (1,\infty)$ depending on q_0, p_-^4, N and $s \ge p_+^4 - p_-^4$ is arbitrary.

Proof. We use Moser's iteration technique and we follow the lines of the proof of Lemma 4.6 in [10] for the treatment of the variable exponent. In our situation we are more careful with the choice of κ below in order to get our result, due to the presence of a right hand side.

In what follows p_+ and p_- stand for the maximum and minimum values of p in $\overline{B}_\rho$.

Let $0<\sigma<\rho\le 4$. Let $\eta\in C_0^\infty(B_\rho)$ such that $\eta\equiv 1$ in B_σ and $|\nabla\eta|\le C\frac{1}{\rho-\sigma}$.

Let $\kappa=\frac{\hat N}{\hat N-p_-^4}$ with $\hat N=N$ when $N>p_-^4$ and, $p_-^4<\hat N<q_0p_-^4$ when $N\le p_-^4$.

Then, for $\gamma\ge\gamma_0>0$ using (2.2), Sobolev inequality and the fact that $\kappa p_-\le p_-^*=\frac{Np_-}{N-p_-}$ when $N>p_-^4$, $u\in W^{1,p_-}(B_\rho)$ and, $W_0^{1,p_-}(B_\rho)\subset L^t(B_\rho)$ continuously for every $1<t<\infty$ when $N\le p_-^4\le p_-$,

$$
\begin{aligned}
&\Big(\fint \big(u^{\frac{\gamma-1+p_-}{p_-}}\eta^{p_+/p_-}\big)^{\kappa p_-}\Big)^{1/\kappa p_-}\\
&\le C\rho\Big(\fint\big|\nabla\big(u^{\frac{\gamma-1+p_-}{p_-}}\eta^{p_+/p_-}\big)\big|^{p_-}\Big)^{1/p_-}\\
&\le C\frac{\gamma-1+p_-}{p_-}\rho\Big(\fint u^{\gamma-1}\eta^{p_+}|\nabla u|^{p_-}\Big)^{1/p_-}+C\rho\frac{p_+}{p_-}\Big(\fint u^{\gamma-1+p_-}\eta^{p_+-p_-}|\nabla\eta|^{p_-}\Big)^{1/p_-}\\
&\le C\rho(1+\gamma)\Big[\Big(\fint u^{\gamma-1}\eta^{p_+}\Big)^{1/p_-}+\Big(\fint u^{\gamma-1+p(x)}\eta^{p_+-p(x)}|\nabla\eta|^{p(x)}\Big)^{1/p_-}\\
&\quad+\Big(\fint H(x)u^{\gamma-1+p(x)}\eta^{p_+}\Big)^{1/p_-}\Big]+C\rho\Big(\fint u^{\gamma-1+p_-}\eta^{p_+-p_-}|\nabla\eta|^{p_-}\Big)^{1/p_-}.
\end{aligned}
$$

Here the constant C depends on p_+^4, p_-^4 and γ_0.

Since, by the choice of $\hat N$, there holds that $q_0>\frac{\hat N}{p_-^4}$, there exists $1<q<\frac{\hat N}{\hat N-p_-^4}$ such that $\frac1q+\frac1{q_0}<1$. Let $r_0\in(1,\infty)$ given by $\frac1q+\frac1{q_0}+\frac1{r_0}=1$. Now we bound

$$
\begin{aligned}
\fint u^{\gamma-1}\eta^{p_+}&\le\fint u^{\gamma-1+p_-}\eta^{p_+}\le\Big(\fint u^{(\gamma-1+p_-)q}\eta^{qp_+}\Big)^{1/q}\\
&\le C\Big(\frac{1}{\rho-\sigma}\Big)^{p_+}\Big(\fint_{B_\rho}u^{(\gamma-1+p_-)q}\Big)^{1/q}
\end{aligned}
$$

since $\eta\le 1\le\frac{4}{(\rho-\sigma)}$. And, with $M_1=\Big(\fint_{B_4}u^{sq'}\Big)^{1/sq'}$, $q'=\frac{q}{q-1}$, $s\ge p_+-p_-$,

$$
\begin{aligned}
\fint u^{\gamma-1+p(x)}\eta^{p_+-p(x)}|\nabla\eta|^{p(x)}&\le C\Big(\frac{1}{\rho-\sigma}\Big)^{p_+}\fint_{B_\rho}u^{\gamma-1+p_-}u^{p(x)-p_-}\\
&\le C\Big(\frac{1}{\rho-\sigma}\Big)^{p_+}\Big(\fint_{B_\rho}u^{(\gamma-1+p_-)q}\Big)^{1/q}\Big(\fint_{B_\rho}u^{(p(x)-p_-)q'}\Big)^{1/q'}\\
&\le C\Big(\frac{1}{\rho-\sigma}\Big)^{p_+}M_1^{p_+-p_-}\Big(\fint_{B_\rho}u^{(\gamma-1+p_-)q}\Big)^{1/q}.
\end{aligned}
$$

Similarly,

$$\fint u^{\gamma-1+p_-}\eta^{p_+-p_-}|\nabla\eta|^{p_-} \le C\Big(\frac{1}{\rho-\sigma}\Big)^{p_+}\Big(\fint_{B_\rho} u^{(\gamma-1+p_-)q}\Big)^{1/q}.$$

Finally, with $M_2 = \big(\fint_{B_4} u^{sr_0}\big)^{1/sr_0}$, $s \ge p_+ - p_-$,

$$\begin{aligned}\fint H(x)u^{\gamma-1+p(x)}\eta^{p_+} &\le \fint_{B_\rho} H(x)u^{\gamma-1+p_-}u^{p_+-p_-}\\ &\le \Big(\fint_{B_\rho} H(x)^{q_0}\Big)^{1/q_0}\Big(\fint_{B_\rho} u^{(\gamma-1+p_-)q}\Big)^{1/q}\Big(\fint_{B_\rho} u^{r_0(p_+-p_-)}\Big)^{1/r_0}\\ &\le CM_2^{p_+-p_-}\Big(\frac{1}{\rho-\sigma}\Big)^{p_+}\Big(\fint_{B_\rho} u^{(\gamma-1+p_-)q}\Big)^{1/q},\end{aligned}$$

with C depending on q_0, p_+, p_- and $\|H\|_{L^{q_0}(B_4)}$. In fact, $\rho^{-\frac{N}{q_0}} \le C\rho^{-p_-^4} \le C\rho^{-p_-} \le C\rho^{-p_+} \le (\rho-\sigma)^{-p_+}$.

Since $M = M_1 + M_2 \ge 1$ we conclude that

$$\Big(\fint \big(u^{\frac{\gamma-1+p_-}{p_-}}\eta^{p_+/p_-}\big)^{\kappa p_-}\Big)^{1/\kappa p_-} \le C\rho(1+\gamma)\frac{M^{\frac{p_+}{p_-}-1}}{(\rho-\sigma)^{p_+/p_-}}\Big(\fint_{B_\rho} u^{(\gamma-1+p_-)q}\Big)^{1/qp_-},$$

with C depending on q_0, p_+^4, p_-^4, $\|H(x)\|_{L^{q_0}(B_4)}$ and γ_0.

Let us now take $\beta > p_- - 1$. Then, $\beta = \gamma - 1 + p_-$, with $\gamma = \beta - (p_- - 1) > 0$. Recalling that $\rho^{p_-} \le C\rho^{p_+}$ for a constant C that depends only on the log-Hölder continuity of p,

$$\Big(\fint_{B_\sigma} u^{\kappa\beta}\Big)^{1/\kappa} \le C\Big(\frac{\rho}{\sigma}\Big)^{\frac{N}{\kappa}} M^{p_+-p_-}\Big(\frac{\rho}{\rho-\sigma}\Big)^{p_+}(1+\beta)^{p_-}\Big(\fint_{B_\rho} u^{q\beta}\Big)^{1/q}. \tag{2.4}$$

Let us call

$$\phi(f,t,E) := \Big(\fint_E |f|^t\Big)^{1/t}.$$

Then, if $\beta > p_- - 1$, $s \ge p_+ - p_-$, we have for a constant C depending on p_+^4, p_-^4, $\|H(x)\|_{L^{q_0}(B_4)}$ and $\gamma_0 > 0$ such that $\beta - (p_- - 1) \ge \gamma_0$,

$$\phi(u,\kappa\beta,B_\sigma) \le C^{1/\beta}M^{\frac{p_+-p_-}{\beta}}(1+\beta)^{p_-/\beta}\Big(\frac{\rho}{\sigma}\Big)^{\frac{N}{\kappa\beta}}\Big(\frac{\rho}{\rho-\sigma}\Big)^{p_+/\beta}\phi(u,q\beta,B_\rho).$$

And we have a result quite similar to Lemma 4.6 in [10]. For the sake of completeness we finish the proof.

To this end, we write $\kappa\beta = \bar\kappa\bar\beta$ with $\bar\kappa = \frac{\kappa}{q}$ and $\bar\beta = q\beta$. Recall that, due to the choice of q, we have $q < \kappa$. So that $\bar\kappa > 1$ and

$$\phi(u,\bar\kappa\bar\beta,B_\sigma) \le C^{q/\bar\beta}M^{\frac{q(p_+-p_-)}{\bar\beta}}(1+\bar\beta)^{qp_-/\bar\beta}\Big(\frac{\rho}{\sigma}\Big)^{\frac{N}{\bar\kappa\bar\beta}}\Big(\frac{\rho}{\rho-\sigma}\Big)^{qp_+/\bar\beta}\phi(u,\bar\beta,B_\rho). \tag{2.5}$$

Let $0<\rho_1<\rho_2\le 4$ and let us call $r_j=\rho_1+2^{-j}(\rho_2-\rho_1)$. We will consider (2.5) with $\sigma=r_{j+1}$ and $\rho=r_j$. Observe that

$$\frac{\rho}{\sigma}=\frac{r_j}{r_{j+1}}\le 2,\qquad \frac{\rho}{\rho-\sigma}=\frac{r_j}{r_j-r_{j+1}}=\frac{\rho_1+2^{-j}(\rho_2-\rho_1)}{2^{-(j+1)}(\rho_2-\rho_1)}\le 2^{j+1}\frac{\rho_2}{\rho_2-\rho_1}.$$

Assume first that $t>q(p_+^4-1)$. Take $\bar\beta_j=\bar\kappa^j t$. There holds that $\bar\beta_j=q\beta_j$ with $\beta_j=\bar\kappa^j\frac{t}{q}$. And, $\gamma_j=\beta_j-(p_-^{r_j}-1)\ge\frac{t}{q}-(p_+^4-1)=\gamma_0>0$.

Then, the constant C in every step of the iteration may be taken depending on γ_0 and independent of j. Thus, we have with C_0 depending on $p_+^4.p_-^4$, $M^{p_+^4-p_-^4}$, ω_4, $\|H(x)\|_{L^{q_0}(B_4)}$, q_0 and t,

$$\begin{aligned}\phi(u,\bar\kappa^{j+1}t,B_{r_{j+1}})&\le C^{qt^{-1}\bar\kappa^{-j}}M^{\frac{q(p_+^4-p_-^4)}{t\bar\kappa^j}}(1+\bar\kappa^j t)^{\bar\kappa^{-j}qt^{-1}p_+^4}\Big(\frac{r_j}{r_{j+1}}\Big)^{Nt^{-1}\bar\kappa^{-(j+1)}}\\&\quad\times\Big(\frac{r_j}{r_j-r_{j+1}}\Big)^{qp_+^4t^{-1}\bar\kappa^{-j}}\phi(u,\bar\kappa^j t,B_{r_j})\\&\le C_0^{\bar\kappa^{-j}}(1+\bar\kappa^j t)^{\bar\kappa^{-j}qt^{-1}p_+^4}\Big(2^{j+1}\frac{\rho_2}{\rho_2-\rho_1}\Big)^{qp_+^4t^{-1}\bar\kappa^{-j}}\phi(u,\bar\kappa^j t,B_{r_j}).\end{aligned}$$

Iterating this inequality we get

$$\begin{aligned}\phi(u,\bar\kappa^{j+1}t,B_{r_{j+1}})&\le C_0^{\sum_{i=0}^j\bar\kappa^{-i}}\Big(\prod_{i=0}^j(1+t\bar\kappa^i)^{t^{-1}\bar\kappa^{-i}}\Big)^{qp_+^4}\Big(\frac{\rho_2}{\rho_2-\rho_1}\Big)^{qp_+^4t^{-1}\sum_{i=0}^j\bar\kappa^{-i}}\\&\quad\times\big(2^{qp_+t^{-1}}\big)^{\sum_{i=0}^j(i+1)\bar\kappa^{-i}}\phi(u,t,B_{\rho_2}).\end{aligned}$$

Letting $j\to\infty$,

$$\begin{aligned}\sup_{B_{\rho_1}}u&\le C_0^{\frac{1}{1-\bar\kappa^{-1}}}\Big(\prod_{i=0}^\infty(1+t\bar\kappa^i)^{t^{-1}\bar\kappa^{-i}}\Big)^{qp_+^4}\big(2^{qp_+^4t^{-1}}\big)^{\sum_{i=0}^\infty(i+1)\bar\kappa^{-i}}\\&\quad\times\Big(\frac{\rho_2}{\rho_2-\rho_1}\Big)^{qp_+t^{-1}\frac{1}{1-\bar\kappa^{-1}}}\Big(\fint_{B_{\rho_2}}u^t\Big)^{1/t},\end{aligned}$$

and the lemma is proved for $t>q(p_+^4-1)$ since $\prod_{i=0}^\infty(1+t\bar\kappa^i)^{t^{-1}\bar\kappa^{-i}}\le C$.

In order to get the result for $0<t\le q(p_+^4-1)$ we proceed again as in [10] and use the extrapolation result Lemma 3.38 in [12] with $s=\infty$, $p>q(p_+^4-1)$ fixed (here q is the one in our paper, s and p the ones in [12]) and $q=t$ (here q is the one in [12] and not the one in our paper) that we state below. □

Lemma 2.3 (Lemma 3.38 in [12]). *Suppose that $0<q<p<s\le\infty$, $\xi\in\mathbb{R}$, and that $B=B_r(x_0)$ is a ball. If a nonnegative function $v\in L^p(B)$ satisfies*

$$\Big(\fint_{\lambda B'}v^s\,dx\Big)^{1/s}\le c_1(1-\lambda)^\xi\Big(\fint_{B'}v^p\,dx\Big)^{1/p}$$

for each ball $B'=B(x_0,r')$ with $r'\le r$ and for all $0\le\lambda<1$, then

$$\Big(\fint_{\lambda B}v^s\,dx\Big)^{1/s}\le c(1-\lambda)^{\xi/\theta}\Big(\fint_{B}v^q\,dx\Big)^{1/q}$$

for all $0 \leq \lambda < 1$. *Here* $c = c(p, q, s, \xi, c_1)$ *and* $\theta \in (0, 1)$ *such that*

$$\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{s}.$$

Remark 2.1. Observe that it is enough to prove Lemma 2.2 for $t \geq t_0 > 0$ with t_0 arbitrary depending only on p_+^4, p_-^4, q, and then use Lemma 2.3 in order to get the result for $0 < t < t_0$. This means that, in order to prove Lemma 2.2, it is enough to get (2.5) for $\bar\beta \geq q\beta_0$ with, for instance, $\beta_0 \geq 2(p_+^4 - 1)$ (this means to have $\gamma_0 \geq p_+^4 - 1$).

Now, we prove a weak Harnack inequality for supersolutions. There holds

Lemma 2.4 (Weak Harnack's inequality)**.** *Let p be log-Hölder continuous in B_4. Let $0 \leq H \in L^{q_0}(B_4)$ with $\max\{1, \frac{N}{p_-^4}\} < q_0 \leq \infty$ and let $s \geq p_+^4 - p_-^4$. There exists $t_0 > 0$ depending only on s, p_-^4, p_+^4, $\|H(x)\|_{L^{q_0}(B_4)}$, ω_4 and $M^{p_+^4 - p_-^4}$, with $M = \big(⨍_{B_4} u^{sq'}\big)^{sq'} + \big(⨍_{B_4} u^{sr_0}\big)^{sr_0}$ for some choice of $1 < q' = \frac{q}{q-1} < \infty$ depending on N, p_-^4, q_0, $1 < r_0 < \infty$, with $\frac{1}{q_0} + \frac{1}{q} + \frac{1}{r_0} = 1$; $C > 0$ depending on the same constants and also on t_0, q_0, q such that, for $u \geq 1$ and bounded with $\Delta_{p(x)} u \leq H(x) u^{p(x)-1}$ in B_4 there holds that*

$$\inf_{B_1} u \geq C \Big(⨍_{B_2} u^{t_0} \Big)^{1/t_0}.$$

Proof. The proof follows the lines of the one of Lemma 2.2. This time we use Caccioppoli's inequality (2.2) with $\gamma < -\gamma_0 = -(p_-^4 - 1) < 0$. We call again $\kappa = \frac{\hat N}{\hat N - p_-^4}$ with $\hat N$ as in the proof of Lemma 2.2 and choose q and r_0 as in that Lemma. Then, we take $0 < \sigma < \rho \leq 4$. For $\beta = \gamma + (p_- - 1) < 0$ we prove that

$$\phi(u, q\beta, B_\rho) \leq C^{1/|\beta|} (1 + |\beta|)^{p_+/|\beta|} \Big(\frac{\rho}{\rho - \sigma} \Big)^{p_+/|\beta|} \phi(u, \kappa\beta, B_\sigma). \tag{2.6}$$

Here C is a constant depending on s, q_0, q, p_+^4, p_-^4, $\gamma_0 = p_-^4 - 1$, $\|H(x)\|_{L^{q_0}(B_4)}$ and $M^{p_+^4 - p_-^4}$.

In fact, we proceed as in the proof of Lemma 2.2 until we get (2.4). Then, since $\beta < 0$ we get (2.6).

Observe that (2.6) holds for any $\beta < 0$ since this is equivalent to $\gamma < -(p_- - 1) \leq -(p_-^4 - 1)$.

In order to finish the proof it is necessary to prove that there exists $t_0 > 0$ and $\bar C > 0$ depending only on p_+^4, p_-^4, $\|H(x)\|_{L^{q_0}(B_4)}$, $M^{p_+^4 - p_-^4}$ and the log-Hölder modulus of continuity of p in B_4 such that

$$\phi(u, t_0, B_2) \leq \bar C \phi(u, -t_0, B_2). \tag{2.7}$$

Then, we choose $\beta = -\frac{t_0}{q}$ in (2.6) in order to start the iterative process.

In order to prove (2.7), we let $0 < r \leq 2$ and we bound by using Caccioppoli's inequality (2.2) with $\gamma = 1 - p_-^{2r}$, $\eta \in C_0^\infty(B_{2r})$ with $\eta \equiv 1$ in B_r, $|\nabla \eta| \leq \frac{C}{r}$,

$$\begin{aligned}
\fint_{B_r} |\nabla \log u|^{p_-^{2r}} &= \fint_{B_r} u^{-p_-^{2r}} |\nabla u|^{p_-^{2r}} \leq C \fint_{B_{2r}} u^{-p_-^{2r}} \eta^{p_+^{2r}} |\nabla u|^{p_-^{2r}} \\
&\leq C \fint_{B_{2r}} u^{-p_-^{2r}} \eta^{p_+^{2r}} + \frac{C}{(p_-^{2r}-1)^{p_+^{2r}}} \fint_{B_{2r}} u^{p(x)-p_-^{2r}} \eta^{p_+^{2r}-p(x)} |\nabla \eta|^{p(x)} \\
&\quad + \frac{C}{p_-^{2r}-1} \fint_{B_{2r}} H(x) u^{p(x)-p_-^{2r}} \eta^{p_+^{2r}-p(x)} \\
&\leq C(p_+^4, p_-^4)\big[1 + r^{-p_+^{2r}} M_1^{p_+^4 - p_-^4}\big] + \frac{C}{p_-^{2r}-1} \fint_{B_{2r}} H(x) u^{p(x)-p_-^{2r}}.
\end{aligned}$$

The last term can be bound in the following way:

$$\begin{aligned}
\fint_{B_{2r}} H(x) u^{p(x)-p_-^{2r}} &\leq \Big(\fint_{B_{2r}} H^{q_0}\Big)^{1/q_0} \Big(\fint_{B_{2r}} u^{(p_+^{2r}-p_-^{2r})q_0'}\Big)^{1/q_0'} \\
&\leq C r^{-N/q_0} \|H\|_{L^{q_0}(B_4)} \Big(\fint_{B_{2r}} u^{(p_+^{2r}-p_-^{2r})r_0}\Big)^{1/r_0} \\
&\leq C r^{-p_+^{2r}} \|H\|_{L^{q_0}(B_4)} M_2^{p_+^{2r}-p_-^{2r}}
\end{aligned}$$

since $q_0' \leq r_0$, $\frac{N}{q_0} < p_-^4 \leq p_-^{2r} \leq p_+^{2r}$, $0 < r \leq 2$.

Gathering all these estimates we get

$$\fint_{B_r} |\nabla \log u|^{p_-^{2r}} \leq C(p_+^4, p_-^4, \|H\|_{L^{q_0}(B_4)}, \omega_4)\; r^{-p_+^{2r}} M^{p_+^4 - p_-^4}.$$

Now the proof follows in a standard way. By Poincaré's inequality applied to $f = \log u$, using that $r^{p_-^{2r}} \leq C r^{p_+^{2r}}$,

$$\fint_{B_r} |f - f_{B_r}|^{p_-^{2r}} \leq C r^{p_-^{2r}} \fint_{B_r} |\nabla f|^{p_-^{2r}} \leq C(p_+^4, p_-^4, \|H\|_{L^{q_0}(B_4)}, \omega_4) M^{p_+^4 - p_-^4}.$$

Since this holds for every ball B_r with $r \leq 2$, by the John–Nirenberg Lemma there exist constants C_1 and C_2 depending only on p_-^4, p_+^4, $\|H\|_{L^{q_0}(B_4)}$, ω_4 and $M^{p_+^4 - p_-^4}$ such that

$$\fint_{B_2} e^{C_1 |f - f_{B_2}|} \leq C_2,$$

where $f_{B_2} = \fint_{B_2} f$.

We conclude that

$$\begin{aligned}
\Big(\fint_{B_2} e^{C_1 f}\Big)\Big(\fint_{B_2} e^{-C_1 f}\Big) &= \Big(\fint_{B_2} e^{C_1(f - f_{B_2})}\Big)\Big(\fint_{B_2} e^{-C_1(f - f_{B_2})}\Big) \\
&\leq \Big(\fint_{B_2} e^{C_1 |f - f_{B_2}|}\Big)^2 \leq C_2^2,
\end{aligned}$$

and we have (2.7) with $t_0 = C_1$.

Now the proof of the lemma ends by an iterative process similar to the one in Lemma 2.2. In fact, we call $\bar\kappa = \frac{\kappa}{q}$, $\bar\beta = q\beta$, and for the iteration we let $\bar\beta_j = -\bar\kappa^j t_0$,

$r_j = 1 + 2^{-j}$. Hence, $\gamma_j = \beta_j - (p_-^{r_j} - 1) = -\bar\kappa^j \frac{t_0}{q} - (p_-^{r_j} - 1) \le -\gamma_0 := -(p_-^4 - 1)$. Then, with $\bar C$ the constant in (2.7), using that $p_-^{r_j}, p_+^{r_j} \le p_+^4$,

$$\bar C^{-1}\phi(u, t_0, B_2) \le \phi(u, -t_0, B_2) \le C_0^{\sum_{i=0}^j \bar\kappa^{-i}} \Big(\prod_{i=0}^j (1 + t_0\bar\kappa^i)^{t^{-1}\kappa^{-i}} \Big)^{q_0 p_+^4} \times \big(2^{qp_+^4 t_0^{-1}}\big)^{\sum_{i=0}^j (i+2)\bar\kappa^{-i}} \phi(u, -\bar\kappa^{j+1} t_0, B_{r_{j+1}}).$$

Thus,

$$\Big(\fint_{B_2} u^{t_0} \Big)^{1/t_0} \le C \lim_{j\to\infty} \phi(u, -\kappa^j t_0, B_{r_j}) = C \inf_{B_1} u,$$

and the lemma is proved. □

We can improve on Lemma 2.4 in the following way (see [14] where this improvement was done in the case of p constant):

Lemma 2.5 (Improved weak Harnack's inequality). *Under the assumptions of Lemma 2.4, let $0 < t < \frac{N}{N - p_-^4}(p_-^4 - 1)$ if $p_-^4 < N$, $t > 0$ arbitrary if $p_-^4 \ge N$. Then, there exists a constant C with the same dependence as the constant in Lemma 2.4 and also depending on t, such that*

$$\Big(\fint_{B_2} u^t \Big)^{1/t} \le C \inf_{B_1} u.$$

Proof. We prove that, for every t in this range, t_0 the one in Lemma 2.4, $0 < \rho_1 < \rho_2 \le 4$, there holds that

$$\Big(\fint_{B_{\rho_1}} u^t \Big)^{1/t} \le \bar C \Big(\fint_{B_{\rho_2}} u^{t_0} \Big)^{1/t_0} \tag{2.8}$$

for a constant $\bar C$ depending on t, t_0, ρ_1, ρ_2, $M^{p_+^4 - p_-^4}$, p_+^4, p_-^4, and q_0.

This will prove the lemma if we replace in the proof of Lemma 2.4 the ball B_2 by B_{ρ_2} with $2 < \rho_2 < 4$ and we take $\rho_1 = 2$ in (2.8).

In order to prove (2.8), we proceed as in Lemma 2.4 but we are more careful with the choice of κ. In fact, as in Lemma 2.4 we choose $\kappa = \frac{\hat N}{\hat N - p_-^4}$, with $\hat N = N$ if $p_-^4 < N$ and $p_-^4 < \hat N < q_0 p_-^4$ if $p_-^4 \ge N$. In this latter case, we choose $\hat N$ close enough to p_-^4 so that $\kappa^{-1} t = t\big(1 - \frac{p_-^4}{\hat N}\big) < p_-^4 - 1$.

Observe that $\kappa^{-1} t < p_-^4 - 1$ also if $p_-^4 < N$. In fact, in this case we have $\kappa = \frac{N}{N - p_-^4}$ and the inequality holds by our hypothesis on t.

Then, we choose q as in Lemma 2.4. That is, $1 \le q_0' < q < \kappa$.

In order to prove (2.8) we go back to (2.4). Recall that we get this inequality if $\gamma \le -\gamma_0 < 0$ and $\beta = \gamma + p_- - 1$.

Then, as in Lemma 2.4, we take $\bar\beta = q\beta$, $\bar\kappa = \frac{\kappa}{q} > 1$.

Now, for $j \in \mathbb{N}$ and $i = 0, 1, \ldots, j$ we let $\bar\beta_{ij} = \bar\kappa^{i-(j+1)} t$. Then, $\beta_{ij} = \bar\kappa^{i-(j+1)} \frac{t}{q}$ and $\gamma_{ij} = \beta_{ij} - (p_- - 1) \le \bar\kappa^{-1}\frac{t}{q} - (p_- - 1) \le \kappa^{-1} t - (p_-^4 - 1) = -\gamma_0 < 0$.

Now, we iterate inequality (2.5) for $i = 0, \dots, j$ with $\rho = r_i$, $\sigma = r_{i+1}$, and $r_i = \rho_1 + 2^{-i}(\rho_2 - \rho_1)$. We get

$$\|u\|_{L^{\bar\kappa\bar\beta_{jj}}(B_{r_{j+1}})} \le \bar C \|u\|_{L^{\bar\beta_{0j}}(B_{r_0})}$$

for a constant $\bar C$ depending on j, q, ρ_1, ρ_2, $M^{p_+^4 - p_-^4}$, p_+^4, p_-^4. Thus, we get (2.8) once we observe that $\rho_1 \le r_{j+1}$, $r_0 = \rho_2$, $\bar\kappa\bar\beta_{jj} = t$, $\bar\beta_{0j} = \bar\kappa^{-(j+1)}t$, and we choose j large so that $\bar\kappa^{-(j+1)}t \le t_0$. $\square$

Now, by modifying the proof of Lemmas 2.1 and 2.2 we will prove that weak subsolutions are locally bounded from above and weak supersolutions are locally bounded from below. This is already known when $p_1 > N$ since weak super- and sub-solutions belong to $W^{1,p_1}(\Omega) \subset L^\infty(\Omega)$ if $p_1 > N$.

We start with a variation of Caccioppoli's inequality:

Lemma 2.6. *Let $u \in W^{1,p(x)}(B)$ such that $\Delta_{p(x)} u \ge -H(x)(1+|u|)^{p(x)-1}$ in a ball B and $\gamma \ge 1$. Here $H \ge 0$ is a measurable function. Then, for $\eta \in C_0^\infty(B)$ there holds that*

$$\begin{aligned}\int_B F_n(u_+ + 1)|\nabla u_+|^{p_-}\eta^{p_+} \le \int_B & F_n(u_+ + 1)\eta^{p_+} \\ &+ C\int_B u_+^{p(x)} F_n(u_+ + 1)\eta^{p_+ - p(x)}|\nabla\eta|^{p(x)} \\ &+ C\int_B H(x)(u_+ + 1)^{p(x)-1} G_n(u_+ + 1)\eta^{p_+},\end{aligned} \tag{2.9}$$

with $u_+ = \max\{u, 0\}$, $C = C(p_+, p_-)$. Here $p_+ = \max_{\overline{B}} p$, $p_- = \min_{\overline{B}} p$.

In (2.9), the functions F_n and G_n are defined, for $s \ge 1$, by

$$G_n(s) = \int_1^s F_n(\tau)\, d\tau,$$

$$F_n(s) = \begin{cases} s^{\gamma-1} & \text{if } 1 \le s \le n, \\ n^{\gamma-1} & \text{if } s \ge n. \end{cases}$$

Proof. We proceed as in the proof of Lemma 2.1. This time we take as test function $\phi = G_n(u_+ + 1)\eta^{p_+} \in W_0^{1,p(x)}(B)$ for every $\gamma \ge 1$. We get

$$\begin{aligned}\int F_n(u_+ + 1)|\nabla u_+|^{p(x)}\eta^{p_+} &\le -p_+ \int G_n(u_+ + 1)\eta^{p_+ - 1}|\nabla u_+|^{p(x)-1}|\nabla\eta| \\ &\quad + \int H(x)(u_+ + 1)^{p(x)-1} G_n(u_+ + 1)\eta^{p_+} \\ &\le C\int u_+ F_n(u_+ + 1)\eta^{p_+ - 1}|\nabla u_+|^{p(x)-1}|\nabla\eta| \\ &\quad + \int H(x)(u_+ + 1)^{p(x)-1} G_n(u_+ + 1)\eta^{p_+},\end{aligned}$$

since $G_n(u_+ + 1) = 0$ if $u_+ = 0$ and $G_n(s) \le F_n(s)(s-1)$, as F_n is a nondecreasing function in $[1, \infty)$.

Now, by applying Young inequality we get

$$\int F_n(u_+ + 1)|\nabla u_+|^{p(x)}\eta^{p_+} \le C\int u_+^{p(x)} F_n(u_+ + 1)\eta^{p_+ - p(x)}|\nabla\eta|^{p(x)} + \int H(x)(u_+ + 1)^{p(x)-1}G_n(u_+ + 1)\eta^{p_+},$$

and the lemma is proved. □

We can now prove the weak maximum principle. There holds

Lemma 2.7. *Let p be log-Hölder continuous in B_4. Let $u \in W^{1,p(x)}(B_4)$ such that $\Delta_{p(x)} u \ge -H(x)(|u|+1)^{p(x)-1}$ in B_4, where $0 \le H \in L^{q_0}(B_4)$ with $\max\{1, \frac{N}{p_-^4}\} < q_0 \le \infty$. Then, there exists $0 < \bar\rho \le 4$ such that, for every $0 < \rho_1 < \rho_2 < \bar\rho$ and for every $0 < t < \infty$, there holds that*

$$\sup_{B_{\rho_1}} u_+ \le C\Big(\frac{\rho_2}{\rho_2 - \rho_1}\Big)^C \Big(\fint_{B_{\rho_2}} (u_+ + 1)^t\Big)^{1/t}. \tag{2.10}$$

The constant C depends only on $p_+^4.p_-^4$, $M^{p_+^4 - p_-^4}$, $\|H(x)\|_{L^{q_0}(B_4)}$, t and q_0. $\bar\rho$ depends on q_0, p_-^4 and the log-Hölder modulus of continuity of p in B_4. Here $M = \big(\fint_{B_4} |u|^{p_-^4}\big)^{1/p_-^4}$.

Proof. We start from (2.9) with $\gamma \ge 1$. Let

$$L_n(s) = \int_1^s \big(F_n(\tau)\big)^{1/p_-}\, d\tau.$$

Then

$$|\nabla L_n(u_+ + 1)|^{p_-} = F_n(u_+ + 1)|\nabla u_+|^{p_-},$$

and we have

$$\begin{aligned}
&\int |\nabla\big(\eta^{p_+/p_-} L_n(u_+ + 1)\big)|^{p_-} \\
&\quad = \int F_n(u_+ + 1)|\nabla u_+|^{p_-}\eta^{p_+} + C\int L_n(u_+ + 1)^{p_-}\eta^{p_+ - p_-}|\nabla\eta|^{p_-} \\
&\quad \le C\Big[\int F_n(u_+ + 1)\eta^{p_+} + \int u_+^p F_n(u_+ + 1)\eta^{p_+ - p}|\nabla\eta|^p \\
&\qquad + \int H(x)(u_+ + 1)^{p-1}G_n(u_+ + 1)\eta^{p_+} + \int L_n(u_+ + 1)^{p_-}\eta^{p_+ - p_-}|\nabla\eta|^{p_-}\Big].
\end{aligned}$$

We bound, for $s \ge 1$,

$$\begin{aligned}
&F_n(s) \le s^{\gamma-1}, \\
&L_n(s) \le F_n(s)^{1/p_-}(s-1) \quad \Rightarrow \quad L_n(u_+ + 1)^{p_-} \le (u_+ + 1)^{\gamma - 1 + p_-}, \\
&s^{p-1}G_n(s) \le s^p F_n(s) \le s^{\gamma-1+p} \quad \Rightarrow \quad (u_+ + 1)^{p-1}G_n(u_+ + 1) \le (u_+ + 1)^{\gamma-1+p}.
\end{aligned}$$

Thus, by the Sobolev inequality with $\kappa = \frac{\hat{N}}{\hat{N}-p_-}$ and $\hat{N}$ as in Lemma 2.2,

$$
\begin{aligned}
\Big(\fint L_n(u_++1)^{\kappa p_-}\eta^{\kappa p_+}\Big)^{1/\kappa}
&\le C\rho^{p_-}\fint |\nabla\big(\eta^{p_+/p_-}L_n(u_++1)\big)|^{p_-}\\
&\le C\rho^{p_-}\Big[\fint (u_++1)^{\gamma-1}\eta^{p_+}+\fint (u_++1)^{\gamma-1+p_-}\eta^{p_+-p_-}|\nabla\eta|^{p_-}\\
&\quad+\fint (u_++1)^{\gamma-1+p}\eta^{p_+-p}|\nabla\eta|^{p}+\fint H(x)(u_++1)^{\gamma-1+p}\eta^{p_+}\Big].
\end{aligned}
$$

We take $\bar\rho \le 4$ such that $p_+^{\bar\rho}-p_-^{\bar\rho} < \min\{p_-^4/q', p_-^4/r_0\}$, with q' and r_0 as in the proof of Lemma 2.2. Let $0<\sigma<\rho\le\bar\rho$, $\eta\in C_0^\infty(B_\rho)$, $0\le\eta\le 1$, $\eta\equiv 1$ in B_σ, $|\nabla\eta|\le \frac{C}{\rho-\sigma}$ and let us proceed as in the proof of Lemma 2.2.

Observe that, by the choice of $\bar\rho$, there exists $s\ge p_+-p_-$ such that $sq'\le p_-^4$ and $sr_0\le p_-^4$, and we fix such an s for the next steps.

We can proceed with the proof as long as $u_+\in L^{q(\gamma-1+p_-)}(B_\rho)$ with q as in the proof of Lemma 2.2. This is the case for any value of $\gamma\ge 1$ if $p_-\ge N$. If instead $p_-<N$, there holds that $\hat N = N$ and $1<q<\frac{N}{N-p_-}$. Therefore, if we take $\gamma=1$ we will have $u_+\in L^{q(\gamma-1+p_-)}(B_\rho)$ as needed in order to continue with the estimates. Thus we get, with $\beta=\gamma-1+p_-$,

$$
\Big(\fint_{B_\sigma} L_n(u_++1)^{\kappa p_-}\Big)^{1/\kappa\beta}\le C\Big(\frac{\rho}{\sigma}\Big)^{N/\kappa\beta}\Big(\frac{\rho}{\rho-\sigma}\Big)^{p_+/\beta}\Big(\fint_{B_\rho}(u_++1)^{q\beta}\Big)^{1/q\beta}.
$$

Since the right hand side is independent of n and finite as long as $u_+\in L^{q\beta}(B_\rho)$ (for instance, if $\beta=p_-$ so that $q\beta\le p_-^*$), we can pass to the limit and get

$$
\begin{aligned}
\Big(\fint_{B_\sigma}(u_++1)^{\kappa\beta}\Big)^{1/\kappa\beta}
&\le C\Big[1+(1+\beta)^{p_-/\beta}\Big(\frac{\rho}{\sigma}\Big)^{N/\kappa\beta}\Big(\frac{\rho}{\rho-\sigma}\Big)^{p_+/\beta}\Big(\fint_{B_\rho}(u_++1)^{q\beta}\Big)^{1/q\beta}\Big].
\end{aligned}
$$

In fact, there holds that

$$
L_n(s)\to\frac{p_-}{\gamma-1+p_-}\big(s^{\frac{\gamma-1+p_-}{p_-}}-1\big)=\frac{p_-}{\beta}\big(s^{\frac{\beta}{p_-}}-1\big).
$$

As in Lemma 2.2 we call $\bar\kappa=\frac{\kappa}{q}$, $\bar\beta=q\beta$ and get

$$
\begin{aligned}
\Big(\fint_{B_\sigma}(u_++1)^{\bar\kappa\bar\beta}\Big)^{1/\bar\kappa\bar\beta}
&\le C\Big[1+(1+\beta)^{qp_-/\bar\beta}\Big(\frac{\rho}{\sigma}\Big)^{N/\bar\kappa\bar\beta}\Big(\frac{\rho}{\rho-\sigma}\Big)^{qp_+/\bar\beta}\Big(\fint_{B_\rho}(u_++1)^{\bar\beta}\Big)^{1/\bar\beta}\Big]\\
&\le 2C(1+\beta)^{qp_-/\bar\beta}\Big(\frac{\rho}{\sigma}\Big)^{N/\bar\kappa\bar\beta}\Big(\frac{\rho}{\rho-\sigma}\Big)^{qp_+/\bar\beta}\Big(\fint_{B_\rho}(u_++1)^{\bar\beta}\Big)^{1/\bar\beta}.
\end{aligned}
$$

Now we can proceed as in Lemma 2.2 with the iterative process. In each step we use that $u_+ \in L^{\bar\beta_j}(B_{r_j})$ in order to deduce that $u_+ \in L^{\bar\beta_{j+1}}(B_{r_{j+1}})$ and continue with the iteration, starting with $\bar\beta_0 = {p^4_-}^*$.

In this way we prove (2.10) for $t = {p^4_-}^*$ if $p^4_- < N$, any positive number if $p^4_- \geq N$. Now, if $p^4_- < N$ and $0 < t < {p^4_-}^*$ we use Lemma 2.3 to get the result. In particular, for $\rho_2 = \bar\rho$ we get (2.10) with $t = p^4_-$. So that, $u \in L^\infty(B_{\widetilde\rho})$ for any $\widetilde\rho < \bar\rho$. Therefore, $u_+ \in L^t(B_{\rho_2})$ for every $t > 0$ if $\rho_2 < \bar\rho$ and we can proceed with the proof without any restriction on t. So that (2.10) holds for every $t > 0$ if $0 < \rho_1 < \rho_2 < \bar\rho$. □

In a similar way, we can prove

Lemma 2.8. *Let $u \in W^{1,p(x)}(B)$ such that $\Delta_{p(x)} u \leq H(x)(|u|+1)^{p(x)-1}$ in a ball B and $\gamma \geq 1$. Here $H \geq 0$ is a measurable function. Then, for $\eta \in C_0^\infty(B)$ there holds that*

$$\begin{aligned}\int_B F_n(u_-+1)|\nabla u_-|^{p_-}\eta^{p_+} \leq \int_B F_n(u_-+1)\eta^{p_+} \\ + C\int_B u_-^{p(x)} F_n(u_-+1)\eta^{p_+-p(x)}|\nabla\eta|^{p(x)} \\ + C\int_B H(x) u_-^{p(x)-1} G_n(u_-+1)\eta^{p_+}\end{aligned} \tag{2.11}$$

with $u_- = \max\{-u, 0\}$, $C = C(p_+, p_-)$. Here $p_+ = \max_{\overline B} p$, $p_- = \min_{\overline B} p$.

In (2.11), the functions F_n and G_n are defined as in Lemma 2.6.

We also have

Lemma 2.9. *Let p be log-Hölder continuous in B_4. Let $u \in W^{1,p(x)}(B_4)$ such that $\Delta_{p(x)} u \leq H(x)(|u|+1)^{p(x)-1}$ in B_4, where $0 \leq H \in L^{q_0}(B_4)$ with $\max\{1, \frac{N}{p^4_-}\} < q_0 \leq \infty$. Then, there exists $\bar\rho$ such that for every $0 < \rho_1 < \rho_2 < \bar\rho < 4$ and any $0 < t < \infty$ there holds that*

$$\sup_{B_{\rho_1}} u_- \leq C\Big(\frac{\rho_2}{\rho_2-\rho_1}\Big)^C \Big(\fint_{B_{\rho_2}} (u_-+1)^t\Big)^{1/t}.$$

The constant C depends on t, $p^4_+ . p^4_-$, $M^{p^4_+ - p^4_-}$, $\|H(x)\|_{L^{q_0}(B_4)}$ and q_0. $\bar\rho$ depends on q, r_0, p^4_- for certain $q, r_0 \in (1,\infty)$ such that $\frac{1}{q_0} + \frac{1}{q} + \frac{1}{r_0} = 1$, and the log-Hölder modulus of continuity of p in B_4. Here $M = \big(\fint_{B_4} |u|^{p^4_-}\big)^{1/p^4_-}$.

We conclude

Proposition 2.1 (Weak maximum principle)**.** *Let $\Omega \subset \mathbb{R}^N$ be bounded and p be log-Hölder continuous in Ω. Let $u \in W^{1,p(x)}(\Omega)$ such that $\Delta_{p(x)} u \geq -H(x)(|u|+1)^{p(x)-1}$ in Ω, with $0 \leq H \in L^{q_0(x)}(\Omega)$, with $q_0 \in C(\Omega)$, $\max\{1, \frac{N}{p(x)}\} < q_0(x)$ for every $x \in \Omega$. Let $\Omega' \subset\subset \Omega$. Then, u is bounded from above in Ω'. More precisely, for every $0 < t < \infty$,*

$$\sup_{\Omega'} u \leq \widetilde C\Big[1 + \|u\|_{L^t(\Omega'')}\Big],$$

where $\Omega'' = \big\{x \in \Omega, \operatorname{dist}(x,\Omega') < \frac{1}{2}\operatorname{dist}(\Omega',\partial\Omega)\big\}$. *Here* $\widetilde{C}$ *depends on* t, Ω', p_1, p_2, $q_0(x)$, $\||H|^{q_0(x)}\|_{L^1(\Omega)}$, *the log-Hölder modulus of continuity of* p *in* Ω, *and* $\||u|^{p(x)}\|_{L^1(\Omega)}$.

If $\Delta_{p(x)}u \le H(x)(|u|+1)^{p(x)-1}$ *in* Ω, *there holds that* u *is bounded from below by* $-\widetilde{C}\Big[1+\|u\|_{L^t(B_{\Omega''})}\Big]$.

Proof. Let $0 < R = \min\{1, \frac14\operatorname{dist}(\Omega',\partial\Omega)\}$. For $x_0 \in \Omega'$, let $\bar u(x) = \frac{u(x_0+Rx)}{R}$, $\bar p(x) = p(x_0+Rx)$ and $\bar H(x) = RH(x_0+Rx)$. Then, $\Delta_{\bar p(x)}\bar u \ge -\bar H(x)(|\bar u|+1)^{\bar p(x)-1}$ in B_4.

We claim that there exists $0 < \bar r < 1$, $\bar q_0 > 0$, possibly depending on x_0, such that $q_0(x_0+Rx) \ge \bar q_0 > \max\{1, \frac{N}{\bar p_-^{4\bar r}}\}$ for every $x \in B_{4\bar r}$. In fact, if $\bar p(0) < N$ we let ρ_1 such that $\bar p(x) < N$ in $B_{4\rho_1}$. Then, let $\varepsilon > 0$ such that $q_0(x_0) \ge \frac{N}{\bar p(0)} + 3\varepsilon$ and $\rho_2 \le \rho_1$ such that $q_0(x_0+Rx) \ge \bar q_0 := \frac{N}{\bar p(0)} + 2\varepsilon$ in $B_{4\rho_2}$. Finally, $\bar r \le \rho_2$ such that $\frac{N}{\bar p(x)} - \frac{N}{\bar p(0)} < \varepsilon$ in $B_{4\bar r}$. So, in $B_{4\bar r}$ we have $q_0(x_0+Rx) \ge \bar q_0 > \max\{1, \frac{N}{p_-^{4\bar r}}\}$.

Now, if $\bar p(0) \ge N$, we let first ρ_1 and $\varepsilon > 0$ such that $q_0(x_0+Rx) \ge \bar q_0 := 1+2\varepsilon$ in $B_{4\rho_1}$ and then, $\bar r \le \rho_1$ such that $\frac{N}{\bar p(x)} \le 1+\varepsilon$ in $B_{4\bar r}$. So we have $q_0(x_0+Rx) \ge \bar q_0 > \max\{1, \frac{N}{\bar p_-^{4\bar r}}\}$ in $B_{4\bar r}$.

We can assume that $\bar r$ is small so that $\bar p_+^{4\bar r} - \bar p_-^{4\bar r} < \min\{p_1/q', p_1/r_0\}$ with q and r_0 as in Lemma 2.2, $(\frac{1}{\bar q_0} + \frac1q + \frac{1}{r_0} = 1)$. Then, by Lemma 2.7 (observe that we may take $\bar\rho = 4\bar r$ in that lemma by the conditions imposed to $\bar r$), for every $0 < t < \infty$,

$$\sup_{B_{\bar r}} \bar u \le C\Big[1 + \|\bar u\|_{L^t(B_{2\bar r})}\Big]$$

with C depending on t, $\bar r$, p_1, p_2, the log-Hölder modulus of continuity of p in Ω'', $\bar q_0$, r_0, $\|\bar H\|_{L^{\bar q_0}(B_{4\bar r})}$ and $M^{p_2-p_1}$, where $M = \|u\|_{L^{p_1}(\Omega'')}$.

Observe that $\|\bar H\|_{L^{\bar q_0}(B_{4\bar r})} \le C\big[1 + \||H|^{q_0(x)}\|_{L^1(\Omega)}\big]^{1/\inf_\Omega q_0}$ with C depending on R, $\bar r$ and q_0.

Thus, any point $x_0 \in \Omega'$ has a neighborhood $B_{\bar r R}(x_0)$ where

$$\sup_{B_{R\bar r}(x_0)} u \le \widetilde{C}\Big[1 + \|u\|_{L^t(B_{2R\bar r})(x_0)}\Big]$$

with $\widetilde{C}$ depending on the neighborhood, on t, $p(x)$, $q(x)$, $\||H(x)|^{q_0(x)}\|_{L^1(\Omega)}^{1/\inf_\Omega q_0}$ and $\||u|^{p(x)}\|_{L^1(\Omega)}^{1/\inf_\Omega p}$.

Since Ω' is compact, we get the result on the upper bound.

Analogously, if $\Delta_{p(x)}u \le H(x)|u|^{p(x)-1}$ in Ω we find a similar uniform bound from above for u_- in Ω'. So, we get the lower bound. $\square$

As a corollary we get local bounds for weak solutions to (1.1). There holds

Corollary 2.1. *Let* $\Omega \subset \mathbb{R}^N$ *be bounded and* p *log-Hölder continuous in* Ω. *Let* $u \in W^{1,p(x)}(\Omega)$ *be a weak solution to*

$$\Delta_{p(x)}u = f \quad \text{in } \Omega,$$

with $f\in L^{q_0(x)}(\Omega)$ *with* $q_0\in C(\Omega)$ *such that* $\max\{1,\frac{N}{p(x)}\}<q_0(x)$ *in* Ω*. Then,* u *is locally bounded.*

Proof. Let $H(x)=|f(x)|$. Then,

$$|\Delta_{p(x)}u|=|f(x)|\le H(x)(|u|+1)^{p(x)-1}.$$

The result follows by applying Propositon 2.1. □

Now, we prove Harnack's inequality for solutions to (1.1).

Proof of Theorem 2.1. Without loss of generality we may assume that $x_0=0$.

Let u and f be as in the statement. Let $\bar p(x)=p(Rx)$.

If $f\not\equiv 0$ in B_{4R}, let $\widetilde H(x)=R|f(Rx)|$,

$$\bar u(x)=1+\|\widetilde H\|_{L^{q_0}(B_4)}^{\frac{1}{p_-^{4R}-1}}+\frac{u(Rx)}{R},$$

and

$$H(x)=\frac{\widetilde H(x)}{\|\widetilde H\|_{L^{q_0}(B_4)}}.$$

If $f\equiv 0$ in B_{4R}, let

$$\bar u(x)=1+\frac{u(Rx)}{R}$$

and

$$H(x)\equiv 0.$$

Then,

$$\max_{B_4}\bar p=\max_{B_{4R}}p,\qquad \min_{B_4}\bar p=\min_{B_{4R}}p,$$

and for $x,y\in B_4$,

$$|\bar p(x)-\bar p(y)|\le\omega_{4R}(R|x-y|)\le\omega_{4R}(|x-y|)$$

if $0<R\le 1$, and

$$\big|\Delta_{\bar p(x)}\bar u(x)\big|=\big|Rf(Rx)\big|\le H(x)\Big(1+\|\widetilde H\|_{L^{q_0}(B_4)}^{\frac{1}{p_-^{4R}-1}}+\big(\frac{u(Rx)}{R}\big)\Big)^{p_-^{4R}-1}\le H(x)\bar u^{\bar p(x)-1}.$$

Therefore, we can apply Lemmas 2.2 and 2.4 (recall that we already know that u is locally bounded and therefore, $\bar u$ is bounded in B_4) with $\rho_1=1$, $\rho_2=2$ and $t=t_0$ to obtain

$$\sup_{B_1}\bar u\le C\Big(\fint_{B_2}\bar u^{t_0}\Big)^{1/t_0}\le C\inf_{B_1}\bar u.$$

Recall that $\|H\|_{L^{q_0}(B_4)}=1$ or $\|H\|_{L^{q_0}(B_4)}=0$. Thus, C is independent of H and so it depends on f only through its dependence on $\bar u$.

Since $\bar u(x)=\frac{u(Rx)+R+R\|\widetilde H\|_{L^{q_0}(B_4)}^{\frac{1}{p_-^{4R}-1}}}{R}$ there holds that

$$\sup_{B_R}u\le C\big[\inf_{B_R}u+R+R\|\widetilde H\|_{L^{q_0}(B_4)}^{\frac{1}{p_-^{4R}-1}}\big].$$

Now, $\|\widetilde{H}\|_{L^{q_0}(B_4)} = R^{1-\frac{N}{q_0}}\|f\|_{L^{q_0}(B_{4R})}$. And

$$\begin{aligned}\bar{M}_1^{\bar{p}_+^4-\bar{p}_-^4} :=& \Big(\fint_{B_4} \bar{u}^{sq'}\Big)^{\frac{\bar{p}_+^4-\bar{p}_-^4}{sq'}}\\ &\le C\Big[R^{-1}\Big(\fint_{B_{4R}} u^{sq'}\Big)^{1/sq'} + 1 + \|\widetilde{H}\|_{L^{q_0}(B_4)}^{\frac{1}{p_-^{4R}-1}}\Big]^{p_+^{4R}-p_-^{4R}}\\ &\le C\Big[\Big(\|u\|_{L^{sq'}(B_{4R})} + 1 + \big(R^{1-\frac{N}{q_0}}\|f\|_{L^{q_0}(B_{4R})}\big)^{\frac{1}{p_-^{4R}-1}}\Big]^{p_+^{4R}-p_-^{4R}},\end{aligned}$$

since $R^{-(p_+^{4R}-p_-^{4R})} \le C$ with C independent of R. In particular, $\bar{M}_1^{\bar{p}_+^4-\bar{p}_-^4}$ is bounded independently of R.

The same kind of bound holds for $\bar{M}_2^{\bar{p}_+^4-\bar{p}_-^4}$. So, the theorem is proved. □

Remark 2.2. Observe that, since $q_0 > \frac{N}{p_-^{4R}}$, there holds that

$$1 + \frac{1-\frac{N}{q_0}}{p_-^{4R}-1} > 1 - \frac{p_-^{4R}-1}{p_-^{4R}-1} = 0.$$

Thus, (2.1) can be stated as:

$$\sup_{B_R(x_0)} u \le C\big[\inf_{B_R(x_0)} u + R + R^\delta L\big] \tag{2.12}$$

for a certain $\delta > 0$.

The power δ can be made independent of R. In fact, we may take $\delta = 1 + \frac{1-\frac{N}{q_0}}{p_1-1} > 0$ if $N \ge q_0 > \frac{N}{p_1}$, with $p_1 = \inf_\Omega p$, and $\delta = 1 + \frac{1-\frac{N}{q_0}}{p_2-1} > 1$ if $q_0 > N$, with $p_2 = \sup_\Omega p$. Here $L := \big(1 + \|f\|_{L^{q_0}(\Omega)}\big)^{\frac{1}{p_1-1}} \ge \|f\|_{L^{q_0}(B_{4R})}^{\frac{1}{p_-^{4R}-1}}$.

Remark 2.3. Observe that, since p is continuous in Ω, if R is small enough, we may choose $s \ge p_+^{4R} - p_-^{4R}$ such that $sq' \le p_-^{4R}$ and $sr_0 \le p_-^{4R}$. So, the constant C in (2.3) depends on u only through $\||u|^{p(x)}\|_{L^1(B_{4R}(x_0))}^{p_+^{4R}-p_-^{4R}}$.

A similar comment applies to (2.1) and (2.12).

From Harnack's inequality we get Hölder continuity of weak solutions. There holds

Corollary 2.2. *Let $\Omega \subset \mathbb{R}^N$ be bounded and p log-Hölder continuous in Ω with $1 < p_1 \le p(x) \le p_2 < \infty$ in Ω. Let $f \in L^{q_0}(\Omega)$ with $\max\{1, \frac{N}{p_1}\} < q_0 \le \infty$. Let u be a weak solution to*

$$\Delta_{p(x)} u = f \quad \text{in } \Omega. \tag{2.13}$$

Then, u is locally Hölder continuous in Ω with constant and exponent depending only on the compact subdomain and on p_1, p_2, q_0, $\|f\|_{L^{q_0}(\Omega)}$, the log-Hölder modulus of continuity of p in Ω and $M^{p_2-p_1}$ and where $M = \||u|^{p(x)}\|_{L^1(\Omega)}$.

Proof. Once we have Harnack's inequality, the proof is standard. Let $\Omega' \subset\subset \Omega$. There exist $L, R_0, \delta > 0$ such that for any nonnegative weak solution v of (2.13), any $x_0 \in \Omega'$ and $0 < R \leq R_0$,

$$\sup_{B_R(x_0)} v \leq C\big[\inf_{B_R(x_0)} v + R + R^\delta L\big]. \tag{2.14}$$

Now, apply (2.14) with $R = 2^{-(j+1)} R_0$ to the functions $v_1 = M_j - u(x)$ and $v_2 = u(x) - m_j$, where $M_j = \sup_{B_{2^{-j}R_0}(x_0)} u$, $m_j = \inf_{B_{2^{-j}R_0}(x_0)} u$, to obtain that

$$\operatorname{osc}_{j+1} u \leq \nu \operatorname{osc}_j u + C(L)R^\delta,$$

with $0 < \nu < 1$, and the result follows (see [9] for the details). The constant and exponent of the Hölder continuity in Ω' depend only on ν, $C(L)$ and δ. □

By applying Corollary 2.2 on small enough neighborhoods of points $x_0 \in \Omega' \subset\subset \Omega$ —as in Proposition 2.1— we get local Hölder continuity with variable q_0. There holds

Corollary 2.3. *Let $\Omega \subset \mathbb{R}^N$ be bounded and p log-Hölder continuous in Ω, with $1 < p_1 \leq p(x) \leq p_2 < \infty$ in Ω. Let $f \in L^{q_0(x)}(\Omega)$, with $q_0 \in C(\Omega)$ and $\max\{1, \frac{N}{p(x)}\} < q_0(x)$ in Ω. Let u be a weak solution to*

$$\Delta_{p(x)} u = f \quad \text{in } \Omega.$$

Then, u is locally Hölder continuous in Ω with constant and exponent depending only on the compact subdomain and on p_1, p_2, $q_0(x)$, $\||f|^{q_0(x)}\|_{L^1(\Omega)}$, the log-Hölder modulus of continuity of p in Ω and $\||u|^{p(x)}\|_{L^1(\Omega)}^{p_2-p_1}$.

3. Harnack's inequality for solutions to general elliptic equations

In this section we will generalize the results of Section 2 to elliptic equations with $p(x)$-type growth. More precisely,

$$\operatorname{div} A(x, u, \nabla u) = B(x, u, \nabla u) \quad \text{in } \Omega. \tag{3.1}$$

We assume that for every $M_0 > 0$ there exist a constant α and nonnegative functions $g_0, C_0 \in L^{q_0}(\Omega)$, $g_1, C_1 \in L^{q_1}(\Omega)$, $f, C_2 \in L^{q_2}(\Omega)$, $K_2^{p(x)} \in L^{t_2}(\Omega)$, $K_1 \in L^\infty(\Omega)$ for some $\max\{1, \frac{N}{p_1-1}\} < q_0, q_1 \leq \infty$ ($p_1 = \inf_\Omega p$), $\max\{1, \frac{N}{p_1}\} < q_2, t_2 \leq \infty$, such that, for every $x \in \Omega$, $|s| \leq M_0$, $\xi \in \mathbb{R}^N$,

(1) $A(x, s, \xi) \cdot \xi \geq \alpha|\xi|^{p(x)} - C_0|s|^{p(x)} - g_0(x)$,
(2) $\big|A(x, s, \xi)\big| \leq g_1(x) + C_1|s|^{p(x)-1} + K_1|\xi|^{p(x)-1}$,
(3) $\big|B(x, s, \xi)\big| \leq f(x) + C_2|s|^{p(x)-1} + K_2|\xi|^{p(x)-1}$.

We start with a Caccioppoli type estimate.

Lemma 3.1. *Let $1 \leq u \in L^\infty(B)$ be such that $\operatorname{div} A(x, u, \nabla u) \geq -\big(H_2(x)u^{p(x)-1} + G_2(x)|\nabla u|^{p(x)-1}\big)$ in a ball B and $\gamma > 0$, or $\operatorname{div} A(x, u, \nabla u) \leq H_2(x)u^{p(x)-1} + G_2(x)|\nabla u|^{p(x)-1}$ in a ball B and $\gamma < 0$. Assume that there exists a positive constant α such that*

(1) *$A(x, u(x), \nabla u(x)) \cdot \nabla u(x) \geq \alpha|\nabla u(x)|^{p(x)} - H_0(x)u(x)^{p(x)}$ in B.*

(2) $\big|A(x,u(x),\nabla u(x))\big| \le H_1(x)u^{p(x)-1} + G_1(x)|\nabla u|^{p(x)-1}$ *in* B

for certain nonnegative measurable functions H_i, G_j, $i=0,1,2$, $j=1,2$.

Let $\eta \in C_0^\infty(B)$, $\eta \ge 0$. *Then, there exists a constant* C *that depends only on* $p_+ = \sup_B p$, $p_- = \inf_B p$ *and* α *such that*

$$\begin{aligned}\int u^{\gamma-1}\eta^{p_+}|\nabla u|^{p_-} \le \int u^{\gamma-1}\eta^{p_+} + C\Big[|\gamma|^{-1}\int (H_0+H_2)u^{\gamma+p(x)-1}\eta^{p_+} \\ + |\gamma|^{-1}\int H_1 u^{\gamma+p(x)-1}\eta^{p_+-1}|\nabla\eta| \\ + |\gamma|^{-p_+}\int G_1^{p(x)}u^{\gamma+p(x)-1}\eta^{p_+-p(x)}|\nabla\eta|^{p(x)} \\ + |\gamma|^{-p_+}\int G_2^{p(x)}u^{\gamma+p(x)-1}\eta^{p_+-p(x)}\Big].\end{aligned} \tag{3.2}$$

Here $p_+ = p_+^B, p_- = p_-^B$.

Proof. Let us consider the case of $\gamma > 0$. As in the proof of Lemma 2.2 we take $u^\gamma\eta^{p_+}$ as test function. Then,

$$\begin{aligned}\alpha\gamma\int u^{\gamma-1}\eta^{p_+}|\nabla u|^{p(x)} \le -p_+\int H_1 u^{\gamma+p(x)-1}\eta^{p_+-1}|\nabla\eta| \\ - p_+\int G_1 u^\gamma\eta^{p_+-1}|\nabla u|^{p(x)-1}|\nabla\eta| \\ + \int H_2 u^{\gamma+p(x)-1}\eta^{p_+} + \int G_2 u^\gamma\eta^{p_+}|\nabla u|^{p(x)-1} \\ + \int H_0 u^{\gamma+p-1}\eta^{p_+}.\end{aligned}$$

As in the proof of Lemma 2.2,

$$\begin{aligned}\int G_1 u^\gamma\eta^{p_+-1}|\nabla u|^{p(x)-1}|\nabla\eta| \le \frac{\alpha\gamma}{4p_+}\int u^{\gamma-1}\eta^{p_+}|\nabla u|^{p(x)} \\ + C\gamma^{-p_++1}\int G_1^{p(x)}u^{\gamma+p(x)-1}\eta^{p_+-p(x)}|\nabla\eta|^{p(x)}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int G_2 u^\gamma\eta^{p_+}|\nabla u|^{p(x)-1} \le \frac{\alpha\gamma}{4}\int u^{\gamma-1}\eta^{p_+}|\nabla u|^{p(x)} \\ + C\gamma^{-p_++1}\int G_2^{p(x)}u^{\gamma+p(x)-1}\eta^{p_+-p(x)}.\end{aligned}$$

Hence, since

$$\int u^{\gamma-1}\eta^{p_+}|\nabla u|^{p_-} \le \int u^{\gamma-1}\eta^{p_+} + \int u^{\gamma-1}\eta^{p_+}|\nabla u|^{p(x)},$$

we have (3.2).

The case of $\gamma < 0$ is done in a similar way. □

Once we have a Caccioppoli type estimate we can get results similar to Lemmas 2.2 and 2.4.

So, we have

Lemma 3.2. *Let p be log-Hölder continuous in B_4. Let $u \geq 1$ and bounded be such that* $\operatorname{div} A(x,u,\nabla u) \geq -\big(H_2(x)u^{p(x)-1} + G_2(x)|\nabla u|^{p(x)-1}\big)$ *in B_4. Assume that there exists a positive constant α such that*

(1) $A(x,u(x),\nabla u(x)) \cdot \nabla u(x) \geq \alpha |\nabla u(x)|^{p(x)} - H_0(x)u(x)^{p(x)}$ *in* B_4,

(2) $\big|A(x,u(x),\nabla u(x))\big| \leq H_1(x)u^{p(x)-1} + G_1(x)|\nabla u|^{p(x)-1}$ *in* B_4.

Here $H_i \in L^{q_i}(B_4)$, $i=0,1,2$, $G_2^{p(x)} \in L^{t_2}(B_4)$ *with* $\max\{1, \frac{N}{p_-^{4R}}\} < q_i, t_2 \leq \infty$ *for* $i=0,2$, $\max\{1, \frac{N}{p_-^{4R}-1}\} < q_1 \leq \infty$, $G_1 \in L^\infty(B_4)$ *and they are nonnegative. Then, for every* $0<\sigma<\rho\leq 4$ *and* $t>0$ *there holds that*

$$\sup_{B_{\rho_1}} u \leq C\Big(\frac{\rho_2}{\rho_2-\rho_1}\Big)^C \Big(⨍_{B_{\rho_2}} u^t\Big)^{1/t}.$$

The constant C depends only on s, p_+^4, p_-^4, ω_4, q_i, t_2, t, α, $\|H_i\|_{L^{q_i}(B_4)}$, $\|G_1^{p(x)}\|_{L^\infty(B_4)}$, $\|G_2^{p(x)}\|_{L^{t_2}(B_4)}$, $\|u\|_{L^{sq'}(B_4)}^{p_+^4-p_-^4}$, $\|u\|_{L^{ss_2}(B_4)}^{p_+^4-p_-^4}$ *and* $\|u\|_{L^{sr_i}(B_4)}^{p_+^4-p_-^4}$ *for certain* $q'=\frac{q}{q-1}$, $r_0 \in (1,\infty)$ *with* $\frac{1}{q_i}+\frac{1}{q}+\frac{1}{r_i}=1$, $i=0,1,2$, $\frac{1}{t_2}+\frac{1}{q}+\frac{1}{s_2}=1$. *Here* $s \geq p_+^4 - p_-^4$ *is arbitrary.*

Proof. We proceed as in the proof of Lemma 2.2. If $p_-^4 \geq N$ we choose $\hat N = N$. If $p_-^4 < N$ we choose $\hat N$ such that $p_-^4 < \hat N < q_i p_-^4$ for $i=0,1,2$ and also $p_-^4 < \hat N < t_2 p_-^4$. Then, we choose $1 < q < \frac{\hat N}{\hat N - p_-^4}$ such that $\frac{1}{q_i}+\frac{1}{q} < 1$ for $i=0,1,2$ and $\frac{1}{t_2}+\frac{1}{q}<1$. Finally, we take $r_i \in (1,\infty)$ such that $\frac{1}{q_i}+\frac{1}{q}+\frac{1}{r_i}=1$ and $s_2 \in (1,\infty)$ such that $\frac{1}{t_2}+\frac{1}{q}+\frac{1}{s_2}=1$.

We will be calling $M_{i+2} = \Big(⨍_{B_4} u^{sr_i}\Big)^{1/sr_i}$, $i=0,1,2$, $M_1 = \Big(⨍_{B_4} u^{sq'}\Big)^{1/sq'}$, $M_5 = \Big(⨍_{B_4} u^{ss_2}\Big)^{1/ss_2}$, $M = \sum_{j=1}^5 M_j$.

The terms involving H_0, H_2 are treated exactly as the term with H in Lemma 2.2. The term involving H_1 is treated similarly. We have

$$\begin{aligned}
&⨍ H_1(x) u^{\gamma+p(x)-1}\eta^{p_+-1}|\nabla\eta| \\
&\qquad \leq \frac{C}{\rho-\sigma}\Big(⨍_{B\rho} H_1^{q_1}\Big)^{1/q_1}\Big(⨍_{B_\rho} u^{q(\gamma+p_--1)}\Big)^{1/q}\Big(⨍ u^{r_1(p^+-p_-)}\Big)^{1/r_1} \\
&\qquad \leq \frac{C}{(\rho-\sigma)^{1+\frac{N}{q_1}}}\|H_1\|_{B_4} M_3^{p_+-p_-}\Big(⨍_{B_\rho} u^{q(\gamma+p_--1)}\Big)^{1/q} \\
&\qquad \leq \frac{C}{(\rho-\sigma)^{p_+}}\|H_1\|_{B_4} M_3^{p_+-p_-}\Big(⨍_{B_\rho} u^{q(\gamma+p_--1)}\Big)^{1/q},
\end{aligned}$$

since $1+\frac{N}{q_1} < p_-^4 \leq p_- \leq p_+$.

And

$$\fint G_2^{p(x)} u^{\gamma+p(x)-1}\eta^{p_+} \le \rho^{-\frac{N}{t_2}} \|G_2^{p(x)}\|_{L^{t_2}(B_4)} \Big(\fint_{B_\rho} u^{q(\gamma+p_--1)}\Big)^{1/q}$$
$$\times \Big(\fint_{B_\rho} u^{s_2(p_+-p_-)}\Big)^{1/s_2}$$
$$\le \frac{C}{(\rho-\sigma)^{-p_+}} M_5^{p_+-p_-} \|G_2^{p(x)}\|_{L^{t_2}(B_4)},$$

since $\frac{N}{t_2} < p_-^4 \le p_- \le p_+$, $0<\rho-\sigma<\rho<4$.

Let us now look at the term involving G_1, which is bounded by

$$\frac{C}{(\rho-\sigma)^{p_+}} \|G_1^{p(x)}\|_{L^\infty(B_4)} M_1^{p_+-p_-} \Big(\fint u^{q(\gamma+p_--1)}\Big)^{1/q}.$$

Now, the proof follows with no change. □

Also, we have

Lemma 3.3 (Weak Harnack). *Let p be log-Hölder continuous in B_4. There exist $t_0>0$ such that, for $s \ge p_+^4 - p_-^4$ there exists C such that, if $u \ge 1$ and bounded is such that* $\operatorname{div} A(x,u,\nabla u) \le H_2(x)u^{p(x)-1} + G_2(x)|\nabla u|^{p(x)-1}$ *in B_4 and there exists a positive constant α such that*

1. $A(x,u(x),\nabla u(x))\cdot\nabla u(x) \ge \alpha|\nabla u(x)|^{p(x)} - H_0(x)u(x)^{p(x)}$ *in B_4,*
2. $\big|A(x,u(x),\nabla u(x))\big| \le H_1(x)u^{p(x)-1} + G_1(x)|\nabla u|^{p(x)-1}$ *in B_4,*

with $H_i \in L^{q_i}(B_4)$, $G_2^{p(x)} \in L^{t_2}(B)$ for some $\max\{1,\frac{N}{p_-^{4R}}\} < q_i, t_2 \le \infty$, *$i=0,2$,* $\max\{1,\frac{N}{p_-^4-1}\} < q_1 \le \infty$ *and $G_1 \in L^\infty(B)$ and they are nonnegative, there holds that*

$$\inf_{B_1} u \ge C\Big(\fint_{B_2} u^{t_0}\Big)^{1/t_0}. \tag{3.3}$$

The constant C depends only on s, p_+^4, p_-^4, ω_4, q_i, t_2, t, α, $\|H_i\|_{L^{q_i}(B_4)}$, $i=0,1,2$, $\|G_2^{p(x)}\|_{L^{t_2}(B_4)}$, $\|G_1^{p(x)}\|_{L^\infty(B_4)}$, $\big(\fint_{B_4} u^{sq'}\big)^{\frac{p_+^4-p_-^4}{sq'}}$, $\big(\fint_{B_4} u^{sr_i}\big)^{\frac{p_+^4-p_-^4}{sr_i}}$, $i=0,1,2$ and $\big(\fint_{B_4} u^{ss_2}\big)^{\frac{p_+^4-p_-^4}{ss_2}}$ for certain $q'=\frac{q}{q-1}$, $r_i\in(1,\infty)$ such that $\frac{1}{q_i}+\frac{1}{q}+\frac{1}{r_i}=1$, $s_2\in(1,\infty)$ such that $\frac{1}{t_2}+\frac{1}{q}+\frac{1}{s_2}=1$. Here $s \ge p_+^4-p_-^4$ is arbitrary.

Proof. We proceed as in the proof of Lemma 2.4 by using (3.2) and the ideas in Lemma 3.2. Recall that in this process we have $\gamma \le -(p_-^4-1)$.

In this way we get (2.6). As in Lemma 2.4, in order to finish the proof we need to find $t_0>0$ such that (2.7) holds for u. So we bound, by using (3.2), for an arbitrary $0<r\le 2$, $\eta\in C_0^\infty(B_{2r})$ with $0\le\eta\le1$, $\eta\equiv1$ in B_r, $|\nabla\eta|\le\frac{C}{r}$ and

$\gamma = 1 - p_-^{2r}$,

$$\begin{aligned}\fint_{B_r} |\nabla \log u|^{p_-^{2r}} &= \fint_{B_r} u^{-p_-^{2r}} |\nabla u|^{p_-^{2r}} \le C \fint_{B_{2r}} u^{-p_-^{2r}} \eta^{p_+^{2r}} |\nabla u|^{p_-^{2r}} \\ &\le C \fint_{B_{2r}} u^{-p_-^{2r}} \eta^{p_+^{2r}} + \frac{C}{(p_-^{2r}-1)} \fint (H_0 + H_2) u^{p(x)-p_-^{2r}} \eta^{p_+^{2r}} \\ &\quad + \frac{C}{(p_-^{2r}-1)} \fint H_1 u^{p(x)-p_-^{2r}} \eta^{p_+^{2r}-1} |\nabla \eta| \\ &\quad + \frac{C}{(p_-^{2r}-1)^{p_+^{2r}}} \fint G_1^{p(x)} u^{p(x)-p_-^{2r}} \eta^{p_+^{2r}-p(x)} |\nabla \eta|^{p(x)} \\ &\quad + \frac{C}{(p_-^{2r}-1)^{p_+^{2r}}} \fint G_2^{p(x)} u^{p(x)-p_-^{2r}} \eta^{p_+^{2r}}.\end{aligned}$$

So that

$$\begin{aligned}\fint_{B_r} |\nabla \log u|^{p_-^{2r}} \le C \Big[1 &+ \|H_0\|_{L^{q_0}(B_4)} r^{-N/q_0} \Big(\fint_{B_r} u^{q_0'(p_+-p_-)} \Big)^{1/q_0'} \\ &+ \|H_2\|_{L^{q_2}(B_4)} r^{-N/q_2} \Big(\fint_{B_r} u^{q_2'(p_+-p_-)} \Big)^{1/q_2'} \\ &+ \|H_1\|_{L^{q_1}(B_4)} r^{-(1+\frac{N}{q_1})} \Big(\fint_{B_r} u^{q_1'(p_+-p_-)} \Big)^{1/q_1'} \\ &+ \|G_1^{p(x)}\|_{L^\infty(B_4)} r^{-p_+^{2r}} \Big(\fint_{B_r} u^{q'(p_+-p_-)} \Big)^{1/q'} \\ &+ \|G_2^{p(x)}\|_{L^{t_2}(B_4)} r^{-N/t_2} \Big(\fint_{B_r} u^{t_2'(p_+-p_-)} \Big)^{1/t_2'} \Big].\end{aligned}$$

Now, since $q_i' < r_i$, $t_2' < s_2$,

$$\begin{aligned}\fint_{B_r} |\nabla \log u|^{p_-^{2r}} \le C \Big[1 &+ \|H_0\|_{L^{q_0}(B_4)} r^{-N/q_0} M_2^{p_+-p_-} + \|H_2\|_{L^{q_2}(B_4)} r^{-N/q_2} M_4^{p_+-p_-} \\ &+ \|H_1\|_{L^{q_1}(B_4)} r^{-(1+\frac{N}{q_1})} M_3^{p_+-p_-} + \|G_1^{p(x)}\|_{L^\infty(B_4)} r^{-p_+^{2r}} M_1^{p_+-p_-} \\ &+ \|G_2^{p(x)}\|_{L^{t_2}(B_4)} r^{-N/t_2} M_5^{p_+-p_-} \Big].\end{aligned}$$

Finally, since $0 < r \le 2$, $\frac{N}{q_i} < p_-^4$, $i = 0, 2$, $\frac{N}{t_2} < p_-^4$, $1 + \frac{N}{q_1} \le p_-^4$ and $p_-^4 \le p_-^{2r} \le p_+^{2r}$,

$$\begin{aligned}&\fint_{B_r} |\nabla \log u|^{p_-^{2r}} \\ &\quad \le C \Big[1 + \sum_{i=0}^{3} \|H_i\|_{L^{q_i}(B_4)} + \|G_1^{p(x)}\|_{L^\infty(B_4)} + \|G_2^{p(x)}\|_{L^{t_2}(B_4)} \Big] r^{-p_+^{2r}} M^{p_+^4 - p_-^4}.\end{aligned}$$

Now the proof follows in a standard way as in Lemma 2.4 □

Remark 3.1 (Improved weak Harnack)**.** With the same proof as that of Lemma 2.5 we can improve on Lemma 3.3. In fact, (3.3) holds for any $t_0 > 0$ if $p_-^4 \geq N$ and for any $0 < t_0 < \frac{N}{N-p_-^4}(p_-^4 - 1)$ if $N > p_-^4$.

Remark 3.2 (Local bounds)**.** As in the previous section, by modifying the proof of Lemmas 3.1 and 3.2, we get that if u satisfies weakly

$$\big|\operatorname{div} A(x,u,\nabla u)\big| \leq H_2(x)(|u|+1)^{p(x)-1} + G_2(x)|\nabla u|^{p(x)-1} \quad \text{in } \Omega$$

and

(1) $A(x,u(x),\nabla u(x)) \cdot \nabla u(x) \geq \alpha |\nabla u(x)|^{p(x)} - H_0(x)(|u(x)|+1)^{p(x)}$ in Ω,
(2) $\big|A(x,u(x),\nabla u(x))\big| \leq H_1(x)(|u|+1)^{p(x)-1} + G_1(x)|\nabla u|^{p(x)-1}$ in Ω,

with $0 \leq H_i \in L^{q_i(x)}(\Omega)$, $0 \leq G_1 \in L^\infty(\Omega)$, $0 \leq G_2^{p(x)} \in L^{t_2(x)}(\Omega)$ with $q_i, t_2 \in C(\Omega)$ and $\max\{1, \frac{N}{p(x)}\} < q_2(x), t_2(x)$ in Ω, $\max\{1, \frac{N}{p(x)-1}\} < q_0(x), q_1(x)$ in Ω, there holds that u is locally bounded.

Then, as in the proof of Corollary 2.2, we get that, if the structure conditions (1), (2), (3) do not depend on M_0, weak solutions to (3.1) are locally bounded. In fact, we let u be a weak solution to (3.1) and

$$\begin{aligned} H_i(x) &= g_i(x) + C_i(x), \quad i = 0,1, \\ H_2(x) &= f(x) + C_2(x), \\ G_j(x) &= K_j(x), \quad j = 1,2. \end{aligned}$$

Then,

$$\big|\operatorname{div} A(x,u,\nabla u)\big| = \big|B(x,u,\nabla u)\big| \leq H_2(x)(|u(x)|+1)^{p(x)-1} + G_2(x)|\nabla u(x)|^{p(x)-1}$$

and

(1) $A(x,u(x),\nabla u(x)) \cdot \nabla u(x) \geq \alpha |\nabla u(x)|^{p(x)} - H_0(x)(|u(x)|+1)^{p(x)}$ in Ω,
(2) $\big|A(x,u(x),\nabla u(x))\big| \leq H_1(x)(|u|+1)^{p(x)-1} + G_1(x)|\nabla u|^{p(x)-1}$ in Ω.

So, we get that u is locally bounded.

We can now prove Harnack's inequality for solutions of general elliptic equations with non-standard growth.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be bounded and let be p log-Hölder continuous in Ω. Let $A(x,s,\xi)$, $B(x,s,\xi)$ satisfy the structure conditions (1), (2) and (3) for certain nonnegative functions $g_0, C_0 \in L^{q_0}(\Omega)$, $g_1, C_1 \in L^{q_1}(\Omega)$, $f, C_2 \in L^{q_2}(\Omega)$, $K_1 \in L^\infty(\Omega)$, $K_2^{p(x)} \in L^{t_2}(\Omega)$, with $\max\{1, \frac{N}{p_1-1}\} < q_0, q_1 \leq \infty$, $\max\{1, \frac{N}{p_1}\} < q_2, t_2 \leq \infty$.*

Let $\Omega' \subset\subset \Omega$. There exists $R \leq \min\{1, \frac{1}{4}\operatorname{dist}(\Omega', \partial\Omega)\}$ such that, if $u \geq 0$ is a bounded weak solution to (3.1) in Ω, there exists and $C > 0$ such that, for every $x_0 \in \Omega'$,

$$\sup_{B_R(x_0)} u \leq C\big[\inf_{B_R(x_0)} u + R + \mu R\big]. \tag{3.4}$$

Here

$$\mu = \Big[R^{1-\frac{N}{q_2}}\|f\|_{L^{q_2}(B_{4R})}\Big]^{\frac{1}{p_-^{4R}-1}} + \Big[R^{-\frac{N}{q_0}}\|g_0\|_{L^{q_0}(B_{4R})}\Big]^{\frac{1}{p_-^{4R}-1}} + \Big[R^{-\frac{N}{q_1}}\|g_1\|_{L^{q_1}(B_{4R})}\Big]^{\frac{1}{p_-^{4R}-1}}.$$

The constant C depends only on s, p_+^{4R}, p_-^{4R}, ω_{4R}, q_i, t_2, α, $\mu^{p_+^{4R}-p_-^{4R}}$, $\|C_i\|_{L^{q_i}(B_{4R}(x_0))}$, $i=0,1,2$, $\|K_2^{p(x)}\|_{L^{t_2}(B_{4R}(x_0))}$, $\|K_1^{p(x)}\|_{L^\infty(B_{4R}(x_0))}$, $\|u\|^{p_+^{4R}-p_-^{4R}}_{L^{sq'}(B_{4R}(x_0))}$, $\|u\|^{p_+^{4R}-p_-^{4R}}_{L^{sr_i}(B_{4R}(x_0))}$ $i=0,1,2$, $\|u\|^{p_+^{4R}-p_-^{4R}}_{L^{ss_2}(B_{4R}(x_0))}$ for certain $q'=\frac{q}{q-1}$, $r_i\in(1,\infty)$ such that $\frac{1}{q_i}+\frac{1}{q}+\frac{1}{r_i}=1$, $s_2\in(1,\infty)$ such that $\frac{1}{t_2}+\frac{1}{q}+\frac{1}{s_2}=1$. Here $s\geq p_+^4-p_-^4$ is arbitrary.

Observe that $\mu^{p_+^{4R}-p_-^{4R}}$ is bounded independently of R.

Proof. Without loss of generality we will assume that $x_0=0$. Let us call

$$\begin{aligned}
H_0(x) &= \frac{g_0(Rx)}{R^{-\frac{N}{q_0}}\|g_0\|_{L^{q_0}(B_{4R})}} + R^{p(Rx)-1}C_0(Rx)\\
H_1(x) &= \frac{g_1(Rx)}{R^{-\frac{N}{q_1}}\|g_1\|_{L^{q_1}(B_{4R})}} + R^{p(Rx)-1}C_1(Rx)\\
H_2(x) &= \frac{f(Rx)}{R^{-\frac{N}{q_2}}\|f\|_{L^{q_2}(B_{4R})}} + R^{p(Rx)}C_2(Rx)\\
G_1(x) &= K_1(Rx)\\
G_2(x) &= RK_2(Rx).
\end{aligned}$$

Let

$$\bar u(x) = 1+\mu+\frac{u(Rx)}{R}, \qquad \bar p(x)=p(Rx).$$

If a function is identically zero in $B_{4R}(x_0)$ the corresponding term does not appear in the definition of the functions H_i.

Then, $\|G_1(x)^{\bar p(x)}\|_{L^\infty(B_4)} \leq \|K_1(x)^{p(x)}\|_{L^\infty(B_{4R})}$ and, for $i=0,1$,

$$\begin{aligned}
\Big(\fint_{B_4} H_i^{q_i}\Big)^{1/q_i} &\leq C_{N,q_i}\Big[1+\Big(\int_{B_{4R}} R^{q_i(p(x)-1)-N}C_i^{q_i}\Big)^{1/q_i}\Big]\\
&\leq C_{N,q_i}\big[1+\|C_i\|_{L^{q_i}(B_{4R})}\big],
\end{aligned}$$

since $q_i>\frac{N}{p_1-1}$ for $i=0,1$ and $0<R\leq 1$.

On the other hand, since $q_2>\frac{N}{p_1}$,

$$\begin{aligned}
\Big(\fint_{B_4} H_2^{q_2}\Big)^{1/q_2} &\leq C_{N,q_2}\Big[1+\Big(\int_{B_{4R}} R^{q_2p(x)-N}C_2^{q_2}\Big)^{1/q_2}\Big]\\
&\leq C_{N,q_2}\big[1+\|C_2\|_{L^{q_2}(B_{4R})}\big],
\end{aligned}$$

and, since $t_2 > \frac{N}{p_1}$,

$$\Big(\fint_{B_4} G_2^{t_2\bar p(x)}\Big)^{1/t_2} \le C_{N,t_2}\Big[1+\Big(\int_{B_{4R}} R^{t_2p(x)-N}K_2^{t_2p(x)}\Big)^{1/t_2}\Big]$$
$$\le C_{N,q_2}\big[1+\|K_2^{p(x)}\|_{L^{t_2}(B_{4R})}\big].$$

On the other hand, for $0<R\le 1$ let

$$\bar A(x,s,\xi) := A\big(Rx, R(s-1-\mu),\xi\big).$$

Then, $\bar A\big(x,\bar u(x),\nabla\bar u(x)\big) = A\big(Rx,u(Rx),\nabla u(Rx)\big)$ and we have

$$\Big|\operatorname{div}\bar A\big(x,\bar u(x),\nabla\bar u(x)\big)\Big| \le Rf(Rx)+RC_2(Rx)u(Rx)^{p(Rx)-1}$$
$$+RK_2(Rx)|\nabla u(Rx)|^{p(Rx)-1}$$
$$\le H_2(x)\bar u(x)^{\bar p(x)-1}+G_2(x)|\nabla\bar u(x)|^{\bar p(x)-1}.$$

Also,

$$\Big|\bar A\big(x,\bar u(x),\nabla\bar u(x)\big)\Big| \le g_1(Rx)+C_1(Rx)u(Rx)^{p(Rx)-1}+K_1(Rx)|\nabla u(Rx)|^{p(Rx)-1}$$
$$\le H_1(x)\bar u(x)^{\bar p(x)-1}+G_1(x)|\nabla\bar u(x)|^{\bar p(x)-1}$$

and

$$\bar A\big(x,\bar u(x),\nabla\bar u(x)\big)\cdot\nabla\bar u(x) \ge \alpha|\nabla u(Rx)|^{p(Rx)}-C_0(Rx)u(Rx)^{p(Rx)-1}-g_0(Rx)$$
$$\ge \alpha|\nabla\bar u(x)|^{\bar p(x)}-H_0(x)\bar u(x)^{\bar p(x)-1}.$$

Thus, since $\bar u\ge 1$ and

$$\|\bar u\|_{L^t(B_4)}^{\bar p_+^4-\bar p_-^4} \le C\big[1+\mu^{p_+^{4R}-p_-^{4R}}+R^{-\frac{N}{t}(p_+^{4R}-p_-^{4R})}\|u\|_{L^t(B_{4R})}^{p_+^{4R}-p_-^{4R}}\big]$$
$$\le C\big[1+\mu^{p_+^{4R}-p_-^{4R}}+\|u\|_{L^t(B_{4R})}^{p_+^{4R}-p_-^{4R}}\big],$$

by applying Lemmas 3.2 and 3.3 to $\bar u$ we get the result. □

Remark 3.3. Since p is continuous in $\overline{\Omega}$ we can choose R small enough in such a way that, by choosing s small enough, $M_j^{p_+^{4R}-p_-^{4R}} \le \big(\fint_{B_{4R}(x_0)} u^{p_1}\big)^{\frac{p_+^{4R}-p_-^{4R}}{p_1}} \le c\big(1+\big(\int_\Omega u^{p(x)}\big)^{\frac{p_2}{p_1}-1}\big)$, $j=1,\dots,5$, where $p_1=\inf_\Omega p$, $p_2=\sup_\Omega p$ and the constant c depends only on the log-Hölder modulus of continuity of p in Ω.

So that, if moreover the constant α and the functions $g_0,g_1,f,C_0,C_1,C_2,K_1$ and K_2 in the structure conditions do not depend on M_0, Harnack's inequality holds —on small enough balls depending only on p— for any nonnegative weak solution, with a constant C depending on u only through $\big(\int_\Omega u^{p(x)}\big)^{\frac{p_2}{p_1}-1}$.

From Harnack's inequality we get Hölder continuity. There holds

Corollary 3.1. *Let $\Omega \subset \mathbb{R}^N$ bounded. Let p be log-Hölder continuous in Ω and $p_1 = \inf_\Omega p(x)$. Let $A(x,s,\xi)$, $B(x,s,\xi)$ satisfy the structure conditions (1), (2), (3) at the beginning of the section. Assume that $g_0, C_0 \in L^{q_0}(\Omega)$, $g_1, C_1 \in L^{q_1}(\Omega)$ and $\max\{1, \frac{N}{p_1-1}\} < q_0, q_1 \leq \infty$, $f, C_2 \in L^{q_2}(\Omega)$, $K_2^{p(x)} \in L^{t_2}(\Omega)$ and $\max\{1, \frac{N}{p_1}\} < q_2, t_2 \leq \infty$. Finally, assume $K_1 \in L^\infty(\Omega)$.*

Then, there holds that any bounded weak solution to (3.1) *is locally Hölder continuous in Ω.*

If the functions in the structure conditions are independent of M_0, any weak solution is locally Hölder continuous and the constant and Hölder exponent are independent of the L^∞ bound.

Proof. Under these assumptions, for every $M_0 > 0$, $\Omega' \subset\subset \Omega$, there exist a universal constant C, a radius $R_0 > 0$ and $\delta > 0$ such that for every $0 < R \leq R_0$, $x_0 \in \Omega'$ and any weak solution $0 \leq v \leq M_0$,

$$\sup_{B_R(x_0)} v \leq C\big[\inf_{B_R(x_0)} v + R^\delta\big]. \tag{3.5}$$

In fact, we apply (3.4) and observe that we are assuming that $q_0, q_1 > \frac{N}{p_1-1}$. So that $1 - \frac{N}{q_0}\frac{1}{p_-^{4R}-1} \geq 1 - \frac{N}{q_0}\frac{1}{p_1-1} := \delta_0 > 0$, $1 - \frac{N}{q_1}\frac{1}{p_-^{4R}-1} \geq 1 - \frac{N}{q_1}\frac{1}{p_1-1} := \delta_1 > 0$. On the other hand, if $q_2 \geq N$, $1 + \big(1 - \frac{N}{q_2}\big)\frac{1}{p_-^{4R}-1} \geq 1 + \big(1 - \frac{N}{q_2}\big)\frac{1}{p_2-1} := \delta_2 \geq 1$, if $\frac{N}{p_1} < q_2 < N$, $1 + \big(1 - \frac{N}{q_2}\big)\frac{1}{p_-^{4R}-1} \geq 1 + \big(1 - \frac{N}{q_2}\big)\frac{1}{p_1-1} := \bar{\delta}_2 > 0$.

Once we have (3.5), we deduce that u is Hölder continuous in a standard way by applying (3.5) with $R = R_0 2^{-(j+1)}$ to $v_1(x) = \sup_{B_{R_0 2^{-j}(x_0)}} u - u(x)$ and to $v_2(x) = u(x) - \inf_{B_{R_0 2^{-j}(x_0)}} u$. Here, $M_0 = \sup_\Omega u$ (see [9] for the details).

Recall that, when the functions in the structure condition are independent of M_0, any weak solution is locally bounded. So that they are locally Hölder continuous and the Hölder exponent and constant are independent of the L^∞ bounds. ☐

Now, we assume that A and B satisfy the following structure conditions: For every $M_0 > 0$ there exist a constant α and nonnegative functions f, g_0, g_1, C_0, C_1, C_2, K_1, K_2 as before and $b \in \mathbb{R}_{>0}$ such that, for every $x \in \Omega$, $|s| \leq M_0$, $\xi \in \mathbb{R}^N$,

(1) $A(x,s,\xi)\cdot\xi \geq \alpha|\xi|^{p(x)} - C_0|s|^{p(x)} - g_0(x)$,
(2) $\big|A(x,s,\xi)\big| \leq g_1(x) + C_1|s|^{p(x)-1} + K_1|\xi|^{p(x)-1}$,
(3') $\big|B(x,s,\xi)\big| \leq f(x) + C_2|s|^{p(x)-1} + K_2|\xi|^{p(x)-1} + b|\xi|^{p(x)}$.

We will prove Harnack's inequality for bounded weak solutions.

In fact, for $0 \leq u \leq M_0$ we can reduce the problem to the case of $b = 0$ treated before since, on one hand, there holds that

$$\operatorname{div} A(x,u,\nabla u) \geq -\big(f(x) + C_2(x)u^{p(x)-1} + K_2(x)|\nabla u|^{p(x)-1} + b|\nabla u|^{p(x)}\big) \quad \text{in } B_r$$
$$\Rightarrow$$
$$\operatorname{div} \widetilde{A}(x,u,\nabla u) \geq -\big(f(x) + C_2(x)u^{p(x)-1} + K_2(x)|\nabla u|^{p(x)-1}\big) \quad \text{in } B_r,$$

with $\widetilde{A}(x,s,\xi) = e^{\frac{b}{\alpha}(s-M_0)}A(x,s,\xi)$ satisfying

(1) $\widetilde{A}(x,u(x),\nabla u(x))\cdot\nabla u(x)\geq \alpha e^{-\frac{b}{\alpha}M_0}|\nabla u(x)|^{p(x)}-C_0(x)u(x)^{p(x)}-g_0(x)$,
(2) $\big|\widetilde{A}(x,u(x),\nabla u(x))\big|\leq g_1(x)+C_1(x)|u(x)|^{p(x)-1}+K_1(x)|\nabla u(x)|^{p(x)-1}$.

On the other hand, again for $0\leq u\leq M_0$ there holds that

$$\operatorname{div} A(x,u,\nabla u)\leq f(x)+C_2(x)u^{p(x)-1}+K_2(x)|\nabla u|^{p(x)-1}+b|\nabla u|^{p(x)}\quad \text{in } B_r$$
$$\Rightarrow$$
$$\operatorname{div}\bar{A}(x,u,\nabla u)\leq e^{\frac{b}{\alpha}M_0}\big(f(x)+C_2(x)u^{p(x)-1}+K_2(x)|\nabla u|^{p(x)-1}\big)\quad \text{in } B_r$$

with $\bar{A}(x,s,\xi)=e^{\frac{b}{\alpha}(M_0-s)}A(x,s,\xi)$ satisfying,

(1) $\bar{A}(x,u(x),\nabla u(x))\cdot\nabla u(x)\geq \alpha|\nabla u(x)|^{p(x)}-e^{\frac{b}{\alpha}M_0}\big(C_0(x)u(x)^{p(x)}+g_0(x)\big)$.
(2) $\big|\bar{A}(x,u(x),\nabla u(x))\big|\leq e^{\frac{b}{\alpha}M_0}\big(g_1(x)+C_1(x)|u(x)|^{p(x)-1}+K_1(x)|\nabla u(x)|^{p(x)-1}$.

Thus, there holds

Theorem 3.2. *Let $\Omega\subset\mathbb{R}^N$ be bounded and let p be log-Hölder continuous in Ω. Let $A(x,s,\xi)$, $B(x,s,\xi)$ satisfy the structure conditions (1), (2), (3'). Let $u\geq 0$ be a bounded weak solution to* (3.1) *and let M_0 be such that $u\leq M_0$ in Ω. Let $\Omega'\subset\subset\Omega$. There exists $R_0\leq\min\{1,\frac14\operatorname{dist}(\Omega',\partial\Omega)\}$ such that if $x_0\in\Omega'$ and $0<R\leq R_0$,*

$$\sup_{B_R(x_0)} u\leq C\big[\inf_{B_R(x_0)} u+R+\mu R\big],$$

where

$$\mu=\Big[R^{1-\frac{N}{q_0}}\|f\|_{L^{q_2}(B_{4R})}\Big]^{\frac{1}{p_-^{4R}-1}}+\Big[R^{-\frac{N}{q_0}}\|g_0\|_{L^{q_0}(B_{4R})}\Big]^{\frac{1}{p_-^{4R}-1}}$$
$$+\Big[R^{-\frac{N}{q_1}}\|g_1\|_{L^{q_1}(B_{4R})}\Big]^{\frac{1}{p_-^{4R}-1}}.$$

The constant C depends only on bM_0, α, s, q_i, $i=0,1,2$, the log-Hölder modulus of continuity of p in Ω, $\mu^{p_+^{4R}-p_-^{4R}}$, and $M^{p_+^{4R}-p_-^{4R}}$, where $p_+=\sup_{B_{4R}(x_0)}p$, $p_-=\inf_{B_{4R}(x_0)}p$, $\|K_1^{p(x)}\|_{L^\infty(B_{4R}(x_0))}$, $\|K_2^{p(x)}\|_{L^{t_2}(B_{4R}(x_0))}$, $M=\sum_{j=1}^4 M_j$ and $M_1=\big(⨍_{B_{4R}(x_0)}u^{sq'}\big)^{1/sq'}$, $M_{i+2}=\big(⨍_{B_{4R}(x_0)}u^{sr_i}\big)^{1/sr_i}$ for certain $q'=\frac{q}{q-1}$ depending on q_i, p_1 and N and $r_i\in(1,\infty)$, $i=0,1,2$ with $\frac{1}{q_i}+\frac{1}{q}+\frac{1}{r_i}=1$. Here $s\geq p_+-p_-$ is arbitrary.

Observe that $\mu^{p_+^{4R}-p_-^{4R}}$ and $M^{p_+^{4R}-p_-^{4R}}$ are bounded independently of R.

Proof. Theorem 3.2 is obtained from Lemmas 3.2 and 3.3 applied to $\bar{u}$ with the operator A replaced by $\widetilde{A}$ and $\bar{A}$ respectively. □

With the same proof as that of Corollary 3.1 we get the following regularity result.

Corollary 3.2. *Let $\Omega\subset\mathbb{R}^N$ be bounded. Let A and B satisfy the structure conditions (1),(2), (3'). Let u be a bounded weak solution to* (3.1) *in Ω with p log-Hölder continuous. Then u is locally Hölder continuous in Ω.*

Remark 3.4. Observe that under condition (3') the constant in Harnack's inequality and the Hölder exponent and constant of a bounded weak solution depend explicitly on the L^∞ bound.

4. Strong maximum principle for $p(x)$-superharmonic functions

In this section we prove the strong maximum principle for $p(x)$-superharmonic functions. As stated at the introduction, the strong maximum principle cannot be deduced from Harnack's inequality as in the case p constant. Instead, we will use some barriers constructed in [8].

Proposition 4.1 (Lemma B.4 in [8])**.** *Suppose that $p(x)$ is Lipschitz continuous. Let $w_\mu = Me^{-\mu|x|^2}$, for $M > 0$ and $r_1 \geq |x| \geq r_2 > 0$. Then there exist $\mu_0, \varepsilon_0 > 0$ such that, if $\mu > \mu_0$ and $\|\nabla p\|_\infty \leq \varepsilon_0$,*

$$\mu^{-1}e^{\mu|x|^2}M^{-1}|\nabla w|^{2-p}\Delta_{p(x)}w_\mu \geq C_1(\mu - C_2\|\nabla p\|_\infty|\log M|) \quad \text{in } B_{r_1}\setminus B_{r_2}.$$

Here C_1, C_2 depend only on r_2, r_1, p_+, p_-, $\mu_0 = \mu_0(p_+, p_-, N, \|\nabla p\|_\infty, r_2, r_1)$, and $\varepsilon_0 = \varepsilon_0(p_+, p_-, r_1, r_2)$.

Then we have

Corollary 4.1. *Suppose that $p(x)$ is Lipschitz continuous. Let $A_0 > 0$. Then, there exists $\delta_0 > 0$ depending on p_+, p_-, $\|\nabla p\|_\infty$ and A_0, and for every $0 < A \leq A_0$ there exists $\mu_0 > 0$ depending on the same constants and also on A such that, if moreover $\delta \leq \delta_0$ and $\mu \geq \mu_0$, the function*

$$w(x) = A\,\frac{e^{-\mu\frac{|x-x_0|^2}{\delta^2}} - e^{-\mu}}{e^{-\frac{\mu}{4}} - e^{-\mu}}$$

satisfies

$$\begin{cases} \Delta_{p(x)}w \geq 0 & \text{in } B_\delta(x_0)\setminus B_{\delta/2}(x_0), \\ w = 0 & \text{on } \partial B_\delta(x_0), \\ w = A & \text{on } \partial B_{\delta/2}(x_0). \end{cases}$$

Proof. Set $\bar{w}(x) = \frac{1}{\delta}w(x_0 + \delta x)$, $\bar{p}(x) = p(x_0 + \delta x)$. Let $M = \frac{A}{e^{-\frac{\mu}{4}} - e^{-\mu}}$. Then,

$$\bar{w}(x) = M\,e^{-\mu|x|^2} + c, \qquad |\nabla\bar{p}(x)| = \delta|\nabla p(x_0 + \delta x)|.$$

Hence, by Proposition 4.1, if δ is small and μ is large depending only on p_+, p_- and $\|\nabla p\|_\infty$,

$$\mu^{-1}e^{\mu|x|^2}M^{-1}|\nabla\bar{w}|^{2-\bar{p}}\Delta_{\bar{p}(x)}\bar{w}(x) \geq C_1(\mu - C_2\|\nabla\bar{p}\|_\infty|\log M|)\text{in } B_1\setminus B_{1/2}.$$

Observe that $M = Ae^{\mu/4}\frac{1}{1-e^{-3\mu/4}}$. Therefore, if μ is large there holds that

$$1 \leq M \leq 4Ae^{\mu/4},$$

so that

$$|\log M| \leq A\mu.$$

Hence, in this situation,

$$\mu^{-1}e^{\mu|x|^2}M^{-1}|\nabla\bar{w}|^{2-\bar{p}}\Delta_{\bar{p}(x)}\bar{w}(x) \geq C_1(1-C_2\delta\|\nabla p\|_\infty A)\mu \geq 0 \quad \text{in } B_1\setminus B_{1/2}$$

if, moreover, δ is small depending on C_1, C_2, A_0 and $\|\nabla p\|_\infty$. □

We can now prove our main result in this section. We follow the ideas of the proof in [17] for the case p constant.

Theorem 4.1. *Suppose that $p(x)$ is Lipschitz continuous. Let $\Omega\subset\mathbb{R}^N$ be connected and $0\leq u\in C^1(\Omega)$ such that $\Delta_{p(x)}u\leq 0$ in Ω. Then, either $u\equiv 0$ in Ω or $u>0$ in Ω.*

Proof. Assume the result is not true. Then, since Ω is connected, $\partial\{u>0\}\cap\Omega\neq\emptyset$. Let $x_1\in\{u>0\}$ such that $\operatorname{dist}(x_1,\partial\{u>0\})<\operatorname{dist}(x_1,\partial\Omega)$, and let $y\in\partial\{u>0\}\cap\Omega$ such that $r=|x_1-y|=\operatorname{dist}(x_1,\partial\{u>0\})$. Let $A_0=\sup_{B_r(x_1)}u$. Let δ_0 be the constant in Corollary 4.1. By choosing x_0 on the line between x_1 and y and taking $\delta=|x_0-y|$ we may assume that $\delta\leq\delta_0$ and $B_\delta(x_0)\subset\{u>0\}$. Let now $A=\inf_{\partial B_{\delta/2}(x_0)}u$. Then, $0<A\leq A_0$. Therefore, by taking w as in Corollary 4.1 we have

$$u(x)\geq w(x)\geq 0 \quad \text{in } B_\delta(x_0)\setminus B_{\delta/2}(x_0).$$

Since $u(y)=w(y)=0$, there holds that

$$|\nabla u(y)|\geq|\nabla w(y)|>0.$$

But this is a contradiction since $y\in\partial\{u>0\}\cap\Omega$, $u\geq 0$ in Ω and $u\in C^1(\Omega)$ so that $\nabla u(y)=0$. □

Remark 4.1. Recall that in [2] it was proved that solutions to $\Delta_{p(x)}u=0$ are $C^{1,\alpha}_{\text{loc}}$. Thus, Theorem 4.1 applies to nonnegative weak solutions.

With a similar proof we get

Theorem 4.2. *Under the assumptions of Theorem 4.1, if, moreover, there exists $y\in\partial\Omega$ such that there is a ball B contained in Ω such that $y\in\partial B$, $u\in C(\overline{B})$, $u>0$ in B and $u(y)=0$, then for $x\in B$ close enough to y there holds that $u(x)\geq c_0(x-y)\cdot\nu$, where $c_0>0$ and ν is the unitary direction from y to the center of the ball B.*

If, moreover, $u\in C^1(\Omega\cup\{y\})$, there holds that either $u\equiv 0$ in Ω or else $\frac{\partial u(y)}{\partial\nu}>0$. Here ν is as above.

References

[1] T. Adamowicz, P. Hästö, *Harnack's inequality and the strong $p(x)$-Laplacian*, J. Differential Equations **250** (2011), 1631–1649. MR 2737220.

[2] E. Acerbi, G. Mingione, *Regularity results for a class of functionals with non-standard growth*, Arch. Ration. Mech. Anal. **156** (2001), 121–140. MR 1814973.

[3] R. Aboulaich, D. Meskine, A. Souissi, *New diffusion models in image processing*, Comput. Math. Appl. **56** (2008), 874–882. MR 2437859.

[4] Y. Chen, S. Levine, M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math. **66** (2006), 1383–1406. MR 2246061.

[5] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Lebesque and Sobolev Spaces with variable exponents, Lecture Notes in Mathematics 2017, Springer, 2011. MR 2790542.

[6] X. Fan, *Global $C^{1,\alpha}$ regularity for variable exponent elliptic equations in divergence form*, J. Differential Equations **235** (2007), 397–417. MR 2317489.

[7] X. Fan, D. Zhao, *A class of De Giorgi type and Hölder continuity*, Nonlinear Anal. TM&A **36** (1999), 295–318. MR 1688232.

[8] J. Fernández Bonder, S. Martínez, N. Wolanski, *A free boundary problem for the $p(x)$-Laplacian*, Nonlinear Anal. **72** (2010), 1078–1103. MR 2579371.

[9] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, 2nd edition, Grundlehren der Mathematischen Wissenschaften, vol. 224, Springer-Verlag, Berlin, 1983. MR 0737190.

[10] P. Harjulehto, P. Hasto, M. Koskenoja, T. Lukkari, N. Marola, *An obstacle problem and superharmonic functions with nonstandard growth*, Nonlinear Anal. **67** (2007), 3424–3440. MR 2350898.

[11] P. Harjulehto, T. Kuusi, T. Lukkari, N. Marola, M. Parviainen, *Harnack's inequality for quasiminimizers with nonstandard growth conditions*, J. Math. Anal. Appl. **344** (2008), 504–520. MR 2416324.

[12] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear potential theory of degenerate elliptic equations, Oxford University Press, New York, 1993. MR 1207810.

[13] O. Kováčik, J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J. **41** (1991), 592–618. MR 1134951.

[14] J. Malý, W. Ziemer, Fine regularity of solutions of elliptic partial differential equations, Math. Surveys and Monographs, vol. 51, American Mathematical Society, 1997. MR 1461542.

[15] M. Růžička, Electrorheological fluids: modeling and mathematical theory, Lecture Notes in Mathematics, 1748, Springer-Verlag, Berlin, 2000. MR 1810360.

[16] J. Serrin, *Local behavior of solutions of quasi-linear equations*, Acta Math. **111** (1964), 247–302. MR 0170096.

[17] Vázquez, J. L., *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12**, (1984) 191–202. MR 0768629.

[18] X. Zhang, X. Liu, *Local boundedness and Harnack inequality of $p(x)$-Laplace equation*, J. Math. Anal. Appl. **332** (2007), 209–218. MR 2319655.

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