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INEQUALITIES FOR SUBMANIFOLDS OF A RIEMANNIAN MANIFOLD OF NEARLY QUASI-CONSTANT CURVATURE WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. By using two new algebraic lemmas we obtain Chen's inequalities for submanifolds of a Riemannian manifold of nearly quasi-constant curvature endowed with a semi-symmetric non-metric connection. Moreover, we correct a result of C. Özgür and A. Mihai's paper (*Chen inequalities for submanifolds of real space forms with a semi-symmetric non-metric connection*, Canad. Math. Bull. 55 (2012), 611–622).

1. Introduction

According to B.-Y. Chen [1], one of the most important problems in submanifold theory is to find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Recently the study of this topic has attracted a lot of attention [1, 2, 3, 4, 5]. Related with the famous Nash embedding theorem [6], B.-Y. Chen introduced a new type of Riemannian invariants, known as δ*-invariants* [3, 4, 5]. The author's original motivation was to provide answers to a question raised by S. S. Chern concerning the existence of minimal isometric immersions into Euclidean space [7]. Therefore, B.-Y. Chen obtained a necessary condition for the existence of minimal isometric immersion from a given Riemannian manifold into Euclidean space and established inequalities for submanifolds in real space forms in terms of the sectional curvature, the scalar curvature and the squared mean curvature [8]. Later, he established general inequalities relating $\delta(n_1, \ldots, n_k)$ and the squared mean curvature for submanifolds in real space forms [9]. Similar inequalities also hold for Lagrangian submanifolds of complex space forms. However, in [10], B.-Y. Chen proved that, for any $\delta(n_1, \ldots, n_k)$, the equality case holds at a single point if and only if the Lagrangian submanifold is minimal at that point. This indicated that the inequality was not optimal and inspired people to look for a more optimal inequality. In 2007, T. Oprea improved the inequality on $\delta(2)$ for Lagrangian submanifolds in complex space forms [11].

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Recently, B.-Y. Chen and F. Dillen established general inequalities for Lagrangian submanifolds in complex space forms and provided some examples showing that these new improved inequalities are the best possible [12]. However, it was pointed out in [13] that the proof of the general inequality given in [12] is incorrect when $\sum_{i=1}^{k} \frac{1}{2+n_i} > \frac{1}{3}$. In [14], B.-Y. Chen, F. Dillen, J. Van der Veken and L. Vrancken corrected the proof of the general inequality in the case $n_1 + \cdots + n_k < n$ and showed that the inequality can be improved in the case $n_1 + \cdots + n_k = n$.

Such invariants and inequalities have many nice applications to several areas in mathematics [15].

Afterwards, many papers studied Chen's inequalities for different submanifolds in various ambient spaces, like generalized S-space forms [16], generalized complex space forms [17], (κ, μ)-contact space forms [18], Riemannian manifold of nearly quasi-constant curvature [19] and Sasakian space forms [20].

Recently, C. Özgür and A. Mihai proved Chen's inequalities for submanifolds of real space forms endowed with a semi-symmetric non-metric connection [21]. In this paper, we obtain Chen's first inequalities and Chen–Ricci inequalities for submanifolds of a Riemannian manifold of nearly quasi-constant curvature endowed with a semi-symmetric non-metric connection by using algebraic lemmas. We should point out that our approaches are different from B.-Y. Chen's. Moreover, we show that a result of C. Özgür and A. Mihai [21, Theorem 4.1] is incorrect. For the sake of correcting the result, we establish Chen–Ricci inequalities for submanifolds of real space forms with a semi-symmetric non-metric connection in Section 5.

2. Preliminaries

To meet the requirements in the next sections, here the basic elements of the theory of a Riemannian manifold endowed with a semi-symmetric non-metric connection are briefly presented.

Let N^{n+p} be an $(n+p)$-dimensional Riemannian manifold with Riemannian metric g and the linear connection $\overline{\nabla}$. For the vector fields $\overline{X}, \overline{Y}$ on N^{n+p} the torsion tensor field $\overline{T}$ of the linear connection $\overline{\nabla}$ is defined by $\overline{T}(\overline{X}, \overline{Y}) = \overline{\nabla}_{\overline{X}}\overline{Y} - \overline{\nabla}_{\overline{Y}}\overline{X} - [\overline{X}, \overline{Y}]$. A linear connection $\overline{\nabla}$ is said to be a semi-symmetric connection if the torsion tensor $\overline{T}$ of the connection $\overline{\nabla}$ satisfies $\overline{T}(\overline{X}, \overline{Y}) = \phi(\overline{Y})\overline{X} - \phi(\overline{X})\overline{Y}$, where ϕ is a 1-form on N^{n+p}. Further, if $\overline{\nabla}$ satisfies $\overline{\nabla} g = 0$, then $\overline{\nabla}$ is called a semi-symmetric metric connection [22]. If $\overline{\nabla} g \neq 0$, then $\overline{\nabla}$ is called a semi-symmetric non-metric connection [23]. Suppose $\hat{\overline{\nabla}}$ is the Levi–Civita connection of N. Following [23], we define a semi-symmetric connection $\overline{\nabla}$ given by

$$\overline{\nabla}_{\overline{X}}\overline{Y} = \hat{\overline{\nabla}}_{\overline{X}}\overline{Y} + \phi(\overline{Y})\overline{X},$$

where ϕ is a 1-form on N. This clearly is a semi-symmetric non-metric connection. As ϕ is a 1-form we can introduce a dual vector field P by

$$g(P, \overline{X}) = \phi(\overline{X}).$$

Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional manifold N^{n+p} with the semi-symmetric non-metric connection $\overline{\nabla}$ and the Levi–Civita connection $\hat{\overline{\nabla}}$. On M^n we consider the induced semi-symmetric non-metric connection denoted by ∇ and the induced Levi–Civita connection denoted by $\hat{\nabla}$. Let $\overline{R}$ be the curvature tensor of N^{n+p} with respect to $\overline{\nabla}$ and $\hat{\overline{R}}$ the curvature tensor of N^{n+p} with respect to $\hat{\overline{\nabla}}$. We also denote by R and $\hat{R}$ the curvature tensors associated to ∇ and $\hat{\nabla}$.

The Gauss formulas with respect to ∇, respectively $\hat{\nabla}$, can be written as follows

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \hat{\overline{\nabla}}_X Y = \hat{\nabla}_X Y + \hat{h}(X,Y),$$

for any vector fields X, Y on M^n, where h is a $(0,2)$ symmetric tensor on M^n and $\hat{h}$ is the second fundamental form associated to the Levi–Civita connection $\hat{\nabla}$. According to formula (3.4) in [24] we have

$$h = \hat{h}. \tag{2.1}$$

We will consider a Riemannian manifold N^{n+p} of *nearly quasi-constant curvature* [25] endowed with a semi-symmetric non-metric connection $\overline{\nabla}$ and the Levi–Civita connection $\hat{\overline{\nabla}}$. From [25], for any vector fields $\overline{X}, \overline{Y}, \overline{Z}, \overline{W}$ on N^{n+p}, the curvature tensor $\hat{\overline{R}}$ with respect to the Levi–Civita connection $\hat{\overline{\nabla}}$ on N^{n+p} is expressed by

$$\begin{aligned}\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = {} & a[g(\overline{X},\overline{Z})g(\overline{Y},\overline{W}) - g(\overline{X},\overline{W})g(\overline{Y},\overline{Z})] \\ & + b[g(\overline{Y},\overline{W})B(\overline{X},\overline{Z}) - g(\overline{Y},\overline{Z})B(\overline{X},\overline{W}) \\ & + g(\overline{X},\overline{Z})B(\overline{Y},\overline{W}) - g(\overline{X},\overline{W})B(\overline{Y},\overline{Z})],\end{aligned} \tag{2.2}$$

where a, b are scalar functions and B is a non-zero symmetric tensor of type $(0,2)$. If $b = 0$, it can be easily seen that the manifold reduces to a space of constant curvature.

A non-flat Riemannian manifold (N^m, g), $m > 2$, is defined to be a nearly quasi-Einstein manifold if its Ricci tensor satisfies the condition

$$S(\overline{X},\overline{Y}) = cg(\overline{X},\overline{Y}) + dE(\overline{X},\overline{Y}),$$

where $\overline{X}, \overline{Y}$ are any vector fields on N and c and d are non zero scalar functions and E is a non-zero symmetric tensor of type $(0,2)$ [26]. It can be easily seen that every Riemannian manifold of nearly quasi-constant curvature is a nearly quasi-Einstein manifold.

Example 2.1 ([26])**.** *Let $(\mathbb{R}^4, g)$ be a Riemannian manifold endowed with the metric given by*

$$ds^2 = g_{ij}dx^i dx^j = (x^4)^{\frac{4}{3}}[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + (dx^4)^2.$$

Then $(\mathbb{R}^4, g)$ is a manifold of nearly quasi-constant curvature. Detailed explanations were given in [26] *(see also* [19]*).*

The *curvature tensor* $\overline{R}$ with respect to the semi-symmetric non-metric connection $\overline{\nabla}$ on N^{n+p} can be written as (see [23])

$$\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) = \hat{\overline{R}}(\overline{X},\overline{Y},\overline{Z},\overline{W}) + s(\overline{Y},\overline{Z})g(\overline{X},\overline{W}) - s(\overline{X},\overline{Z})g(\overline{Y},\overline{W}), \tag{2.3}$$

for any vector fields $\overline{X},\overline{Y},\overline{Z},\overline{W}$ on N, where s is a $(0,2)$-tensor field defined by

$$s(\overline{X},\overline{Y}) = (\hat{\overline{\nabla}}_{\overline{X}}\phi)\overline{Y} - \phi(\overline{X})\phi(\overline{Y}).$$

From (2.2) and (2.3) it follows that the *curvature tensor* $\overline{R}$ can be expressed as

$$\begin{aligned}\overline{R}(\overline{X},\overline{Y},\overline{Z},\overline{W}) &= a[g(\overline{X},\overline{Z})g(\overline{Y},\overline{W}) - g(\overline{X},\overline{W})g(\overline{Y},\overline{Z})]\\ &\quad + b[g(\overline{Y},\overline{W})B(\overline{X},\overline{Z}) - g(\overline{Y},\overline{Z})B(\overline{X},\overline{W})\\ &\quad + g(\overline{X},\overline{Z})B(\overline{Y},\overline{W}) - g(\overline{X},\overline{W})B(\overline{Y},\overline{Z})]\\ &\quad + s(\overline{Y},\overline{Z})g(\overline{X},\overline{W}) - s(\overline{X},\overline{Z})g(\overline{Y},\overline{W}).\end{aligned} \tag{2.4}$$

Decomposing the vector field P on M uniquely into its tangent and normal components P^T and $P^\perp$, respectively, we have $P = P^T + P^\perp$. For any vector fields X,Y,Z,W on M, the Gauss equation with respect to the semi-symmetric non-metric connection is (see [24])

$$\begin{aligned}R(X,Y,Z,W) &= \overline{R}(X,Y,Z,W) + g(h(X,Z),h(Y,W)) - g(h(X,W),h(Y,Z))\\ &\quad + g(P^\perp,h(Y,Z))g(X,W) - g(P^\perp,h(X,Z))g(Y,W).\end{aligned} \tag{2.5}$$

In N^{n+p} we can choose a local orthonormal frame

$$e_1,\ldots,e_n,e_{n+1},\ldots,e_{n+p}, \tag{2.6}$$

such that, restricting to M^n, $e_1,e_2,\ldots,e_n$ are tangent to M^n. We write $h^r_{ij} = g(h(e_i,e_j),e_r)$. The squared length of h is $\|h\|^2 = \sum\limits_{i,j=1}^{n} g(h(e_i,e_j),h(e_i,e_j))$ and the *mean curvature vector* of M associated to ∇ is $H = \frac{1}{n}\sum\limits_{i=1}^{n} h(e_i,e_i)$. Similarly, the *mean curvature vector* of M^n associated to $\hat{\nabla}$ is $\hat{H} = \frac{1}{n}\sum\limits_{i=1}^{n}\hat{h}(e_i,e_i)$. Combining with (2.1) we have

$$H = \hat{H}.$$

Let $\pi \subset T_xM$ and $\pi^\perp \subset T_x^\perp M$ be plane sections for any x in M^n and $K(\pi)$ the sectional curvature of M^n associated to the induced semi-symmetric non-metric connection ∇. The *scalar curvature* τ at x is defined by

$$\tau(x) = \sum_{1\le i<j\le n} K_{ij}. \tag{2.7}$$

Suppose L is an l-dimensional subspace of T_xM, $x \in M$, $l \geq 2$ and $\{e_1,\ldots,e_l\}$ an orthonormal basis of L. We define the *scalar curvature* $\tau(L)$ of the l-plane L by

$$\tau(L) = \sum_{1\le\mu<\nu\le l} K(e_\mu \wedge e_\nu). \tag{2.8}$$

For simplicity we put

$$\begin{aligned} \Psi_1(L) &= \sum_{1\le i<j\le l} [s(e_i,e_i)+\phi(h(e_i,e_i))], \\ \Psi_2(L) &= \sum_{1\le i<j\le l} [B(e_i,e_i)^2+B(e_j,e_j)^2]. \end{aligned} \tag{2.9}$$

For an integer $k \ge 0$ we denote by $S(n,k)$ the set of k-tuples $(n_1,\dots,n_k)$ of integers ≥ 2 satisfying $n_1 < n$ and $n_1+\cdots+n_k \le n$. We denote by $S(n)$ the set of unordered k-tuples with $k \ge 0$ for a fixed n. For each k-tuples $(n_1,\dots,n_k) \in S(n)$, B.-Y. Chen [9] defined a *Riemannian invariant* $\delta(n_1,\dots,n_k)$ as follows:

$$\delta(n_1,\dots,n_k)(x) = \tau(x) - S(n_1,\dots,n_k)(x), \tag{2.10}$$

where

$$S(n_1,\dots,n_k)(x) = \inf\{\tau(L_1)+\cdots+\tau(L_k)\},$$

and $L_1,\dots,L_k$ run over all k mutually orthogonal subspaces of T_xM such that $\dim L_j = n_j,\ j \in \{1,\dots,k\}$. In particular, we have $\delta(2) = \tau(x) - \inf K$, where K is the sectional curvature.

For each $(n_1,\dots,n_k) \in S(n)$, we put

$$c(n_1,\dots,n_k) = \frac{n^2\big(n+k-1-\sum_{j=1}^{k} n_j\big)}{2\big(n+k-\sum_{j=1}^{k} n_j\big)},$$

$$d(n_1,\dots,n_k) = \frac{1}{2}[n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)].$$

We recall the well-known Chen's lemma:

Lemma 2.2 ([8])**.** *Let $a_1,a_2,\dots,a_n,b$ be $(n+1)(n \ge 2)$ real numbers such that*

$$(\sum_{i=1}^{n} a_i)^2 = (n-1)(\sum_{i=1}^{n} a_i^2 + b).$$

Then $2a_1a_2 \ge b$, with equality holding if and only if $a_1+a_2 = a_3 = \cdots = a_n$.

Most of the geometers (cf. [8, 16, 17, 18, 19, 20, 21]) established inequalities relating $\delta(2)$ and the squared mean curvature for different submanifolds in various ambient spaces by using the above algebraic lemma, except for T. Oprea (cf. [11]). In [11], T. Oprea obtained Chen's inequalities for Lagrangian submanifolds in complex space forms by using optimization techniques applied in the setup of Riemannian geometry. We will use another algebraic lemma to obtain inequalities relating $\delta(2)$ and the squared mean curvature in Section 3.

Lemma 2.3. *Let $f(x_1,x_2,\dots,x_n)$, $n \ge 3$, be a function in $\mathbb{R}^n$ defined by*

$$f(x_1,x_2,\dots,x_n) = (x_1+x_2)\sum_{i=3}^{n} x_i + \sum_{3\le i<j\le n} x_ix_j.$$

If $x_1 + x_2 + \cdots + x_n = (n-1)\varepsilon$, then we have

$$f(x_1, x_2, \ldots, x_n) \leq \frac{(n-1)(n-2)}{2}\varepsilon^2,$$

with the equality holding if and only if $x_1 + x_2 = x_3 = \cdots = x_n = \varepsilon$.

Proof. By simple calculation, we have

$$\begin{aligned} f(x_1, x_2, \ldots, x_n) &= (x_1 + x_2)\sum_{i=3}^{n} x_i + \sum_{3\leq i<j\leq n} x_i x_j \qquad (2.11) \\ &= \frac{1}{2}\{(x_1 + x_2 + \cdots + x_n)^2 - [(x_1+x_2)^2 + x_3^2 + \cdots + x_n^2]\} \\ &= \frac{1}{2}\{(n-1)^2\varepsilon^2 - [(x_1+x_2)^2 + x_3^2 + \cdots + x_n^2]\}. \end{aligned}$$

On the other hand, by the Cauchy–Schwarz inequality we have

$$[(x_1 + x_2) + x_3 + \cdots + x_n]^2 \leq (n-1)[(x_1+x_2)^2 + x_3^2 + \cdots + x_n^2], \qquad (2.12)$$

with the equality holding if and only if $x_1 + x_2 = x_3 = \cdots = x_n$.

Noting that $(x_1 + x_2) + \cdots + x_n = (n-1)\varepsilon$, from (2.12) we have

$$(x_1+x_2)^2 + x_3^2 + \cdots + x_n^2 \geq (n-1)\varepsilon^2. \qquad (2.13)$$

Using (2.11) and (2.13) we derive

$$f(x_1, x_2, \ldots, x_n) \leq \frac{1}{2}[(n-1)^2\varepsilon^2 - (n-1)\varepsilon^2] = \frac{(n-1)(n-2)}{2}\varepsilon^2,$$

which proves the lemma. □

In Section 5, we use a more simple way to obtain the relation between the Ricci curvature and the squared mean curvature. We need the following lemma.

Lemma 2.4. *Let $f(x_1, x_2, \ldots, x_n)$ be a function in $\mathbb{R}^n$ defined by*

$$f(x_1, x_2, \ldots, x_n) = x_1 \sum_{i=2}^{n} x_i.$$

If $x_1 + x_2 + \cdots + x_n = 2\varepsilon$, then we have

$$f(x_1, x_2, \ldots, x_n) \leq \varepsilon^2,$$

with the equality holding if and only if $x_1 = x_2 + x_3 + \cdots + x_n = \varepsilon$.

Proof. From $x_1 + x_2 + \cdots + x_n = 2\varepsilon$, we have

$$\sum_{i=2}^{n} x_i = 2\varepsilon - x_1.$$

It follows that

$$f(x_1, x_2, \ldots, x_n) = x_1(2\varepsilon - x_1) = -(x_1 - \varepsilon)^2 + \varepsilon^2,$$

which proves the lemma. □

3. Chen's first inequality

According to equation (3.1) in [21], denote by

$$\Omega(X) = s(X, X) + g(P^{\perp}, h(X, X)), \tag{3.1}$$

for a unit vector X tangent to M^n at a point x. We remark that Ω doesn't depend on X and denote it simply by Ω. Detailed explanations were given in the proof of Theorem 3.1 in [21]. From the definition of vector field P, we have

$$\phi(H) = \frac{1}{n}\sum_{i=1}^{n} \phi(h(e_i, e_i)) = g(P^{\perp}, H). \tag{3.2}$$

For submanifolds of a Riemannian manifold of nearly quasi-constant curvature endowed with a semi-symmetric non-metric connection we establish the following optimal inequality relating $\delta(2)$ and squared mean curvature, which we will call Chen's first inequality. For simplicity, we denote by

$$\chi = \sum_{i=1}^{n} B(e_i, e_i), \quad \lambda = \sum_{i=1}^{n} s(e_i, e_i). \tag{3.3}$$

Theorem 3.1. *Let M^n, $n \geq 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold of nearly quasi-constant curvature N^{n+p} endowed with a semi-symmetric non-metric connection. Then we have*

$$\begin{aligned} \tau(x) - K(\pi) \leq{}& \frac{(n+1)(n-2)}{2}a + b[(n-2)\chi + \operatorname{trace} B|_{\pi^{\perp}}] \\ &+ \frac{n^2(n-2)}{2(n-1)}\|H\|^2 - \frac{n-1}{2}\lambda - \frac{n^2-n}{2}\phi(H) + \Omega, \end{aligned} \tag{3.4}$$

where π is a 2-plane section of T_xM^n, $x \in M^n$. The equality case of (3.4) *holds at a point $x \in M$ if and only if, with respect to a suitable orthonormal basis $\{e_A\}$ at x, the shape operators $A_r = A_{e_r}$ take the following forms:*

$$A_{n+1} = \begin{pmatrix} h_{11}^{n+1} & 0 & 0 & \cdots & 0 \\ 0 & h_{22}^{n+1} & 0 & \cdots & 0 \\ 0 & 0 & h_{11}^{n+1} + h_{22}^{n+1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & h_{11}^{n+1} + h_{22}^{n+1} \end{pmatrix}$$

and

$$A_r = \begin{pmatrix} h_{11}^{r} & h_{12}^{r} & 0 & \cdots & 0 \\ h_{12}^{r} & -h_{11}^{r} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad r = n+2, \ldots, n+p.$$

Remark 3.2. *For $b = 0$, Theorem 3.1 is due to C. Özgür and A. Mihai* [21, Theorem 3.1].

Proof. We consider the point $x \in M^n$, and choose a local orthonormal frame (2.6) such that $\{e_1, e_2\}$ is an orthonormal frame in the 2-plane which minimizes the sectional curvature at the point x. We remark that

$$B(e_1,e_1) + B(e_2,e_2) = \chi - \operatorname{trace} B|_{\pi^\perp}. \tag{3.5}$$

Using (2.4), (2.5) and (3.2) we have

$$\begin{aligned} R_{ijij} &= a + b[B(e_i,e_i) + B(e_j,e_j)] - s(e_i,e_i) - \phi(h(e_i,e_i)) \\ &\quad + \sum_{r=n+1}^{n+p} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2], \end{aligned} \tag{3.6}$$

and it follows that

$$\begin{aligned} \tau(x) = \sum_{1\le i<j\le n} R_{ijij} &= \frac{n^2-n}{2}a + b(n-1)\chi - \frac{(n-1)}{2}\lambda - \frac{n^2-n}{2}\phi(H) \\ &+ \sum_{r=n+1}^{n+p}\sum_{1\le i<j\le n} [h^r_{ii}h^r_{jj} - (h^r_{ij})^2]. \end{aligned} \tag{3.7}$$

Using (3.1), (3.3), (3.5) and (3.6) we have

$$R_{1212} = a + b(\chi - \operatorname{trace} B|_{\pi^\perp}) - \Omega + \sum_{r=n+1}^{n+p} [h^r_{11}h^r_{22} - (h^r_{12})^2]. \tag{3.8}$$

From (3.7) and (3.8) one gets

$$\begin{aligned} &\tau(x) - K(\pi) \\ &= \frac{(n+1)(n-2)}{2}a + b[(n-2)\chi + \operatorname{trace} B|_{\pi^\perp}] - \frac{(n-1)}{2}\lambda - \frac{n^2-n}{2}\phi(H) + \Omega \\ &+ \sum_{r=n+1}^{n+p}\left[\sum_{1\le i<j\le n} h^r_{ii}h^r_{jj} - h^r_{11}h^r_{22} - \sum_{1\le i<j\le n}(h^r_{ij})^2 + (h^r_{12})^2\right] \\ &= \frac{(n+1)(n-2)}{2}a + b[(n-2)\chi + \operatorname{trace} B|_{\pi^\perp}] - \frac{(n-1)}{2}\lambda - \frac{n^2-n}{2}\phi(H) + \Omega \\ &+ \sum_{r=n+1}^{n+p}\left[(h^r_{11}+h^r_{22})\sum_{3\le i\le n} h^r_{ii} + \sum_{3\le i<j\le n} h^r_{ii}h^r_{jj} - \sum_{3\le j\le n}(h^r_{1j})^2 - \sum_{2\le i<j\le n}(h^r_{ij})^2\right] \\ &\le \frac{(n+1)(n-2)}{2}a + b[(n-2)\chi + \operatorname{trace} B|_{\pi^\perp}] - \frac{(n-1)}{2}\lambda - \frac{n^2-n}{2}\phi(H) + \Omega \\ &+ \sum_{r=n+1}^{n+p}\left[(h^r_{11}+h^r_{22})\sum_{3\le i\le n} h^r_{ii} + \sum_{3\le i<j\le n} h^r_{ii}h^r_{jj}\right]. \end{aligned} \tag{3.9}$$

Let us consider the quadratic forms $f_r : \mathbb{R}^n \to \mathbb{R}$, defined by

$$f_r(h_{11}^r, h_{22}^r, \dots, h_{nn}^r) = (h_{11}^r + h_{22}^r) \sum_{3 \le i \le n} h_{ii}^r + \sum_{3 \le i < j \le n} h_{ii}^r h_{jj}^r.$$

We consider the problem $\max f_r$, subject to $\Xi : h_{11}^r + h_{22}^r + \cdots + h_{nn}^r = k^r$, where k^r is a real constant. From Lemma 2.3, we see that the solution $(h_{11}^r, h_{22}^r, \dots, h_{nn}^r)$ of the problem in question must satisfy

$$h_{11}^r + h_{22}^r = h_{ii}^r = \frac{k^r}{n-1}, \quad i = 3, \dots, n, \tag{3.10}$$

which implies

$$f_r \le \frac{n-2}{2(n-1)} (k^r)^2. \tag{3.11}$$

From (3.9) and (3.11) we have

$$\begin{aligned} \tau(x) - K(\pi) \le{}& \frac{(n+1)(n-2)}{2} a + b[(n-2)\chi + \operatorname{trace} B|_{\pi^\perp}] - \frac{(n-1)}{2} \lambda \\ &- \frac{n^2-n}{2} \phi(H) + \Omega + \sum_r \frac{n-2}{2(n-1)} (k^r)^2 \\ ={}& \frac{(n+1)(n-2)}{2} a + b[(n-2)\chi + \operatorname{trace} B|_{\pi^\perp}] - \frac{(n-1)}{2} \lambda \\ &- \frac{n^2-n}{2} \phi(H) + \Omega + \frac{n^2(n-2)}{2(n-1)} \|H\|^2, \end{aligned}$$

which is the inequality to prove.

Next we will study the equality case.

If the equality case of (3.4) holds at a point $x \in M$, then the equality cases of (3.9) and (3.11) hold, and it follows that

$$\sum_{3 \le i \le n} (h_{1i}^r)^2 = 0, \qquad \sum_{2 \le i < j \le n} (h_{ij}^r)^2 = 0, \qquad \forall r,$$

$$h_{11}^r + h_{22}^r = h_{ii}^r, \qquad 3 \le i \le n, \ \forall r.$$

After choosing a suitable orthonormal basis, the shape operators take the desired form due to (2.1). □

4. Chen's general inequality

Next we prove a generalization of Theorem 3.1 in terms of Chen's invariant $\delta(n_1, \dots, n_k)$.

Theorem 4.1. *If M^n, $n \ge 3$, is a submanifold of a Riemannian manifold of nearly quasi-constant curvature N^{n+p} endowed with a semi-symmetric non-metric* *connection, then we have*

$$\begin{aligned}\delta(n_1,\dots,n_k) \le c(n_1,\dots,n_k)\|H\|^2 + d(n_1,\dots,n_k)a - \frac{(n-1)}{2}\lambda \\ - \frac{n^2-n}{2}\phi(H) + \sum_{j=1}^{k}\Psi_1(L_j) + b[(n-1)\chi - \sum_{j=1}^{k}\Psi_2(L_j)],\end{aligned} \tag{4.1}$$

for any k-tuple $(n_1,\dots,n_k) \in S(n)$, where λ and χ are given by (3.3)*. The equality case of* (4.1) *holds at $x \in M^n$ if and only if there exist an orthonormal basis $\{e_1,\dots,e_n\}$ of T_xM and an orthonormal basis $\{e_{n+1},\dots,e_{n+p}\}$ of $T_x^{\perp}M$ such that the shape operators of M^n in N^{n+p} at x have the following forms:*

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix},$$

$$A_{e_r} = \begin{pmatrix} A_1^r & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_k^r & 0 \\ 0 & \cdots & 0 & \varsigma_r I \end{pmatrix}, \quad r = n+2,\dots,n+p,$$

where $a_1,\dots,a_n$ satisfy

$$a_1+\cdots+a_{n_1} = \cdots = a_{n_1+\cdots+n_{k-1}+1}+\cdots+a_{n_1+\cdots+n_k} = a_{n_1+\cdots+n_k+1} = \cdots = a_n$$

and each A_j^r is a symmetric $n_j \times n_j$ submatrix satisfying $\operatorname{trace}(A_1^r) = \cdots = \operatorname{trace}(A_k^r) = \varsigma_r$*. I is an identity matrix.*

Remark 4.2. *For $\delta(2)$, inequality* (4.1) *is due to Theorem 3.1.*

Proof. Choose an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ for T_xM^n and $\{e_{n+1}, e_{n+2}, \dots, e_{n+p}\}$ for the normal space $T_x^{\perp}M^n$ such that the mean curvature vector H is in the direction of the normal vector to e_{n+1}. For convenience, we set

$$\begin{aligned}
&a_i = h_{ii}^{n+1}, \quad i = 1,2,\dots,n, \\
&b_1 = a_1, \quad b_2 = a_2+\cdots+a_{n_1}, \quad b_3 = a_{n_1+1}+\cdots+a_{n_1+n_2}, \\
&\cdots \\
&b_{k+1} = a_{n_1+\cdots+n_{k-1}+1}+\cdots+a_{n_1+n_2+\cdots+n_{k-1}+n_k}, \\
&b_{k+2} = a_{n_1+\cdots+n_k+1},\dots,b_{\gamma+1} = a_n, \\
&\Delta_1 = \{1,\dots,n_1\}, \\
&\cdots \\
&\Delta_k = \{(n_1+\cdots+n_{k-1})+1,\dots,n_1+\cdots+n_k\}, \\
&\Delta_{k+1} = (\Delta_1\times\Delta_1)\cup\cdots\cup(\Delta_k\times\Delta_k).
\end{aligned}$$

Let $L_1,\dots,L_k$ be mutually orthogonal subspaces of T_xM with $\dim L_j = n_j$, defined by

$$L_j = \operatorname{Span}\{e_{n_1+\cdots+n_{j-1}+1},\dots,e_{n_1+\cdots+n_j}\}, \quad j = 1,\dots,k.$$

From (2.4), (2.5), (2.7), (2.8) and (2.9) we have

$$\tau(L_j) = \frac{n_j(n_j-1)}{2}a + b\Psi_2(L_j) - \Psi_1(L_j) + \sum_{r=n+1}^{n+p}\sum_{\mu_j<\nu_j}[h^r_{\mu_j\mu_j}h^r_{\nu_j\nu_j} - (h^r_{\mu_j\nu_j})^2], \quad (4.2)$$

$$2\tau = n(n-1)a + 2b(n-1)\chi - (n-1)\lambda - (n^2-n)\phi(H) + n^2\|H\|^2 - \|h\|^2. \quad (4.3)$$

We can rewrite (4.3) as

$$n^2\|H\|^2 = (\|h\|^2 + \eta)\gamma,$$

or equivalently,

$$(\sum_{i=1}^{n} h^{n+1}_{ii})^2 = \gamma[\sum_{i=1}^{n}(h^{n+1}_{ii})^2 + \sum_{i\neq j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h^r_{ij})^2 + \eta], \quad (4.4)$$

where

$$\eta = 2\tau - 2c(n_1,\ldots,n_k)\|H\|^2 - n(n-1)a - 2(n-1)b\chi + (n-1)\lambda + (n^2-n)\phi(H), \quad (4.5)$$

and

$$\gamma = n + k - \sum_{j=1}^{k} n_j.$$

From (4.4) we deduce

$$(\sum_{i=1}^{\gamma+1} b_i)^2 = \gamma[\eta + \sum_{i=1}^{\gamma+1} b_i^2 + \sum_{i\neq j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h^r_{ij})^2 - 2\sum_{j=1}^{k}\sum_{\mu_j<\nu_j} a_{\mu_j}a_{\nu_j}],$$

where $\mu_j, \nu_j \in \Delta_j$, for all $j = 1, \ldots, k$. Applying Lemma 2.2, we derive

$$\sum_{j=1}^{k}\sum_{\mu_j<\nu_j} a_{\mu_j}a_{\nu_j} \geq \frac{1}{2}[\eta + \sum_{i\neq j}(h^{n+1}_{ij})^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n}(h^r_{ij})^2].$$

It follows that

$$\begin{aligned}\sum_{j=1}^{k}\sum_{r=n+1}^{n+p}\sum_{\mu_j<\nu_j}[h^r_{\mu_j\mu_j}h^r_{\nu_j\nu_j} - (h^r_{\mu_j\nu_j})^2] &\geq \frac{\eta}{2} + \frac{1}{2}\sum_{r=n+1}^{n+p}\sum_{(\mu,\nu)\notin\Delta_{k+1}}(h^r_{\mu\nu})^2 \\ &\quad + \sum_{r=n+2}^{n+p}\sum_{\mu_j\in\Delta_j}(h^r_{\mu_j\mu_j})^2 \qquad (4.6)\\ &\geq \frac{\eta}{2}.\end{aligned}$$

From (4.2) and (4.6) we have

$$\sum_{j=1}^{k}\tau(L_j) \geq \sum_{j=1}^{k}[\frac{n_j(n_j-1)}{2}a + b\Psi_2(L_j) - \Psi_1(L_j)] + \frac{1}{2}\eta. \quad (4.7)$$

Using (2.10), (4.5) and (4.7), we derive the desired inequality.

The equality case of (4.1) at a point $x \in M$ holds if and only if we have the equality in all the previous inequalities and also in Lemma 2.2, and thus from (2.1) the shape operators take the desired forms. □

From Theorem 4.1, we have

Corollary 4.3. *If M^n, $n \geq 3$, is a submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ of constant curvature c endowed with a semi-symmetric non-metric connection, then we have*

$$\begin{aligned}\delta(n_1,\dots,n_k) \leq c(n_1,\dots,n_k)\|H\|^2 + d(n_1,\dots,n_k)c - \frac{(n-1)}{2}\lambda \\ - \frac{n^2-n}{2}\phi(H) + \sum_{j=1}^{k}\Psi_1(L_j),\end{aligned} \tag{4.8}$$

for any k-tuple $(n_1,\dots,n_k) \in S(n)$. The equality case of (4.8) holds at $x \in M^n$ if and only if there exist an orthonormal basis $\{e_1,\dots,e_n\}$ of T_xM and an orthonormal basis $\{e_{n+1},\dots,e_{n+p}\}$ of $T_x^{\perp}M$ such that the shape operators of M^n in N^{n+p} at x have the following forms:

$$A_{e_{n+1}} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix},$$

$$A_{e_r} = \begin{pmatrix} A_1^r & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & A_k^r & 0 \\ 0 & \cdots & 0 & \varsigma_r I \end{pmatrix}, \quad r = n+2,\dots,n+p,$$

where $a_1,\dots,a_n$ satisfy

$$a_1 + \cdots + a_{n_1} = \cdots = a_{n_1+\cdots+n_{k-1}+1} + \cdots + a_{n_1+\cdots+n_k} = a_{n_1+\cdots+n_k+1} = \cdots = a_n$$

and each A_j^r is a symmetric $n_j \times n_j$ submatrix satisfying $\operatorname{trace}(A_1^r) = \cdots = \operatorname{trace}(A_k^r) = \varsigma_r$. *$I$ is an identity matrix.*

Remark 4.4. *For $\delta(2)$, inequality* (4.8) *is due to C. Özgür and A. Mihai* [21, Theorem 3.1].

5. Chen–Ricci inequality

In [27], B.-Y. Chen established a sharp relationship between the Ricci curvature and the squared mean curvature for any n-dimensional Riemannian submanifold of a real space form $R^m(c)$ of constant sectional curvature c as follows:

Theorem 5.1 ([27], Theorem 4)**.** *Let M be an n-dimensional submanifold of a real space form $R^m(c)$. Then the following statements are true:*

(i) For each unit vector $X \in T_xM$, we have

$$\|H\|^2 \geq \frac{4}{n^2}[\mathrm{Ric}(X) - (n-1)c]. \tag{5.1}$$

(ii) If $H(x) = 0$, then a unit vector $X \in T_xM$ satisfies the equality case of (5.1) *if and only if X belongs to the relative null space $\mathcal{N}(x)$ given by*

$$\mathcal{N}(x) = \{X \in T_xM \mid h(X,Y) = 0,\ \forall Y \in T_xM\}.$$

(iii) The equality case of (5.1) *holds for all unit vectors $X \in T_xM$ if and only if either x is a geodesic point or $n = 2$ and x is an umbilical point.*

Afterwards, many papers studied similar Chen–Ricci inequalities for different submanifolds in various ambient manifolds; see for example [28, 29, 30].

In this section, we establish Chen–Ricci inequalities for submanifolds of a Riemannian manifold of nearly quasi-constant curvature endowed with a semi-symmetric non-metric connection.

Theorem 5.2. *Let M^n, $n \geq 2$, be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold of nearly quasi-constant curvature N^{n+p} endowed with a semi-symmetric non-metric connection $\overline{\nabla}$. Then:*

(i) For each unit vector X in T_xM we have

$$\mathrm{Ric}(X) \leq (n-1)(a-\Omega) + b[(n-2)B(X,X) + \chi] + \frac{n^2}{4}\|H\|^2, \tag{5.2}$$

where Ω and χ are given by (3.1), (3.3), *respectively.*

(ii) If $H(x) = 0$, then a unit tangent vector X at x satisfies the equality case of (5.2) *if and only if $X \in \mathcal{N}(x) = \{X \in T_xM \mid h(X,Y) = 0, \forall Y \in T_xM\}$.*

(iii) The equality case of (5.2) *holds for all unit vectors $X \in T_xM$ if and only if either x is a geodesic point or $n = 2$ and x is an umbilical point.*

Proof. (i) Let$X \in T_xM$ be a unit tangent vector at x. We choose the local field of orthonormal frames (2.6) at x such that $e_1 = X$. From equation (3.6) we have

$$\begin{aligned}\mathrm{Ric}(X) = \sum_{i=2}^{n} R_{1i1i} &= (n-1)a + (n-1)bB(e_1,e_1) + b\sum_{i=2}^{n} B(e_i,e_i) \\ &- (n-1)s(X,X) - (n-1)\phi(h(e_1,e_1)) \\ &+ \sum_{r=n+1}^{n+p}\sum_{i=2}^{n}[h_{11}^r h_{ii}^r - (h_{1i}^r)^2] \\ \leq (n-1)(a-\Omega) &+ (n-2)bB(X,X) + b\chi + \sum_{r=n+1}^{n+p}\sum_{i=2}^{n} h_{11}^r h_{ii}^r.\end{aligned} \tag{5.3}$$

Let us consider the quadratic forms $f_r : \mathbb{R}^n \to \mathbb{R}$, defined by

$$f_r(h_{11}^r, h_{22}^r, \ldots, h_{nn}^r) = \sum_{i=2}^{n} h_{11}^r h_{ii}^r.$$

We consider the problem $\max f_r$, subject to $\Xi : h^r_{11} + h^r_{22} + \cdots + h^r_{nn} = k^r$, where k^r is a real constant. From Lemma 2.4, we can see that the solution $(h^r_{11}, h^r_{22}, \ldots, h^r_{nn})$ of the problem in question must satisfy

$$h^r_{11} = \sum_{i=2}^{n} h^r_{ii} = \frac{k^r}{2}, \tag{5.4}$$

which implies

$$f_r \leq \frac{(k^r)^2}{4}. \tag{5.5}$$

From (5.3) and (5.5) we have

$$\begin{aligned}\mathrm{Ric}(X) &\leq (n-1)(a-\Omega) + (n-2)bB(X,X) + b\chi + \sum_{r=n+1}^{n+p} \frac{(k^r)^2}{4} \\ &= (n-1)(a-\Omega) + b[(n-2)B(X,X) + \chi] + \frac{n^2}{4}\|H\|^2.\end{aligned}$$

(ii) For each unit vector X at x, if the equality case of inequality (5.2) holds, from (5.3), (5.4) and (5.5) we have

$$h^r_{1i} = 0, \quad i \neq 1, \ \forall r, \tag{5.6}$$

$$h^r_{11} + h^r_{22} + \cdots + h^r_{nn} - 2h^r_{11} = 0, \ \forall r. \tag{5.7}$$

Noting that $H(x) = 0$, we have $h^r_{11} = 0$, then $h^r_{1i} = 0$, $\forall i, r$, i.e. $X \in \mathcal{N}(x)$.

(iii) If the equality case of inequality (5.2) holds for all unit tangent vectors at x, noting that X is arbitrary, by computing $\mathrm{Ric}(e_j)$, $j = 2, 3, \ldots, n$, and combining (5.6) and (5.7), we have

$$h^r_{ij} = 0, \quad i \neq j, \ \forall r,$$

$$h^r_{11} + h^r_{22} + \cdots + h^r_{nn} - 2h^r_{ii} = 0, \quad \forall i, r.$$

We can distinguish two cases:

(1) $n \neq 2$, $h^r_{ij} = 0$, $i, j = 1, 2, \ldots, n$, $r = n+1, \ldots, n+p$;

(2) $n = 2$, $h^r_{11} = h^r_{22}$, $h^r_{12} = 0$, $r = 3, \ldots, 2+p$.

The converse is trivial. □

Noting that the Riemannian manifold of nearly quasi-constant curvature N^{n+p} reduces to a real space form of constant curvature when $b = 0$, from Theorem 5.2 we have

Corollary 5.3. *Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ of constant curvature c endowed with a semi-symmetric non-metric connection. Then:*

(i) For each unit vector X in T_xM we have

$$\mathrm{Ric}(X) \leq (n-1)(c-\Omega) + \frac{n^2}{4}\|H\|^2. \tag{5.8}$$

(ii) If $H(x) = 0$, then a unit tangent vector X at x satisfies the equality case of (5.8) if and only if $X \in \mathcal{N}(x) = \{X \in T_xM \mid h(X,Y) = 0, \forall Y \in T_xM\}$.

(iii) The equality case of inequality (5.8) *holds for all unit tangent vectors at x if and only if either x is a geodesic point, or $n = 2$ and x is an umbilical point.*

In [21], C. Özgür and A. Mihai also established Chen–Ricci inequalities for submanifolds of a real space form endowed with a semi-symmetric non-metric connection. They proved:

Theorem 5.4 ([21], Theorem 4.1). *Let M^n be an n-dimensional submanifold of an $(n+p)$-dimensional real space form $N^{n+p}(c)$ endowed with a semi-symmetric non-metric connection. Then*

(i) For each unit vector X in T_xM we have

$$\mathrm{Ric}(X) \leq (n-1)c + \frac{n^2\|H\|^2}{4} - \frac{n-1}{2}\lambda + \frac{(n-1)(n-2)}{2}s(X,X) - \frac{n^2-n}{2}\phi(H), \tag{5.9}$$

where λ is given by (3.3)*.*

(ii) If $H(x) = 0$, then a unit tangent vector X at x satisfies the equality case of (5.9) *if and only if $X \in \mathcal{N}(x) = \{X \in T_xM \mid h(X,Y) = 0, \forall Y \in T_xM\}$.*

(iii) The equality case of inequality (5.9) *holds for all unit tangent vectors at x if and only if either x is a totally geodesic point, or $n = 2$ and x is a totally umbilical point.*

We will show that the inequality (5.9) is incorrect.

Remark 5.5. *For $n \neq 2$, if the equality case of* (5.9) *holds for all unit tangent vectors at x, from Theorem 5.4 we know that $h^r_{ij} = 0$, $\forall i,j,r$. Further, using* (2.5) *we have*

$$\mathrm{Ric}(X) = \sum_{i=2}^{n} R_{1i1i} = (n-1)c - (n-1)s(X,X) - (n-1)\phi(h(X,X)),$$

which is a contradiction with the equality case of (5.9)*.*

Remark 5.6. *For $n = 2$, if the equality case of* (5.9) *holds for all unit tangent vectors at x, from Theorem 5.4 we know that $h^r_{11} = h^r_{22}$, $h^r_{12} = 0$, $r = 3,\ldots,2+p$. Further, using* (2.5) *we have*

$$\mathrm{Ric}(X) = R_{1212} = c - s(X,X) - \phi(h(X,X)) + \|H\|^2,$$

which is also a contradiction with the equality case of (5.9)*.*

Remark 5.7. *In the proof of Theorem 4.1 in* [21]*, the authors wrote*

$$\begin{aligned} K_{ij} &= \widetilde{R}(e_i,e_j,e_j,e_i) + g(h(e_i,e_i),h(e_j,e_j)) - g(h(e_i,e_j),h(e_i,e_j)) \\ &= c - s(e_j,e_j) + \sum_{r=n+1}^{n+p}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2]. \end{aligned}$$

But according to formulas (3.2) *and* (3.3) *in* [21], *we get*

$$\begin{aligned} K_{ij} &= \widetilde{R}(e_i,e_j,e_j,e_i) + g(h(e_i,e_i),h(e_j,e_j)) - g(h(e_i,e_j),h(e_i,e_j)) - \phi(h(e_j,e_j)) \\ &= c - s(e_j,e_j) + \sum_{r=n+1}^{n+p} [h_{ii}^r h_{jj}^r - (h_{ij}^r)^2] - \phi(h(e_j,e_j)). \end{aligned}$$

This is the reason they made a mistake. We should note that the Gauss equation with respect to the semi-symmetric non-metric connection is very different from the Gauss equation with respect to the Levi–Civita connection.

6. k-Ricci curvature

Let L be a k-plane section of T_xM^n, $x \in M$, and X a unit vector in L. We choose an orthonormal frame $e_1, \dots, e_k$ of L such that $e_1 = X$. In [27], B.-Y. Chen defined the k-Ricci curvature of L at X by

$$\mathrm{Ric}_L(X) = K_{12} + K_{13} + \cdots + K_{1k}. \tag{6.1}$$

The scalar curvature of a k-plane section L is given by

$$\tau(L) = \sum_{1 \le i < j \le n} K_{ij}. \tag{6.2}$$

For an integer k, $2 \le k \le n$, the Riemannian invariant Θ_k on M^n defined by

$$\Theta_k(x) = \frac{1}{k-1} \inf\{\mathrm{Ric}_L(X) \mid L, X\}, \quad x \in M, \tag{6.3}$$

where L runs over all k-plane sections in T_xM and X runs over all unit vectors in L.

From (2.7), (2.8), (6.1), and (6.2) it follows that for any k-plane section $L_{i_1\cdots i_k}$ spanned by $\{e_{i_1}, \dots, e_{i_k}\}$ one has

$$\tau(L_{i_1\cdots i_k}) = \frac{1}{2} \sum_{i \in \{i_1,\dots,i_k\}} \mathrm{Ric}_{L_{i_1\cdots i_k}}(e_i) \tag{6.4}$$

and

$$\tau(x) = \frac{1}{C_{n-2}^{k-2}} \sum_{1 \le i_1 < \cdots < i_k \le n} \tau(L_{i_1\cdots i_k}). \tag{6.5}$$

From (6.3), (6.4) and (6.5) we obtain

$$\tau(x) \ge \frac{n(n-1)}{2} \Theta_k(x). \tag{6.6}$$

In this section, we prove a relationship between the k-Ricci curvature of M^n (intrinsic invariant) and the mean curvature $\|H\|$ (extrinsic invariant), as another answer to the basic problem in submanifold theory which we have mentioned in the introduction.

Theorem 6.1. *Let M^n, $n \geq 3$, be an n-dimensional submanifold of an $(n+p)$-dimensional Riemannian manifold of nearly quasi-constant curvature N^{n+p} endowed with a semi-symmetric non-metric connection $\overline{\nabla}$, then for any integer k, $2 \leq k \leq n$, and any point $x \in M^n$, we have*

$$\|H\|^2(x) \geq \Theta_k(x) - a - \frac{2b}{n}\chi + \frac{1}{n}\lambda + \phi(H),$$

where λ and χ are given by (3.3).

Remark 6.2. *For $b = 0$, Theorem 6.1 is due to C. Özgür and A. Mihai* [21, Theorem 5.2].

Proof. We choose the orthonormal frame (2.6) at x such that e_{n+1} is in the direction of the mean curvature vector $H(x)$ and $\{e_1, \dots, e_n\}$ diagonalize the shape operator A_{n+1}. Then the shape operators take the following forms:

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & a_n \end{pmatrix}, \tag{6.7}$$

$$\operatorname{trace} A_r = 0, \quad r = n+2, \dots, n+p.$$

From (4.3) and (6.7) we have

$$\begin{aligned} n^2\|H\|^2 &= 2\tau + \|h\|^2 - (n^2-n)a - 2b(n-1)\chi + (n-1)\lambda + (n^2-n)\phi(H) \\ &= 2\tau + \sum_{i=1}^{n} a_i^2 + \sum_{r=n+2}^{n+p}\sum_{i,j=1}^{n} (h_{ij}^r)^2 - (n^2-n)a - 2b(n-1)\chi \\ &\quad + (n-1)\lambda + (n^2-n)\phi(H). \end{aligned} \tag{6.8}$$

Using the Cauchy–Schwarz inequality we have

$$(n\|H\|)^2 = (\sum_{i=1}^{n} a_i)^2 \leq n\sum_{i=1}^{n} a_i^2;$$

it follows that

$$\sum_{i=1}^{n} a_i^2 \geq n\|H\|^2. \tag{6.9}$$

From (6.8) and (6.9) we have

$$n^2\|H\|^2 \geq 2\tau + n\|H\|^2 - (n^2-n)a - 2b(n-1)\chi + (n-1)\lambda + (n^2-n)\phi(H),$$

which implies

$$\|H\|^2 \geq \frac{2\tau}{n(n-1)} - a - \frac{2b}{n}\chi + \frac{1}{n}\lambda + \phi(H). \tag{6.10}$$

Using (6.6) and (6.10) we have

$$\|H\|^2(x) \geq \Theta_k(x) - a - \frac{2b}{n}\chi + \frac{1}{n}\lambda + \phi(H).$$

□

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