REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 56, No. 2, 2015, Pages 39–55
Published online: August 27, 2015

PRIME IDEALS OF SKEW PBW EXTENSIONS

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Abstract. We describe the prime ideals of some important classes of skew PBW extensions, using the classical technique of extending and contracting ideals. Skew PBW extensions include as particular examples Weyl algebras, enveloping algebras of finite-dimensional Lie algebras (and their quantizations), Artamonov quantum polynomials, diffusion algebras, and Manin algebra of quantum matrices, among many others.

1. Introduction

In this paper we describe the prime ideals of some important classes of skew PBW extensions. For this purpose we will consider the techniques that we found in [3], [6] and [7]. However, in several of our results, we needed to introduce some modifications to these techniques. In this section we recall the definition of skew PBW (Poincaré-Birkhoff-Witt) extensions defined firstly in [5], and we will review also some elementary properties about the polynomial interpretation of this kind of non-commutative rings. Weyl algebras, enveloping algebras of finite-dimensional Lie algebras (and their quantizations), Artamonov quantum polynomials, diffusion algebras, and Manin algebra of quantum matrices, are particular examples of skew PBW extensions (see [9]).

Definition 1.1. *Let R and A be rings. We say that A is a* skew PBW extension *of R (also called a σ-PBW extension of R) if the following conditions hold:*

(i) $R \subseteq A$.
(ii) *There exist finitely many elements $x_1, \dots, x_n \in A$ such A is a left R-free module with basis* $\mathrm{Mon}(A) := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n\}$*, with* $\mathbb{N} := \{0, 1, 2, \dots\}$*. The set* $\mathrm{Mon}(A)$ *is called the set of standard monomials of A.*
(iii) *For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that*
$$x_i r - c_{i,r} x_i \in R.$$

2010 *Mathematics Subject Classification.* Primary: 16S80; Secondary: 16D25, 16N60, 16S36, 16U20.

Key words and phrases. Prime ideals, semiprime rings, nilradical, skew polynomial rings, PBW extensions, quantum algebras, skew PBW extensions.

(iv) *For every* $1 \leq i, j \leq n$ *there exists* $c_{i,j} \in R - \{0\}$ *such that*

$$x_j x_i - c_{i,j} x_i x_j \in R + Rx_1 + \cdots + Rx_n.$$

Under these conditions we will write $A := \sigma(R)\langle x_1, \ldots, x_n \rangle$.

Recall that if σ is an endomorphism of a ring R, a *σ-derivation* of R is an additive function $\delta : R \to R$ such that $\delta(rr') = \sigma(r)\delta(r') + \delta(r)r'$. Associated to a skew PBW extension $A = \sigma(R)\langle x_1, \ldots, x_n \rangle$, there are n injective endomorphisms $\sigma_1, \ldots, \sigma_n$ of R and σ_i-derivations, as the following proposition shows.

Proposition 1.2. *Let A be a skew PBW extension of R. Then, for every $1 \leq i \leq n$, there exist an injective ring endomorphism $\sigma_i : R \to R$ and a σ_i-derivation $\delta_i : R \to R$ such that*

$$x_i r = \sigma_i(r) x_i + \delta_i(r),$$

for each $r \in R$.

Proof. See [5, Proposition 3]. □

A particular case of skew PBW extension is when all derivations δ_i are zero. Another interesting case is when all σ_i are bijective and the constants c_{ij} are invertible. We recall the following definition (cf. [5]).

Definition 1.3. *Let A be a skew PBW extension.*

(a) *A is* quasi-commutative *if the conditions* (iii) *and* (iv) *in Definition 1.1 are replaced by*

(iii') *For every $1 \leq i \leq n$ and $r \in R - \{0\}$ there exists $c_{i,r} \in R - \{0\}$ such that*

$$x_i r = c_{i,r} x_i.$$

(iv') *For every $1 \leq i, j \leq n$ there exists $c_{i,j} \in R - \{0\}$ such that*

$$x_j x_i = c_{i,j} x_i x_j.$$

(b) *A is* bijective *if σ_i is bijective for every $1 \leq i \leq n$ and $c_{i,j}$ is invertible for any $1 \leq i < j \leq n$.*

Some extra notation will be used in the paper.

Definition 1.4. *Let A be a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$, as in Proposition 1.2.*

(i) *For* $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\sigma^{\alpha} := \sigma_1^{\alpha_1} \cdots \sigma_n^{\alpha_n}$, $|\alpha| := \alpha_1 + \cdots + \alpha_n$. *If* $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, *then* $\alpha + \beta := (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$.

(ii) *For* $X = x^{\alpha} \in \mathrm{Mon}(A)$, $\exp(X) := \alpha$ *and* $\deg(X) := |\alpha|$.

(iii) *If* $f = c_1 X_1 + \cdots + c_t X_t$, *with* $X_i \in \mathrm{Mon}(A)$ *and* $c_i \in R - \{0\}$, *then* $\deg(f) := \max\{\deg(X_i)\}_{i=1}^{t}$.

A characterization of skew PBW extensions is obtained in [5]. This characterization is similar to the one obtained in [4] for PBW rings.

Theorem 1.5. *Let A and R be rings that satisfy the conditions (i) and (ii) of Definition 1.1. A is a skew PBW extension of R with endomorphisms σ_i, $1 \leq i \leq n$, if and only if the following conditions hold:*

(a) *For every* $x^\alpha \in \operatorname{Mon}(A)$ *and every* $0 \neq r \in R$ *there exist unique elements* $r_\alpha := \sigma^\alpha(r) \in R - \{0\}$ *and* $p_{\alpha,r} \in A$ *such that*

$$x^\alpha r = r_\alpha x^\alpha + p_{\alpha,r},$$

where $p_{\alpha,r} = 0$ *or* $\deg(p_{\alpha,r}) < |\alpha|$ *if* $p_{\alpha,r} \neq 0$. *Moreover, if* r *is left invertible, then* r_α *is left invertible.*

(b) *For every* $x^\alpha, x^\beta \in \operatorname{Mon}(A)$ *there exist unique elements* $c_{\alpha,\beta} \in R$ *and* $p_{\alpha,\beta} \in A$ *such that*

$$x^\alpha x^\beta = c_{\alpha,\beta} x^{\alpha+\beta} + p_{\alpha,\beta},$$

where $c_{\alpha,\beta}$ *is left invertible,* $p_{\alpha,\beta} = 0$ *or* $\deg(p_{\alpha,\beta}) < |\alpha + \beta|$ *if* $p_{\alpha,\beta} \neq 0$.

Proof. See [5, Theorem 7]. □

Remark 1.6. Recall that if R is a ring, σ an endomorphism of R and δ is a σ-derivation, then the skew polynomial ring $R[x;\sigma,\delta]$ is the free left R-module with basis $\{x^k | k \geq 0\}$ and the multiplication is given by distributive law and the rule $xr = \sigma(r)x + \delta(r)$, $r \in R$. It says that $R[x;\sigma,\delta]$ is of endomorphism type if $\delta = 0$, and of derivation type if $\sigma = i_R$. Skew polynomial rings in several variables can be defined by iteration (see [10]). Observe that skew PBW extensions are not the same as iterated skew polynomial rings; for example, the universal enveloping algebra $\mathcal{U}(\mathcal{G})$ of a finite-dimensional Lie algebra $\mathcal{G}$ is a skew PBW extension but is not an iterated skew polynomial ring.

We recall also the following facts from [5].

Remark 1.7. (i) We observe that if A is quasi-commutative, then $p_{\alpha,r} = 0$ and $p_{\alpha,\beta} = 0$ for every $0 \neq r \in R$ and every $\alpha, \beta \in \mathbb{N}^n$.

(ii) If A is bijective, then $c_{\alpha,\beta}$ is invertible for any $\alpha, \beta \in \mathbb{N}^n$.

(iii) Let $\theta, \gamma, \beta \in \mathbb{N}^n$ and $c \in R$, then we have the following identities:

$$\sigma^\theta(c_{\gamma,\beta}) c_{\theta,\gamma+\beta} = c_{\theta,\gamma} c_{\theta+\gamma,\beta},$$
$$\sigma^\theta(\sigma^\gamma(c)) c_{\theta,\gamma} = c_{\theta,\gamma} \sigma^{\theta+\gamma}(c).$$

(iv) In $\operatorname{Mon}(A)$ we define

$$x^\alpha \succeq x^\beta \Leftrightarrow \begin{cases} x^\alpha = x^\beta, \text{ or} \\ x^\alpha \neq x^\beta \text{ but } |\alpha| > |\beta|, \text{ or} \\ x^\alpha \neq x^\beta, |\alpha| = |\beta| \text{ but } \exists i \text{ with } \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i. \end{cases}$$

It is clear that this is a total order on $\operatorname{Mon}(A)$. If $x^\alpha \succeq x^\beta$ but $x^\alpha \neq x^\beta$, we write $x^\alpha \succ x^\beta$. Each element $f \in A$ can be represented in a unique way as $f = c_1 x^{\alpha_1} + \cdots + c_t x^{\alpha_t}$, with $c_i \in R - \{0\}$, $1 \leq i \leq t$, and $x^{\alpha_1} \succ \cdots \succ x^{\alpha_t}$. We say that x^{α_1} is the *leading monomial* of f and we write $\operatorname{lm}(f) := x^{\alpha_1}$; c_1 is the *leading coefficient* of f, $\operatorname{lc}(f) := c_1$, and $c_1 x^{\alpha_1}$ is the *leading term* of f denoted by $\operatorname{lt}(f) := c_1 x^{\alpha_1}$.

Some properties of skew PBW extensions have been studied in previous works (see [8, 9]). For example, the global, Krull and Goldie dimensions of bijective skew PBW extensions were estimated in [9]. The next theorem establishes three classical ring theoretic results for skew PBW extensions.

Theorem 1.8. *Let A be a bijective skew PBW extension of a ring R.*

(i) *(Hilbert Basis Theorem) If R is a left (right) Noetherian ring then A is also left (right) Noetherian.*

(ii) *(Ore's theorem) If R is a left Ore domain, then A is also a left Ore domain, and hence A has left total division ring of fractions such that*

$$Q_l(A) \cong Q_l(\sigma(Q_l(R))\langle x_1, \dots, x_n\rangle), \text{ with } \sigma_i\left(\frac{a}{s}\right) := \frac{\sigma_i(a)}{\sigma_i(s)}, \quad i = 1, 2, \dots, n.$$

(iii) *(Goldie's theorem) If R is a semiprime left Goldie ring, then A is semiprime left Goldie, and hence, $Q_l(A)$ exists and it is semisimple.*

2. Invariant ideals

Let $A = \sigma(R)\langle x_1, \dots, x_n\rangle$ be a skew PBW extension of a ring R. By Proposition 1.2, we know that $x_i r - \sigma_i(r)x_i = \delta_i(r)$ for all $r \in R$, where σ_i is an injective endomorphism of R and δ_i is a σ_i-derivation of R, $1 \leq i \leq n$. This motivates the following general definition.

Definition 2.1. *Let R be a ring, $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ be a finite set of endomorphisms of R and $\Delta := \{\delta_1, \dots, \delta_n\}$ be a finite set such that δ_i is a σ_i-derivation of R, $1 \leq i \leq n$. In this situation we say that (Σ, Δ) is a system of endomorphisms and Σ-derivations of R.*

(i) *If I is an ideal of R, I is called Σ-invariant if $\sigma_i(I) \subseteq I$, for every $1 \leq i \leq n$. Δ-invariant ideals are defined similarly. If I is both Σ and Δ-invariant, we say that I is (Σ, Δ)-invariant.*

(ii) *A proper Σ-invariant ideal I of R is Σ-prime if whenever a product of two Σ-invariant ideals is contained in I, one of the ideals is contained in I. R is a Σ-prime ring if the ideal 0 is Σ-prime. Δ-prime and (Σ, Δ)-prime ideals and rings are defined similarly.*

(iii) *The system Σ is commutative if $\sigma_i\sigma_j = \sigma_j\sigma_i$ for every $1 \leq i, j \leq n$. The commutativity for Δ is defined similarly. The system (Σ, Δ) is commutative if both Σ and Δ are commutative.*

The following proposition describes the behavior of these properties when we pass to a quotient ring.

Proposition 2.2. *Let R be a ring, (Σ, Δ) a system of endomorphisms and Σ-derivations of R, I a proper ideal of R and $\overline{R} := R/I$.*

(i) *If I is (Σ, Δ)-invariant, then over $\overline{R} := R/I$ it is induced a system $(\overline{\Sigma}, \overline{\Delta})$ of endomorphisms and $\overline{\Sigma}$-derivations defined by $\overline{\sigma_i}(\overline{r}) := \overline{\sigma_i(r)}$ and $\overline{\delta_i}(\overline{r}) := \overline{\delta_i(r)}$, $1 \leq i \leq n$. If σ_i is bijective and $\sigma_i(I) = I$, then $\overline{\sigma_i}$ is bijective.*

(ii) *If I be Σ-invariant and Σ is commutative, then $\overline{\Sigma}$ is commutative. This property holds for Δ and (Σ, Δ).*
(iii) *Let I be Σ-invariant. I is Σ-prime if and only if $\overline{R}$ is $\overline{\Sigma}$-prime. Similar properties hold for Δ and (Σ, Δ).*

Proof. All statements follow directly from the definitions. □

According to the properties of Σ and Δ, we need to introduce some special classes of skew PBW extensions.

Definition 2.3. *Let A be a skew PBW extension of a ring R with a set of endomorphisms $\Sigma := \{\sigma_1, \dots, \sigma_n\}$ and Σ-derivations $\Delta := \{\delta_1, \dots, \delta_n\}$.*
(i) *If $\sigma_i = i_R$ for every $1 \leq i \leq n$, we say that A is a skew PBW extension of derivation type.*
(ii) *If $\delta_i = 0$ for every $1 \leq i \leq n$, we say that A is a skew PBW extension of endomorphism type. In addition, if every σ_i is bijective, A is a skew PBW extension of automorphism type.*
(iii) *A is Σ-commutative if the set Σ is commutative. Δ and (Σ, Δ)-commutativity of A are defined similarly.*

Related to the previous definition, we have the following two interesting results. The second one extends Lemma 1.5 (c) in [7].

Proposition 2.4. *Let A be a skew PBW extension of derivation type of a ring R. Then, for any $\theta, \gamma, \beta \in \mathbb{N}^n$ and $c \in R$, the following identities hold:*

$$c_{\gamma,\beta} c_{\theta,\gamma+\beta} = c_{\theta,\gamma} c_{\theta+\gamma,\beta},$$
$$c c_{\theta,\gamma} = c_{\theta,\gamma} c.$$

In particular, the system of constants $c_{i,j}$ is central.

Proof. This is a direct consequence of Remark 1.7, part (iii). □

Proposition 2.5. *Let A be a skew PBW extension of a ring R. If for every $1 \leq i \leq n$, δ_i is inner, then A is a skew PBW extension of R of endomorphism type.*

Proof. Let $a_i \in R$ such that $\delta_i = \delta_{a_i}$ is inner, $1 \leq i \leq n$. We will prove that $A = \sigma(R)\langle z_1, \dots, z_n \rangle$, where $z_i := x_i - a_i$, the set of endomorphisms coincides with the original set Σ and every σ_i-derivation is equal to zero. We will check the conditions in Definition 1.1. It is clear that $R \subseteq A$. Let $r \in R$, then $z_i r = (x_i - a_i)r = x_i r - a_i r = \sigma_i(r)x_i + \delta_{a_i}(r) - a_i r = \sigma_i(r)x_i + a_i r - \sigma_i(r)a_i - a_i r = \sigma_i(r)(x_i - a_i) = \sigma_i(r)z_i$. Thus, the set of constants $c_{i,r}$ of $\sigma(R)\langle z_1, \dots, z_n \rangle$ coincides with the original one, and the same is true for the set of endomorphisms. Note that the set of Σ-derivations is trivial, i.e., each one is equal to zero. This means that $\sigma(R)\langle z_1, \dots, z_n \rangle$ is of endomorphism type.

Now we consider the commutation of variables: $z_j z_i = (x_j - a_j)(x_i - a_i) = x_j x_i - x_j a_i - a_j x_i + a_j a_i = c_{ij} x_i x_j + r_0 + r_1 x_1 + \cdots + r_n x_n - x_j a_i - a_j x_i + a_j a_i$, for some $r_0, r_1, \dots, r_n \in R$. Replacing x_i by $z_i + a_i$ for every $1 \leq i \leq n$, we conclude that $z_j z_i - c_{ij} z_i z_j \in R + R z_1 + \cdots + R z_n$.

Finally, note that $\mathrm{Mon}\{z_1,\dots,z_n\} := \{\, z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n} \mid \alpha = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n \,\}$ is a left R-basis of A. In fact, it is clear that $\mathrm{Mon}\{z_1,\dots,z_n\}$ generates A as left R-module. Let $c_1,\dots,c_t \in R$ such that $c_1 z^{\alpha_1} + \cdots + c_t z^{\alpha_t} = 0$ with $z^{\alpha_i} \in \mathrm{Mon}\{z_1,\dots,z_n\}$, $1 \le i \le n$. Then, using the deglex order in Remark 1.7, we conclude that $c_1 x^{\alpha_1} + \cdots + c_t x^{\alpha_t}$ should be zero, whence $c_1 = \cdots = c_t = 0$. □

In the next proposition we study quotients of skew PBW extensions by (Σ,Δ)-invariant ideals.

Proposition 2.6. *Let A be a skew PBW extension of a ring R and I a (Σ,Δ)-invariant ideal of R. Then,*

- (i) *IA is an ideal of A and $IA \cap R = I$. IA is proper if and only if I is proper. Moreover, if for every $1 \le i \le n$, σ_i is bijective and $\sigma_i(I) = I$, then $IA = AI$.*
- (ii) *If I is proper and $\sigma_i(I) = I$ for every $1 \le i \le n$, then A/IA is a skew PBW extension of R/I. Moreover, if A is of automorphism type, then A/IA is of automorphism type. If A is bijective, then A/IA is bijective. In addition, if A is Σ-commutative, then A/IA is $\overline{\Sigma}$-commutative. Similar properties are true for the $\overline{\Delta}$ and $(\overline{\Sigma},\overline{\Delta})$ commutativity.*
- (iii) *Let A be of derivation type and I proper. Then, $IA = AI$ and A/IA is a skew PBW extension of derivation type of R/I.*
- (iv) *Let R be left (right) Noetherian and σ_i bijective for every $1 \le i \le n$. Then, $\sigma_i(I) = I$ for every i and $IA = AI$. If I is proper and A is bijective, then A/IA is a bijective skew PBW extension of R/I.*

Proof. (i) It is clear that IA is a right ideal, but since I is (Σ,Δ)-invariant, then IA is also a left ideal of A. It is obvious that $IA \cap R = I$. From this last equality we get also that IA is proper if and only if I is proper. Using again that I is (Σ,Δ)-invariant, we get that $AI \subseteq IA$. Assuming that σ_i is bijective and $\sigma_i(I) = I$ for every i, then $IA \subseteq AI$.

(ii) According to (i), we only have to show that $\overline{A} := A/IA$ is a skew PBW extension of $\overline{R} := R/I$. For this we will verify the four conditions of Definition 1.1. It is clear that $\overline{R} \subseteq \overline{A}$. Moreover, $\overline{A}$ is a left $\overline{R}$-module with generating set $\mathrm{Mon}\{\overline{x_1},\dots,\overline{x_n}\}$. Next we show that $\mathrm{Mon}\{\overline{x_1},\dots,\overline{x_n}\}$ is independent. Consider the expression $\overline{r_1}\,\overline{X_1} + \cdots + \overline{r_n}\overline{X_n} = \overline{0}$, where $X_i \in \mathrm{Mon}(A)$ for each i. We have $r_1X_1 + \cdots + r_nX_n \in IA$ and hence

$$r_1X_1 + \cdots + r_nX_n = r_1'X_1 + \cdots + r_n'X_n, \quad \text{for some } r_i' \in I,\ i = 1,\dots,n.$$

Thus, $(r_1 - r_1')X_1 + \cdots + (r_n - r_n')X_n = 0$, so $r_i \in I$, i.e., $\overline{r_i} = \overline{0}$ for $i = 1,\dots,n$.

Let $\overline{r} \ne \overline{0}$ with $r \in R$. Then $r \notin IA$, and hence, $r \notin I$, in particular, $r \ne 0$ and there exists $c_{i,r} := \sigma_i(r) \ne 0$ such that $x_i r = c_{i,r}x_i + \delta_i(r)$. Thus, $\overline{x_i}\,\overline{r} = \overline{c_{i,r}}\,\overline{x_i} + \overline{\delta_i(r)}$. Observe that $\overline{c_{i,r}} \ne \overline{0}$; otherwise $c_{i,r} = \sigma_i(r) \in IA \cap R = I = \sigma_i(I)$, i.e., $r \in I$, a contradiction. This completes the proof of condition (iii) in Definition 1.1.

In A we have $x_jx_i - c_{i,j}x_ix_j \in R + \sum_{t=1}^n Rx_t$, with $c_{i,j} \in R - \{0\}$, so in $\overline{A}$ we get that $\overline{x_j}\,\overline{x_i} - \overline{c_{i,j}}\,\overline{x_i}\,\overline{x_j} \in \overline{R} + \sum_{t=1}^n \overline{R}\,\overline{x_t}$. Since I is proper and $c_{i,j}$ is left invertible for $i < j$ and right invertible for $i > j$, then $\overline{c_{i,j}} \neq \overline{0}$. This completes the proof of condition (iv) in Definition 1.1.

By Proposition 2.2, if σ_i is bijective, then $\overline{\sigma_i}$ is bijective. It is obvious that if every constant c_{ij} is invertible, then $\overline{c_{ij}}$ is invertible..

The statements about the commutativity follow from Proposition 2.2.

(iii) This is a direct consequence of (i) and (ii).

(iv) Considering the Noetherian condition and the ascending chain $I \subseteq \sigma_i^{-1}(I) \subseteq \sigma_i^{-2}(I) \subseteq \cdots$ we get that $\sigma_i(I) = I$ for every i. The rest follows from (i) and (ii). ☐

3. Extensions of derivation type

Now we pass to describe the prime ideals of skew PBW extensions of derivation type. Two technical propositions are needed first. The total order introduced in Remark 1.7 will be used in what follows.

Proposition 3.1. *Let A be a skew PBW extension of a ring R such that σ_i is bijective for every $1 \leq i \leq n$. Let J be a nonzero ideal of A. If f is a nonzero element of J of minimal leading monomial x^{α_t} and $\sigma^{\alpha_t}(r) = r$ for any $r \in \operatorname{rann}_R(\operatorname{lc}(f))$, then* $\operatorname{rann}_A(f) = (\operatorname{rann}_R(\operatorname{lc}(f)))A$.

Proof. Consider $0 \neq f = m_1X_1 + \cdots + m_tX_t$ an element of J of minimal leading monomial $X_t = x^{\alpha_t}$, with $X_t \succ X_{t-1} \succ \cdots \succ X_1$. By definition of the right annihilator, $\operatorname{rann}_R(\operatorname{lc}(f)) := \{r \in R \mid m_tr = 0\}$. From Theorem 1.5 we have

$$fr = m_1X_1r + \cdots + m_t(\sigma^{\alpha_t}(r)x^{\alpha_t} + p_{\alpha_t,r}),$$

where $p_{\alpha_t,r} = 0$ or $\deg(p_{\alpha_t,r}) < |\alpha_t|$ if $p_{\alpha_t,r} \neq 0$. Note that if $r \in \operatorname{rann}_R(\operatorname{lc}(f))$, then $fr = 0$. In fact, if the contrary is assumed, since $\sigma^{\alpha_t}(r) = r$, we get $\operatorname{lm}(fr) \prec X_t$ with $fr \in J$, but this is a contradiction since f has a leading minimal monomial. Thus, $f\operatorname{rann}_R(\operatorname{lc}(f)) = 0$ and $f\operatorname{rann}_R(\operatorname{lc}(f))A = 0$. Therefore $\operatorname{rann}_R(\operatorname{lc}(f))A \subseteq \operatorname{rann}_A(f)$.

Next we will show that $\operatorname{rann}_A(f) \subseteq \operatorname{rann}_R(\operatorname{lc}(f))A$. Let $u = r_1Y_1 + \cdots + r_kY_k$ be an element of $\operatorname{rann}_A(f)$, with $Y_k \succ Y_{k-1} \succ \cdots \succ Y_1$, then

$$fu = (m_1X_1 + \cdots + m_tX_t)(r_1Y_1 + \cdots + r_kY_k) = 0,$$

which implies that $m_tX_tr_kY_k = 0$, whence $m_t\sigma^{\alpha_t}(r_k)X_tY_k = 0$, that is, $m_t\sigma^{\alpha_t}(r_k) = 0$ which means $\sigma^{\alpha_t}(r_k) \in \operatorname{rann}_R(m_t)$. Let $\sigma^{\alpha_t}(r_k) := s$. Then $s \in \operatorname{rann}_R(\operatorname{lc}(f))$. Note that $r_k = \sigma^{-\alpha_t}(s) = s$; moreover $\sigma^{\alpha_t}(s) = s$ implies $s = \sigma^{-\alpha_t}(s)$ and hence $r_k \in \operatorname{rann}_R(\operatorname{lc}(f))$. This shows that $r_kY_k \in \operatorname{rann}_R(\operatorname{lc}(f))A \subseteq \operatorname{rann}_A(f)$, but since $u \in \operatorname{rann}_A(f)$ then $u - r_kY_k \in \operatorname{rann}_A(f)$. Continuing in this way we obtain that $r_{k-1}Y_{k-1}, r_{k-2}Y_{k-2}, \ldots, r_1Y_1 \in \operatorname{rann}_R(\operatorname{lc}(f))A$, which guarantees that $u \in \operatorname{rann}_R(\operatorname{lc}(f))A$. Thus, we have proved that $\operatorname{rann}_A(f) \subseteq \operatorname{rann}_R(\operatorname{lc}(f))A$. ☐

Proposition 3.2. *Let A be a skew PBW extension of derivation type of a ring R and let K be a non zero ideal of A. Let K' be the ideal of R generated by all coefficients of all terms of all polynomials of K. Then K' is a Δ-invariant ideal of R.*

Proof. Let $k \in K'$, then k is a finite sum of elements of the form rcr', with $r, r' \in R$ and c is the coefficient of one term of some polynomial of K. It is enough to prove that for every $1 \leq i \leq n$, $\delta_i(rcr') \in K'$. We have $\delta_i(rcr') = rc\delta_i(r') + r\delta_i(c)r' + \delta_i(r)cr'$. Note that $rc\delta_i(r'), \delta_i(r)cr' \in K'$, so it only rests to prove that $\delta_i(c) \in K'$. There exists $p \in K$ such that $p = cx^\alpha + p'$, with $p' \in A$ and x^α does not appear in p'. Note that the coefficients of all terms of p' are also in K'. Observe that $x_ip \in K$ and we have

$$x_ip = x_icx^\alpha + x_ip' = cx_ix^\alpha + \delta_i(c)x^\alpha + x_ip'.$$

From the previous expression we conclude that the coefficient of x^α in x_ip is $\delta_i(c) + cr + r'$, where r is the coefficient of x^α in x_ix^α and r' is the coefficient of x^α in x_ip'. Since $c \in K'$ we only have to prove that $r' \in K'$. Let $p' = c_1x^{\beta_1} + \cdots + c_tx^{\beta_t}$, then $c_1, \ldots, c_t \in K'$ and $x^\alpha \notin \{x^{\beta_1}, \ldots, x^{\beta_t}\}$. We have

$$\begin{aligned} x_ip' &= (c_1x_i + \delta_i(c_1))x^{\beta_1} + \cdots + (c_tx_i + \delta_i(c_t))x^{\beta_t} \\ &= c_1x_ix^{\beta_1} + \delta_i(c_1)x^{\beta_1} + \cdots + c_tx_ix^{\beta_t} + \delta_i(c_t)x^{\beta_t}; \end{aligned}$$

from the previous expression we get that r' has the form $r' = c_1r_1 + \cdots + c_tr_t$, where r_j is the coefficient of x^α in $x_ix^{\beta_j}$, $1 \leq j \leq t$. This proves that $r' \in K'$. □

The following theorem gives a description of prime ideals of skew PBW extensions of derivation type without assuming any conditions on the ring of coefficients. This result generalizes the description of prime ideals of classical PBW extensions given in Proposition 6.2 of [3]. Compare also with [10], Proposition 14.2.5 and Corollary 14.2.6.

Theorem 3.3. *Let A be a skew PBW extension of derivation type of a ring R. Let I be a proper Δ-invariant ideal of R. I is a Δ-prime ideal of R if and only if IA is a prime ideal of A. In such case, $IA = AI$ and $IA \cap R = I$.*

Proof. By Proposition 2.6 we know that A/IA is a skew PBW extension of R/I of derivation type, $IA = AI$ and $IA \cap R = I$. Then we may assume that $I = 0$. Note that if R is not Δ-prime, then A is not prime. Indeed, there exist $I, J \neq 0$ Δ-invariant ideals of R such that $IJ = 0$, so $IA, JA \neq 0$ and $IAJA = IJA = 0$, i.e., A is not prime.

Suppose that R is Δ-prime. We need to show that if J, K are nonzero ideals of A, then $JK \neq 0$. Let K' be as in Proposition 3.2, then $K' \neq 0$ and it is Δ-invariant. Now let j be a nonzero element of J of minimal leading monomial. If $jK = 0$, then taking $f = j$ in Proposition 3.1 we get

$$K \subseteq \mathrm{rann}_A(j) = \mathrm{rann}_R(\mathrm{lc}(j))A.$$

Therefore $\mathrm{lc}(j)K' = 0$ and hence $\mathrm{lann}_R(K') \neq 0$. We have $\mathrm{lann}_R(K')K' = 0$, and note that $\mathrm{lann}_R(K')$ is also Δ-invariant. In fact, let $a \in \mathrm{lann}_R(K')$ and $k' \in K'$, then $\delta_i(a)k' = \delta_i(ak') - a\delta_i(k') = \delta_i(0) - a\delta_i(k') = 0$ since $\delta_i(k') \in K'$. Thus, R is not Δ-prime, a contradiction. In this way $jK \neq 0$ and so $JK \neq 0$, which concludes the proof. □

4. Extensions of automorphism type

In this section we consider the characterization of prime ideals for extensions of automorphism type over commutative Noetherian rings.

Proposition 4.1. *Let A be a bijective skew PBW extension of a ring R. Suppose that given $a, b \in R - \{0\}$ there exists $\theta \in \mathbb{N}^n$ such that either $aR\sigma^{\theta}(b) \neq 0$ or $aR\delta^{\theta}(b) \neq 0$. Then, A is a prime ring.*

Proof. Suppose that A is not a prime ring, then there exist nonzero ideals I, J of A such that $IJ = 0$. We can assume that $I := \mathrm{lann}_A(J)$ and $J := \mathrm{rann}_A(I)$. Let u be a nonzero element of I with minimal leading monomial x^{α} and leading coefficient c_u. We will prove first that $\sigma^{-\alpha}(c_u) \in I$, i.e., $\sigma^{-\alpha}(c_u)J = 0$. Since $\mathrm{rann}_A(I) \subseteq \mathrm{rann}_A(u)$, then it is enough to show that $\sigma^{-\alpha}(c_u)\mathrm{rann}_A(u) = 0$. Suppose that $\sigma^{-\alpha}(c_u)\mathrm{rann}_A(u) \neq 0$, let $v \in \mathrm{rann}_A(u)$ of minimal leading monomial x^{β} and leading coefficient c_v such that $\sigma^{-\alpha}(c_u)v \neq 0$. Since $uv = 0$ and $c_{\alpha,\beta}$ is invertible (see Theorem 1.5 and Remark 1.7), then $c_u\sigma^{\alpha}(c_v) = 0$, whence $\mathrm{lm}(uc_v) \prec x^{\alpha}$. The minimality of x^{α} implies that $uc_v = 0$, and hence $u(v - c_v x^{\beta}) = 0$. Moreover, $v - c_v x^{\beta} \in \mathrm{rann}_A(u)$ and $\mathrm{lm}((v - c_v x^{\beta}) \prec x^{\beta}$, so $\sigma^{-\alpha}(c_u)(v - c_v x^{\beta}) = 0$. However, $c_u\sigma^{\alpha}(c_v) = 0$, so we have $\sigma^{-\alpha}(c_u)c_v = 0$ and hence $\sigma^{-\alpha}(c_u)v = 0$, a contradiction.

Thus, $I \cap R \neq 0$, and by symmetry, $J \cap R \neq 0$. Let $0 \neq a \in I \cap R$ and $0 \neq b \in J \cap R$, by the hypothesis there exist $\theta \in \mathbb{N}^n$ and $r \in R$ such that $ar\sigma^{\theta}(b) \neq 0$ or $ar\delta^{\theta}(b) \neq 0$. If $\theta = (0, \dots, 0)$, then $arb \neq 0$, and hence $IJ \neq 0$, a contradiction. If $\theta \neq (0, \dots, 0)$, then $arx^{\theta}b = ar(\sigma^{\theta}(b)x^{\theta} + p_{\theta,b})$, but note that the independent term of $p_{\theta,b}$ is $\delta^{\theta}(b)$ (see Theorem 1.5, part (b)). Thus, $arx^{\theta}b \neq 0$, i.e., $IJ \neq 0$, a contradiction. □

Corollary 4.2. *If R is a prime ring and A is a bijective skew PBW extension of R, then A is prime and* $\mathrm{rad}(A) = 0$.

Proof. Since R is prime and A is bijective, given $a, b \in R - \{0\}$, then $aR\sigma^{\theta}(b) \neq 0$ for every $\theta \in \mathbb{N}^n$. Thus, from the previous Proposition, A is prime. □

Lemma 4.3. *Let A be a bijective Σ-commutative skew PBW extension of automorphism type of a left (right) Noetherian ring R. Let I be a proper ideal of R Σ-invariant. I is a Σ-prime ideal of R if and only if IA is a prime ideal of A. In such case, $IA = AI$ and $IA \cap R = I$.*

Proof. By Proposition 2.6, $IA = AI$ is a proper ideal of A, $I = IA \cap R$ and $\overline{A} := A/IA$ is a bijective skew PBW extension of $\overline{R} := R/I$ of automorphism type. In addition, observe that I is Σ-prime if and only if $\overline{R}$ is $\overline{\Sigma}$-prime (Proposition 2.2). Thus, we can assume that $I = 0$, and hence, we have to prove that R is Σ-prime if and only if A is prime.

$\Rightarrow$): Suppose that R is Σ-prime, i.e., 0 is Σ-prime. According to Proposition 4.1, we have to show that given $a, b \in R - \{0\}$ there exists $\theta \in \mathbb{N}^n$ such that $aR\sigma^{\theta}(b) \neq 0$. Let L be the ideal generated by the elements $\sigma^{\theta}(b)$, $\theta \in \mathbb{N}^n$; observe that $L \neq 0$, and since A is Σ-commutative, L is Σ-invariant. But R is Noetherian and A is bijective, then $\sigma_i(L) = L$ for every $1 \leq i \leq n$ (see Proposition 2.6). This implies that $\mathrm{Ann}_R(L)$ is Σ-invariant, but 0 is Σ-prime, therefore $\mathrm{Ann}_R(L) = 0$. Thus, $aL \neq 0$, so there exists $\theta \in \mathbb{N}^n$ such that $a\sigma^{\theta}(b) \neq 0$.

$\Leftarrow$): Note that if R is not Σ-prime, then A is not prime. Indeed, there exist $K, J \neq 0$ Σ-invariant ideals of R such that $KJ = 0$, so $KA, JA \neq 0$ and since J is Σ-invariant, then $AJ = JA$ and hence $KAJA = KJA = 0$, i.e., A is not prime. □

Proposition 4.4. *Any bijective skew PBW extension A of a commutative ring R is Σ-commutative.*

Proof. For $i = j$ it is clear that $\sigma_j\sigma_i = \sigma_i\sigma_j$. Let $i \neq j$, say $i < j$; then for any $r \in R$ we have $\mathrm{lc}(x_jx_ir) = \sigma_j\sigma_i(r)c_{ij} = c_{ij}\sigma_i\sigma_j(r)$, but since R is commutative and c_{ij} is invertible, then $\sigma_j\sigma_i(r) = \sigma_i\sigma_j(r)$. □

Theorem 4.5. *Let A be a bijective skew PBW extension of automorphism type of a commutative Noetherian ring R. Let I be a proper Σ-invariant ideal of R. I is a Σ-prime ideal of R if and only if IA is a prime ideal of A. In such case, $IA = AI$ and $IA \cap R = I$.*

Proof. This follows from Lemma 4.3 and Propositions 2.6 and 4.4. □

5. Extensions of mixed type

Our next task is to give a description of prime ideals of bijective skew PBW extensions of mixed type, i.e., when both systems Σ and Δ could be non trivial. We will assume that the ring R is commutative, Noetherian and semiprime. The proof of the main theorem (Theorem 5.7) is as in Lemma 4.3, but anyway we have to show first some preliminary technical propositions.

Definition 5.1. *Let R be a commutative ring, I a proper ideal of R and $\overline{R} := R/I$. We define*

$$S(I) := \{\, a \in R \mid \overline{a} := a + I \ \textit{is regular} \}.$$

By regular element we mean a non zero divisor. Note that $S(0)$ is the set of regular elements of R. Next we will describe the behavior of the properties introduced in Definition 2.1 when we pass to the total ring of fractions.

Proposition 5.2. *Let R be a commutative ring with total ring of fractions $Q(R)$, and let (Σ, Δ) be a system of automorphisms and Σ-derivations of R. Then,*

(i) *Over $Q(R)$ is induced a system $(\widetilde{\Sigma}, \widetilde{\Delta})$ of automorphisms and $\widetilde{\Sigma}$-derivations defined by $\widetilde{\sigma_i}(\frac{a}{s}) := \frac{\sigma_i(a)}{\sigma_i(s)}$ and $\widetilde{\delta_i}(\frac{a}{s}) := -\frac{\delta_i(s)}{\sigma_i(s)}\frac{a}{s} + \frac{\delta_i(a)}{\sigma_i(s)}$.*

(ii) *$Q(R)$ is $\widetilde{\Sigma}$-prime if and only if R is Σ-prime. The same is valid for Δ and (Σ, Δ).*

Proof. (i) This part can be proved not only for commutative rings but also in the noncommutative case (see [8]).

(ii) $\Rightarrow$): Let I, J be Σ-invariant ideals of R such that $IJ = 0$, then $IJS(0)^{-1} = IS(0)^{-1}JS(0)^{-1} = 0$, but note that $IS(0)^{-1}, JS(0)^{-1}$ are $\widetilde{\Sigma}$-invariant, so $IS(0)^{-1} = 0$ or $JS(0)^{-1} = 0$, i.e., $I = 0$ or $J = 0$.

$\Leftarrow$): Let K, L be two $\widetilde{\Sigma}$-invariant ideals of $Q(R)$ such that $KL = 0$, then $K = IS(0)^{-1}$ and $L = JS(0)^{-1}$, with $I := \{a \in R \mid \frac{a}{1} \in K\}$ and $J := \{b \in R \mid \frac{b}{1} \in L\}$. Note that $IJ = 0$; moreover, I, J are Σ-invariant. In fact, if $a \in I$, then $\frac{a}{1} \in K$ and hence $\widetilde{\sigma_i}(\frac{a}{1}) = \frac{\sigma_i(a)}{1} \in K$, i.e., $\sigma_i(a) \in I$, for every $1 \leq i \leq n$. Analogously for L. Since R is Σ-prime, then $I = 0$ or $J = 0$, i.e., $K = 0$ or $L = 0$. The proofs for Δ and (Σ, Δ) are analogous. □

Next we will use the following special notation: Let $m \geq 0$ be an integer, then $\sigma(m)$ will denote the product of m endomorphisms taken from Σ in any order, and possibly with repetitions, i.e., $\sigma(m) = \sigma_{i_1} \cdots \sigma_{i_m}$, with $i_1, \ldots, i_m \in \{1, \ldots, n\}$. For $m = 0$ we will understand that this product is the identical isomorphism of R.

Proposition 5.3. *Let R be a commutative ring and (Σ, Δ) a system of endomorphisms and Σ-derivations of R. Let I be a Σ-invariant ideal of R. Set $I_0 := R$, $I_1 := I$, and for $j \geq 2$,*

$$I_j := \{\, r \in I \mid \delta_{i_1}\sigma(m(1))\delta_{i_2}\sigma(m(2)) \cdots \delta_{i_l}\sigma(m(l))(r) \in I, \\ \text{for all } l = 1, \ldots, j-1;\ i_1, \ldots, i_l \in \{1, \ldots, n\} \text{ and } m(1), \ldots, m(l) \geq 0 \,\}.$$

Then,

(i) $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$.
(ii) $\delta_i(I_j) \subseteq I_{j-1}$, *for every $1 \leq i \leq n$ and any $j \geq 1$.*
(iii) I_j *is a Σ-invariant ideal of R, for any $j \geq 0$.*
(iv) $II_j \subseteq I_{j+1}$, *for any $j \geq 0$.*

Proof. (i) This is evident.

(ii) It is clear that for every i, $\delta_i(I_1) \subseteq I_0$. Let $j \geq 2$ and let $r \in I_j$; then $\delta_{i_1}\sigma(m(1))\delta_{i_2}\sigma(m(2)) \cdots \delta_{i_l}\sigma(m(l))(r) \in I$, for all $l = 1, \ldots, j-1$. From this we obtain that $\delta_{i_1}\sigma(m(1))\delta_{i_2}\sigma(m(2)) \cdots \delta_{i_l}\sigma(m(l))\delta_i\sigma(0)(r) \in I$, for all $l = 1, \ldots, j-2$. This means that $\delta_i(r) \in I_{j-1}$.

(iii) It is clear that I_0, I_1 are Σ-invariant ideals. Let $j \geq 2$ and let $r \in I_j$; then for every σ_i we have

$$\delta_{i_1}\sigma(m(1))\delta_{i_2}\sigma(m(2)) \cdots \delta_{i_l}\sigma(m(l))(\sigma_i(r)) \\ = \delta_{i_1}\sigma(m(1))\delta_{i_2}\sigma(m(2)) \cdots \delta_{i_l}\sigma(m(l)+1)(r) \in I.$$

This means that $\sigma_i(r) \in I_j$, i.e., I_j is Σ-invariant. Let us prove that I_j is an ideal of R. By induction we assume that I_j is an ideal. It is obvious that if $a, a' \in I_{j+1}$, then $a + a' \in I_{j+1}$; let $r \in R$, then $a \in I_j$ and hence $ra \in I_j$, therefore $\delta_{i_1}\sigma(m(1))\delta_{i_2}\sigma(m(2)) \cdots \delta_{i_k}\sigma(m(k))(ra) \in I$ for all $k < j$. Consider any $\sigma(m(j))$, we have $\sigma(m(j))(a) \in I_{j+1}$, so $\sigma(m(j))(a), \delta_i(\sigma(m(j))(a)) \in I_j$ for every i. Therefore,

$$\delta_i(\sigma(m(j))(ra)) = \delta_i(\sigma(m(j))(r)\sigma(m(j))(a)) \\ = \sigma_i\sigma(m(j))(r)\delta_i(\sigma(m(j))(a)) + \delta_i(\sigma(m(j))(r))\sigma(m(j))(a) \in I_j.$$

This implies that $\delta_{i_1}\sigma(m(1))\delta_{i_2}\sigma(m(2)) \cdots \delta_{i_j}\sigma(m(j))(ra) \in I$. This indicates that $ra \in I_{j+1}$, and hence I_{j+1} is an ideal.

(iv) Of course $II_0 \subseteq I_1$. Let $j \geq 1$, and suppose that for $II_{j-1} \subseteq I_j$; we have to prove that $II_j \subseteq I_{j+1}$. Let $a \in I$ and $b \in I_j$, then $b \in I_{j-1}$ and $ab \in I_j$. Therefore, $\delta_{i_1}\sigma(m(1))\delta_{i_2}\sigma(m(2))\cdots\delta_{i_k}\sigma(m(k))(ab) \in I$ for $i < j$. For any $\sigma(m(j))$ and every δ_i we have, as above,

$$\begin{aligned}\delta_i(\sigma(m(j))(ab)) &= \delta_i(\sigma(m(j))(a)\sigma(m(j))(b)) \\ &= \sigma_i\sigma(m(j))(a)\delta_i(\sigma(m(j))(b)) \\ &\quad + \delta_i(\sigma(m(j))(a))\sigma(m(j))(b) \in II_{j-1} + RI_j \subseteq I_j.\end{aligned}$$

From this we conclude that $\delta_{i_1}\sigma(m(1))\delta_{i_2}\sigma(m(2))\cdots\delta_{i_j}\sigma(m(j))(ab) \in I$, and this means that $ab \in I_{j+1}$. This completes the proof. □

Proposition 5.4. *Let R be a commutative Noetherian ring and Σ a system of automorphisms of R. Then, any Σ-prime ideal of R is semiprime.*

Proof. Let I be a Σ-prime ideal of R. Since R is Noetherian, $\sqrt{I}$ is finitely generated, so there exists $m \geq 1$ such that $(\sqrt{I})^m \subseteq I$. Since I is Σ-invariant and R is Noetherian, $\sqrt{I}$ is Σ-invariant, whence $\sqrt{I} \subseteq I$, i.e., $\sqrt{I} = I$. This says that I is intersection of prime ideals, i.e., I is semiprime. □

Proposition 5.5. *Let R be a commutative Noetherian ring and (Σ, Δ) a system of automorphisms and Σ-derivations of R. If R is (Σ, Δ)-prime, then*

(i) $\operatorname{rad}(R)$ *is Σ-prime.*
(ii) $S(0) = S(\operatorname{rad}(R))$.
(iii) $Q(R)$ *is Artinian.*

Proof. (i) The set of Σ-invariant ideals I of R such that $\operatorname{Ann}_R(I) \neq 0$ is not empty since 0 satisfies these conditions. Since R is assumed to be Noetherian, let I be maximal with these conditions. Let K, L be Σ-invariant ideals of R such that $I \subsetneq K$ and $I \subsetneq L$, then $\operatorname{Ann}_R(K) = 0 = \operatorname{Ann}_R(L)$, and hence $\operatorname{Ann}_R(KL) = 0$. This implies that $KL \nsubseteq I$. This proves that I is Σ-prime.

We will prove that $I = \operatorname{rad}(R)$. By Proposition 5.3, we have the descending chain of Σ-invariant ideals $I_0 \supseteq I_1 \supseteq \cdots$, and the ascending chain $\operatorname{Ann}_R(I_0) \subseteq \operatorname{Ann}_R(I_1) \subseteq \cdots$. There exists $m \geq 1$ such that $\operatorname{Ann}_R(I_m) = \operatorname{Ann}_R(I_{m+1})$ (since $I_0 = R$ and $I_1 = I$, see Proposition 5.3, $m \neq 0$). Note that $\operatorname{Ann}_R(I_m)$ is Σ-invariant since $\sigma_i(I_m) = I_m$ for every i (here we have used again that R is Noetherian, Proposition 2.6). Let $b \in \operatorname{Ann}_R(I_m)$. For $a \in I_{m+1}$ we have $a \in I_m$; moreover, by Proposition 5.3, for every i, $\delta_i(a) \in I_m$, so $ab = 0 = \delta_i(a)b$, therefore $0 = \delta_i(ab) = \sigma_i(a)\delta_i(b)$. From this we obtain that $\sigma_i(I_{m+1})\delta_i(b) = 0$, i.e., $I_{m+1}\delta_i(b) = 0$. Thus, $\delta_i(b) \in \operatorname{Ann}_R(I_{m+1}) = \operatorname{Ann}_R(I_m)$. We have proved that $\operatorname{Ann}_R(I_m)$ is (Σ, Δ)-invariant.

Let $H := \operatorname{Ann}_R(\operatorname{Ann}_R(I_m))$; we shall see that H is also (Σ, Δ)-invariant. In fact, let $x \in H$, then $x \operatorname{Ann}_R(I_m) = 0$ and for every i we have $\sigma_i(x)\sigma_i(\operatorname{Ann}_R(I_m)) = 0 = \sigma_i(x)\operatorname{Ann}_R(I_m)$, thus $\sigma_i(x) \in H$. Now let $y \in \operatorname{Ann}_R(I_m)$, then $xy = 0$ and for every i we have $\delta_i(xy) = 0 = \sigma_i(x)\delta_i(y) + \delta_i(x)y$, but $\delta_i(y) \in \operatorname{Ann}_R(I_m)$, so $\sigma_i(x)\delta_i(y) = 0$. Thus, $\delta_i(x)y = 0$, i.e., $\delta_i(x) \in H$.

Since R is (Σ, Δ)-prime and $H \operatorname{Ann}_R(I_m) = 0$, then $H = 0$ or $\operatorname{Ann}_R(I_m) = 0$. From Proposition 5.3, $\operatorname{Ann}_R(I_m) \supseteq \operatorname{Ann}_R(I) \neq 0$, whence $H = 0$. Since $I_m \subseteq H$, then $I_m = 0$. Again, from Proposition 5.3 we obtain that $I^m \subseteq I_m$, so $I^m = 0$, and hence $I \subseteq \operatorname{rad}(R)$. On the other hand, since I is Σ-prime, then I is semiprime (Proposition 5.4), but $\operatorname{rad}(R)$ is the smallest semiprime ideal of R, whence $I = \operatorname{rad}(R)$. Thus, $\operatorname{rad}(R)$ is Σ-prime.

(ii) The inclusion $S(0) \subseteq S(\operatorname{rad}(R))$ is well known (see [10], Proposition 4.1.3). The other inclusion is equivalent to proving that R is $S(\operatorname{rad}(R))$-torsion free. Since $I_m = 0$, it is enough to prove that every factor I_j/I_{j+1} is $S(\operatorname{rad}(R))$-torsion free. In fact, in general, if M is an R-module and N is a submodule of M such that M/N and N are S-torsion free (with S an arbitrary system of R), then M is S-torsion free. Thus, the assertion follows from

$$R = I_0/I_m, \quad I_0/I_2/I_1/I_2 \cong I_0/I_1, \quad I_0/I_3/I_2/I_3 \cong I_0/I_2, \quad \ldots, \quad I_0/I_m/I_{m-1}/I_m \cong I_0/I_{m-1}.$$

$I_0/I_1 = R/\operatorname{rad}(R)$ is clearly $S(\operatorname{rad}(R))$-torsion free. By induction, we assume that I_{j-1}/I_j is $S(\operatorname{rad}(R))$-torsion free. Let $a \in I_j$ and $r \in S(\operatorname{rad}(R))$ such that $r\widehat{a} = \widehat{0}$ in I_j/I_{j+1}, i.e., $ra \in I_{j+1}$. From Proposition 5.3 we get that for every i and any $\sigma(m)$, $\sigma(m)(a) \in I_j$, $\sigma(m)(ra) \in I_{j+1}$ and $\delta_i(\sigma(m)(ra)) \in I_j$, then

$$\begin{aligned}\delta_i(\sigma(m)(ra)) &= \delta_i(\sigma(m)(r)\sigma(m)(a)) \\ &= \sigma_i(\sigma(m)(r))\delta_i(\sigma(m)(a)) + \delta_i(\sigma(m)(r))\sigma(m)(a) \in I_j,\end{aligned}$$

whence $\sigma_i(\sigma(m)(r))\delta_i(\sigma(m)(a)) \in I_j$. For every k, $\sigma_k(\operatorname{rad}(R)) = \operatorname{rad}(R)$, then we can prove that $\sigma_k(S(\operatorname{rad}(R))) = S(\operatorname{rad}(R))$, so $\sigma_i(\sigma(m)(r)) \in S(\operatorname{rad}(R))$; moreover, $\delta_i(\sigma(m)(a)) \in I_{j-1}$, then by induction $\delta_i(\sigma(m)(a)) \in I_j$. But this is valid for any i and any $\sigma(m)$, then $a \in I_{j+1}$. This proves that I_{j-1}/I_j is $S(\operatorname{rad}(R))$-torsion free.

(iii) This follows from (ii) and Small's theorem (see [10, Corollary 4.1.4]). □

Corollary 5.6. *Let R be a commutative Noetherian semiprime ring and (Σ, Δ) a system of automorphisms and Σ-derivations of R. If R is (Σ, Δ)-prime, then R is Σ-prime.*

Proof. By Proposition 5.5, the total ring of fractions $Q(R)$ of R is Artinian. Then, by Proposition 5.2, we can assume that R is Artinian. Applying again Proposition 5.5, we get that $0 = \operatorname{rad}(R)$ is Σ-prime, i.e., R is Σ-prime. □

Theorem 5.7. *Let R be a commutative Noetherian semiprime ring and A a bijective skew PBW extension of R. Let I be a semiprime (Σ, Δ)-invariant ideal of R. I is a (Σ, Δ)-prime ideal of R if and only if IA is a prime ideal of A. In such case, $IA = AI$ and $I = IA \cap R$.*

Proof. The proof is exactly as in Lemma 4.3; anyway we will repeat it. By Proposition 2.6, $IA = AI$ is a proper ideal of A, $I = IA \cap R$ and $\overline{A} := A/IA$ is a bijective skew PBW extension of the commutative Noetherian semiprime ring $\overline{R} := R/I$. In addition, observe that I is (Σ, Δ)-prime if and only if $\overline{R}$ is $(\overline{\Sigma}, \overline{\Delta})$-prime (see

Propositions 2.2 and 2.6). Thus, we can assume that $I = 0$, and hence, we have to prove that R is (Σ, Δ)-prime if and only if A is prime.

$\Rightarrow$): From Corollary 5.6, we know that R is Σ-prime, i.e., 0 is Σ-prime. Let L be the ideal generated by the elements $\sigma^{\theta}(b)$, $\theta \in \mathbb{N}^n$; observe that $L \neq 0$, and since A is Σ-commutative (Proposition 4.4), L is Σ-invariant. But R is Noetherian and A is bijective, then $\sigma_i(L) = L$ for every $1 \leq i \leq n$ (see Proposition 2.6). This implies that $\mathrm{Ann}_R(L)$ is Σ-invariant, therefore $\mathrm{Ann}_R(L) = 0$. Thus, $aL \neq 0$, so there exists $\theta \in \mathbb{N}^n$ such that $a\sigma^{\theta}(b) \neq 0$. From Proposition 4.1 we get that A is prime.

$\Leftarrow$): Note that if R is not (Σ, Δ)-prime, then A is not prime. Indeed, there exist $I, J \neq 0$ (Σ, Δ)-invariant ideals of R such that $IJ = 0$, so $IA, JA \neq 0$ and $IAJA = IJA = 0$, i.e., A is not prime. □

6. Examples

Some important examples of skew PBW extensions covered by Theorems 3.3, 4.5 or 5.7 are given in the following table. The definition of each ring or algebra can be found in [9]. For each example we have marked with ✓ the theorem that can be applied.

Example 6.1. Theorem 4.5 gives a description of prime ideals for the ring of skew quantum polynomials over commutative Noetherian rings. Skew quantum polynomials were defined in [9], and represent a generalization of Artamonov's quantum polynomials (see [1, 2]). They can be defined as a localization of a quasi-commutative bijective skew PBW extension. We recall next its definition. Let R be a ring with a fixed matrix of parameters $\mathbf{q} := [q_{ij}] \in M_n(R)$, $n \geq 2$, such that $q_{ii} = 1 = q_{ij}q_{ji} = q_{ji}q_{ij}$ for every $1 \leq i, j \leq n$, and suppose also that it is given a system $\sigma_1, \dots, \sigma_n$ of automorphisms of R. The ring of *skew quantum polynomials over* R, denoted by $R_{\mathbf{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$, is defined as follows:

(i) $R \subseteq R_{\mathbf{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$;

(ii) $R_{\mathbf{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$ is a free left R-module with basis
$$\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \in \mathbb{Z} \text{ for } 1 \leq i \leq r \text{ and } \alpha_i \in \mathbb{N} \text{ for } r+1 \leq i \leq n\};$$

(iii) the variables $x_1, \dots, x_n$ satisfy the defining relations
$$x_i x_i^{-1} = 1 = x_i^{-1} x_i, \quad 1 \leq i \leq r,$$
$$x_j x_i = q_{ij} x_i x_j, \quad x_i r = \sigma_i(r) x_i, \ r \in R, \ 1 \leq i, j \leq n.$$

When all automorphisms are trivial, we write $R_{\mathbf{q}}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$, and this ring is called the ring of *quantum polynomials over* R. If $R = K$ is a field, then $K_{\mathbf{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$ is the *algebra of skew quantum polynomials*. For trivial automorphisms we get the *algebra of quantum polynomials*, simply denoted by $\mathcal{O}_{\mathbf{q}}$ (see [1]). When $r = 0$, $R_{\mathbf{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n] = R_{\mathbf{q},\sigma}[x_1, \dots, x_n]$ is the *n-multiparametric skew quantum space over R*, and when $r = n$, it coincides with $R_{\mathbf{q},\sigma}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, i.e., with the *$n$-multiparametric skew quantum torus over R*.

Ring	**3.3**	**4.5**	**5.7**
Habitual polynomial ring $R[x_1, \dots, x_n]$	✓		
Ore extension of bijective type $R[x_1; \sigma_1, \delta_1] \cdots [x_n; \sigma_n, \delta_n]$, R commutative Noetherian semiprime, $\delta_i\delta_j = \delta_j\delta_i$			✓
Weyl algebra $A_n(K)$	✓		
Extended Weyl algebra $B_n(K)$	✓		
Universal enveloping algebra of a Lie algebra $\mathcal{G}$, $\mathcal{U}(\mathcal{G})$	✓		
Tensor product $R \otimes_K \mathcal{U}(\mathcal{G})$	✓		
Crossed product $R * \mathcal{U}(\mathcal{G})$	✓		
Algebra of q-differential operators $D_{q,h}[x, y]$			✓
Algebra of shift operators S_h		✓	
Mixed algebra D_h			✓
Discrete linear systems $K[t_1, \dots, t_n][x_1, \sigma_1] \cdots [x_n; \sigma_n]$		✓	
Linear partial shift operators $K[t_1, \dots, t_n][E_1, \dots, E_n]$		✓	
Linear partial shift operators $K(t_1, \dots, t_n)[E_1, \dots, E_n]$		✓	
L. P. Differential operators $K[t_1, \dots, t_n][\partial_1, \dots, \partial_n]$	✓		
L. P. Differential operators $K(t_1, \dots, t_n)[\partial_1, \dots, \partial_n]$	✓		
L. P. Difference operators $K[t_1, \dots, t_n][\Delta_1, \dots, \Delta_n]$			✓
L. P. Difference operators $K(t_1, \dots, t_n)[\Delta_1, \dots, \Delta_n]$			✓
L. P. q-dilation operators $K[t_1, \dots, t_n][H_1^{(q)}, \dots, H_m^{(q)}]$		✓	
L. P. q-dilation operators $K(t_1, \dots, t_n)[H_1^{(q)}, \dots, H_m^{(q)}]$		✓	
L. P. q-differential operators $K[t_1, \dots, t_n][D_1^{(q)}, \dots, D_m^{(q)}]$			✓
L. P. q-differential operators $K(t_1, \dots, t_n)[D_1^{(q)}, \dots, D_m^{(q)}]$			✓
Diffusion algebras	✓		
Additive analogue of the Weyl algebra $A_n(q_1, \dots, q_n)$			✓
Multiplicative analogue of the Weyl algebra $\mathcal{O}_n(\lambda_{ji})$		✓	
Quantum algebra $\mathcal{U}'(\mathfrak{so}(3, K))$	✓		
3-dimensional skew polynomial algebras	✓		
Dispin algebra $\mathcal{U}(osp(1, 2))$,	✓		
Woronowicz algebra $\mathcal{W}_\nu(\mathfrak{sl}(2, K))$	✓		
Complex algebra $V_q(\mathfrak{sl}_3(\mathbb{C}))$		✓	
Algebra $\mathbf{U}$		✓	
Manin algebra $\mathcal{O}_q(M_2(K))$		✓	
Coordinate algebra of the quantum group $SL_q(2)$		✓	
q-Heisenberg algebra $\mathbf{H}_n(q)$		✓	
Quantum enveloping algebra of $\mathfrak{sl}(2, K)$, $\mathcal{U}_q(\mathfrak{sl}(2, K))$		✓	
Hayashi algebra $W_q(J)$	✓		

TABLE 1. Prime ideals of skew PBW extensions.

Note that $R_{\mathbf{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$ can be viewed as a localization of the n-multiparametric skew quantum space, which, in turn, is a skew PBW extension. In fact, we have the quasi-commutative bijective skew PBW extension

$$A := \sigma(R)\langle x_1, \dots, x_n\rangle, \text{ with } x_i r = \sigma_i(r)x_i \text{ and } x_j x_i = q_{ij}x_i x_j,\ 1 \leq i, j \leq n;$$

observe that $A = R_{\mathbf{q},\sigma}[x_1, \dots, x_n]$. If we set

$$S := \{rx^\alpha \mid r \in R^*, x^\alpha \in \mathrm{Mon}\{x_1, \dots, x_r\}\},$$

then S is a multiplicative subset of A and

$$S^{-1}A \cong R_{\mathbf{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n] \cong AS^{-1}.$$

Thus, if R is a commutative Noetherian ring, then Theorem 4.5 gives a description of prime ideals for A. With this, we get a description of prime ideals for $R_{\mathbf{q},\sigma}[x_1^{\pm 1}, \dots, x_r^{\pm 1}, x_{r+1}, \dots, x_n]$ since it is well known that there exists a bijective correspondence between the prime ideals of $S^{-1}A$ and the prime ideals of A with empty intersection with S (recall that A is left (right) Noetherian, Theorem 1.8).

Acknowledgements

The authors are grateful to the editors and the referee for valuable suggestions and corrections.

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Received: June 25, 2014
Accepted: July 23, 2015