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A GENERALIZED VERSION OF FUBINI'S THEOREM ON $C_{a,b}[0,T]$ AND APPLICATIONS

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ABSTRACT. We define a transform with respect to the Gaussian process, the $\diamond$-product and the first variation on function space. We then establish a generalized Fubini theorem rather than the Fubini theorem introduced in H. S. Chung, J. G. Choi and S. J. Chang, Banach J. Math. Anal. **7** (2013), 173–185. Also, we examine the various relationships of the transform with respect to the Gaussian process, the $\diamond$-product and the first variation for functionals on function space.

1. Introduction

Let $C_0[0,T]$ denote one-parameter Wiener space, i.e., the space of real-valued continuous functions $x(t)$ on $[0,T]$ with $x(0) = 0$. In [13, 14], Huffman, Skoug and Storvick established Fubini theorems for various analytic Wiener and Feynman integrals. In [17], the authors used a weight function to establish a Fubini theorem for certain finite-dimensional functionals. Recently in [12] the authors studied the Fubini theorem on function space and established several relationships as applications of the Fubini theorem. The function space $C_{a,b}[0,T]$ induced by a generalized Brownian motion was introduced by J. Yeh in [19] and was used extensively by Chang and Chung [2].

In this paper, we establish a generalized Fubini theorem for function space. We then introduce a transform with respect to the Gaussian process, $\diamond$-product and the first variation. Also, we establish relationships involving the transform of $\diamond$-product and the first variation. In Section 3, we use the Gaussian process to establish a generalized Fubini theorem for function space. In Section 4, we establish the relationships involving exactly two of the three concepts of transform, the $\diamond$-product, and first variation of the functionals. In Section 5, we establish all the relationships involving all three of these concepts.

The stochastic process used in this paper as well as in [10, 12, 17, 19], is nonstationary in time, is subject to a drift $a(t)$, and can be used to explain the position of the Ornstein-Uhlenbeck process in an external force field [18]. However, when

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$a(t) \equiv 0$ and $b(t) = t$ on $[0, T]$, the general function space $C_{a,b}[0, T]$ reduces to the Wiener space $C_0[0, T]$.

2. Definitions and preliminaries

Let $D = [0, T]$ and let $(\Omega, \mathcal{B}, P)$ be a probability measure space. A real-valued stochastic process Y on $(\Omega, \mathcal{B}, P)$ and D is called *a generalized Brownian motion process* if $Y(0, \omega)=0$ almost everywhere and for $0 = t_0 < t_1 < \cdots < t_n \le T$, the n-dimensional random vector $(Y(t_1, \omega), \ldots, Y(t_n, \omega))$ is normally distributed with the density function

$$K(\vec{t}, \vec{\eta}) = \Big((2\pi)^n \prod_{j=1}^{n} \big(b(t_j) - b(t_{j-1})\big)\Big)^{-1/2} \cdot \exp\bigg\{-\frac{1}{2}\sum_{j=1}^{n} \frac{((\eta_j - a(t_j)) - (\eta_{j-1} - a(t_{j-1})))^2}{b(t_j) - b(t_{j-1})}\bigg\},$$

where $\vec{\eta} = (\eta_1, \ldots, \eta_n)$, $\eta_0 = 0$, $\vec{t} = (t_1, \ldots, t_n)$, $a(t)$ is an absolutely continuous real-valued function on $[0, T]$ with $a(0) = 0$, $a'(t) \in L^2[0, T]$, and $b(t)$ is a strictly increasing, continuously differentiable real-valued function with $b(0) = 0$ and $b'(t) > 0$ for each $t \in [0, T]$.

In [20], Yeh showed that the generalized Brownian motion process Y determined by $a(\cdot)$ and $b(\cdot)$ is a Gaussian process with mean function $a(t)$ and covariance function $r(t, s) = \min\{b(s), b(t)\}$, and that the probability measure μ induced by Y, taking a separable version, is supported by $C_{a,b}[0, T]$ (which is equivalent to the Banach space of continuous function x on $[0, T]$ with $x(0) = 0$ under the sup norm). Hence $(C_{a,b}[0, T], \mathcal{B}(C_{a,b}[0, T]), \mu)$ is the function space induced by Y, where $\mathcal{B}(C_{a,b}[0, T])$ is the Borel σ-algebra of $C_{a,b}[0, T]$. We then complete this function space to obtain $(C_{a,b}[0, T], \mathcal{W}(C_{a,b}[0, T]), \mu)$, where $\mathcal{W}(C_{a,b}[0, T])$ is the set of all Wiener measurable subsets of $C_{a,b}[0, T]$.

Let $L^2_{a,b}[0, T]$ be the set of functions on $[0, T]$ which are Lebesgue measurable and square integrable with respect to the Lebesgue-Stieltjes measures on $[0, T]$ induced by $a(\cdot)$ and $b(\cdot)$; i.e.,

$$L^2_{a,b}[0, T] = \bigg\{v : \int_0^T |v^2|(s)db(s) < \infty \text{ and } \int_0^T |v^2|(s)d|a|(s) < \infty\bigg\},$$

where $|a|(t)$ denotes the total variation of the function $a(\cdot)$ on the interval $[0, t]$.

For $u, v \in L^2_{a,b}[0, T]$, let

$$(u, v)_{a,b} = \int_0^T u(t)\overline{v(t)}d[b(t) + |a|(t)].$$

Then $(\cdot, \cdot)_{a,b}$ is an inner product on $L^2_{a,b}[0, T]$ and $\|u\|_{a,b} = \sqrt{(u, u)_{a,b}}$ is a norm on $L^2_{a,b}[0, T]$. In particular, note that $\|u\|_{a,b} = 0$ if and only if $u(t) = 0$ a.e. on $[0, T]$. Furthermore, $(L^2_{a,b}[0, T], \|\cdot\|_{a,b})$ is a separable Hilbert space.

Let $\{\phi_j\}_{j=1}^{\infty}$ be a complete orthogonal set of real-valued functions of bounded variation on $[0,T]$ such that

$$(\phi_j,\phi_k)_{a,b} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}.$$

Then for each $v \in L^2_{a,b}[0,T]$, the Paley-Wiener-Zygmund (PWZ) stochastic integral $\langle v,x\rangle$ is defined by the formula

$$\langle v,x\rangle = \lim_{n\to\infty}\int_0^T \sum_{j=1}^n (v,\phi_j)_{a,b}\phi_j(t)dx(t)$$

for all $x \in C_{a,b}[0,T]$ for which the limit exists. For $u \in L^2_{a,b}[0,T]$, let

$$(u,a') = \int_0^T u(t)a'(t)dt = \int_0^T u(t)da(t)$$

and

$$(u^2,b') = \int_0^T u^2(t)b'(t)dt = \int_0^T u^2(t)db(t).$$

For $h \in L^2_{a,b}[0,T]$, we define the Gaussian process Z_h by

$$Z_h(x,t) = \int_0^t h(s)\tilde{d}x(s), \tag{2.1}$$

where $\int_0^t h(s)\tilde{d}x(s)$ denotes the PWZ integral. For each $v \in L^2_{a,b}[0,T]$, let $\langle v,x\rangle = \int_0^T v(t)\tilde{d}x(t)$. From [9], we note that $\langle v, Z_h(x,\cdot)\rangle = \langle vh, x\rangle$ for $h \in L_\infty[0,T]$ and s-a.e. $x \in C_{a,b}[0,T]$. Thus, throughout this paper, we require h to be in $L_\infty[0,T]$ rather than simply in $L^2_{a,b}[0,T]$.

Let $K_{a,b}[0,T]$ be the set of all complex-valued continuous functions $x(t)$ defined on $[0,T]$ which vanish at $t=0$ and whose real and imaginary parts are elements of $C_{a,b}[0,T]$. Thus $C_{a,b}[0,T]$ is the subspace of all real-valued functions in $K_{a,b}[0,T]$.

Now, we state the definitions of the transform with respect to the Gaussian process, the $\diamond$-product and the first variation.

Definition 2.1. Let F and G be functionals on $K_{a,b}[0,T]$ and let γ, β, ρ and τ be non-zero complex numbers. Then the transform with respect to the Gaussian process, the $\diamond$-product and the first variation are defined by formulas

$$T^{h_1,h_2}_{\gamma,\beta}(F)(y) = \int_{C_{a,b}[0,T]} F(\gamma Z_{h_1}(x,\cdot) + \beta Z_{h_2}(y,\cdot))d\mu(x), \tag{2.2}$$

$$\begin{aligned}((F\diamond G)^{s_1,s_2}_{\rho,\tau})(y) = \int_{C_{a,b}[0,T]} & F(\tau Z_{s_2}(y,\cdot) + \rho Z_{s_1}(x,\cdot)) \\ & \cdot G(\tau Z_{s_2}(y,\cdot) - \rho Z_{s_1}(x,\cdot))d\mu(x),\end{aligned} \tag{2.3}$$

$$\delta F(Z_h(x,\cdot)|Z_s(z,\cdot)) = \frac{\partial}{\partial k}F(Z_h(x,\cdot) + kZ_s(z,\cdot))\Big|_{k=0}, \tag{2.4}$$

if they exist.

Remark 2.2. (1) In a unifying paper [16], Lee defined an integral transform $\mathcal{F}_{\gamma,\beta}$ of analytic functionals on abstract Wiener spaces. In [4, 5, 10, 11], the authors introduced the generalized version of the integral transform given by Lee [16]. Let F be a functional on $K_{a,b}[0,T]$. For all non-zero complex numbers γ and β

$$\mathcal{F}_{\gamma,\beta}(F)(y) = \int_{C_{a,b}[0,T]} F(\gamma x + \beta y) d\mu(x).$$

If $h_1(t) = h_2(t) = 1$ on $[0,T]$, then the transform with respect to the Gaussian process $T^{h_1,h_2}_{\gamma,\beta}(F)$ coincides with $\mathcal{F}_{\gamma,\beta}(F)$. In particular, if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, then $T^{1,1}_{\gamma,\beta}(F)$ is the integral transform used by Kim, Skoug, Yoo and Lee [7, 15, 16].

(2) In various fields of mathematics, in particular in functional analysis, convolution is a powerful tool that is used to operate on two functions, f and g, to produce a third function. Mathematically, convolution is described using an integral that expresses the amount of overlap of one function, g, with respect to a second function, f. In view of this description, this classical concept is extremely useful in a variety of research applications. The convolution product with respect to the Fourier transform $\hat{f}$ of a function f is defined by

$$(f * g)(\vec{u}) = \int_{\mathbb{R}^n} f(\vec{v}) g(\vec{u} - \vec{v}) d\vec{v}, \quad \text{for } \vec{u}, \vec{v} \in \mathbb{R}^n. \tag{2.5}$$

The homomorphism property holds, namely, $\widehat{f * g} = \hat{f}\hat{g}$.

In [5, 11], the authors examined the generalized integral transform on function space; they introduced a convolution product. Let F and G be functionals on $K_{a,b}[0,T]$. For any non-zero complex number γ

$$(F * G)_\gamma(y) = \int_{C_{a,b}[0,T]} F\Big(\frac{y + \gamma x}{\sqrt{2}}\Big) G\Big(\frac{y - \gamma x}{\sqrt{2}}\Big) d\mu(x).$$

From this, they can obtain the homomorphism property of the convolution. If $s_1(t) = s_2(t) = 1$ on $[0,T]$, $\tau = \frac{1}{\sqrt{2}}$ and $\rho = \frac{\gamma}{\sqrt{2}}$, then $(F \diamond G)^{s_1,s_2}_{\rho,\tau} = (F * G)_\gamma$. In particular, if $a(t) \equiv 0$ and $b(t) = t$ on $[0,T]$, $\diamond$-product is the convolution product used by Chang, Kim and Yoo [6, 7]. The convolution product has been studied widely as the notion has important applications in several branches of mathematics. Due to the relation established above many results for convolution product are corollaries of the results for $\diamond$-product.

(3) In [1, 3, 5], the authors studied a first variation of F. Let F be a functional on $C_{a,b}[0,T]$ and let $z \in C_{a,b}[0,T]$. Then the first variation is defined by

$$\delta F(x|z) = \frac{\partial}{\partial k} F(x + kz)\Big|_{k=0}.$$

$\delta F(x|z)$ acts like a directional derivative in the direction of z.

3. A generalized Fubini theorem on function space

In [12], the authors introduced the Fubini theorem on function space. In this paper, we establish a more generalized Fubini theorem for the function space integral.

The following lemma is the Fubini theorem on function space which plays a key role in this paper, see [12].

Lemma 3.1. *Let F be a complex-valued Borel measurable functional on $C_{a,b}[0,T]$ such that*

$$\int_{C^2_{a,b}[0,T]} |F(px+qy)| d(\mu\times\mu)(x,y) < \infty$$

for all non-zero real-numbers p and q. Then

$$\int_{C^2_{a,b}[0,T]} F(px+qy) d(\mu\times\mu)(x,y) \\ = \int_{C_{a,b}[0,T]} F(\sqrt{p^2+q^2}z + (p+q-\sqrt{p^2+q^2})a)\, d\mu(z). \quad (3.1)$$

Now we state a definition and notations that are needed to understand this paper.

(I) Let $\{\phi_j\}_{j=1}^{\infty}$ be a complete orthonormal system in $L^2_{a,b}[0,T]$ of functions of bounded variation. Then every element h of $L^2_{a,b}[0,T]$ can be written as

$$h(t) = \sum_{j=1}^{\infty} (h,\phi_j)_{a,b}\phi_j(t). \quad (3.2)$$

(II) Throughout this paper, for any nonzero complex number γ, $\gamma^{\frac{1}{2}}$ is always chosen to have positive real part.

(III) For any non-zero complex numbers γ and β, let

$$\nu(t) \equiv \nu^{h_1,h_2}_{\gamma,\beta}(t) = \sum_{j=1}^{\infty} (\gamma^2(h_1,\phi_j)^2_{a,b} + \beta^2(h_2,\phi_j)^2_{a,b})^{\frac{1}{2}}\phi_j(t), \quad (3.3)$$

$$\Phi^{\pm}_{\phi_j} = \gamma(h_1,\phi_j)_{a,b} \pm \beta(h_2,\phi_j)_{a,b} - (\gamma^2(h_1,\phi_j)^2_{a,b} + \beta^2(h_2,\phi_j)^2_{a,b})^{\frac{1}{2}},$$

$$W^{h_1,h_2}_{\gamma,\beta}(\cdot) = \sum_{j=1}^{\infty} \Phi^{+}_{\phi_j} Z_{\phi_j}(a,\cdot) \text{ and } B^{h_1,h_2}_{\gamma,\beta}(\cdot) = \sum_{j=1}^{\infty} \Phi^{-}_{\phi_j} Z_{\phi_j}(a,\cdot). \quad (3.4)$$

Since $|\gamma^2+\beta^2|^{\frac{1}{2}} \le |\gamma^2|^{\frac{1}{2}} + |\beta^2|^{\frac{1}{2}}$ for all complex numbers γ and β, we have

$$\begin{aligned}
|\nu(t)| &\le \sum_{j=1}^{\infty} \Big|\gamma^2(h_1,\phi_j)_{a,b}^2 + \beta^2(h_2,\phi_j)_{a,b}^2\Big|^{\frac{1}{2}} |\phi_j(t)| \\
&\le \sum_{j=1}^{\infty} \Big|\gamma^2(h_1,\phi_j)_{a,b}^2\phi_j^2(t)\Big|^{\frac{1}{2}} + \sum_{j=1}^{\infty} \Big|\beta^2(h_2,\phi_j)_{a,b}^2\phi_j^2(t)\Big|^{\frac{1}{2}} \\
&= |\gamma^2|^{\frac{1}{2}} \sum_{j=1}^{\infty}(h_1,\phi_j)_{a,b}\phi_j(t) + |\beta^2|^{\frac{1}{2}} \sum_{j=1}^{\infty}(h_2,\phi_j)_{a,b}\phi_j(t) \\
&= |\gamma^2|^{\frac{1}{2}} h_1(t) + |\beta^2|^{\frac{1}{2}} h_2(t) \le |\gamma^2|^{\frac{1}{2}}\|h_1\|_\infty + |\beta^2|^{\frac{1}{2}}\|h_2\|_\infty.
\end{aligned}$$

Hence ν is an element of $L_\infty[0,T]$.

(IV) Let F be a functional defined on $K_{a,b}[0,T]$ and let

$$F_y(x) = F(x+y) \quad \text{for } x, y \in K_{a,b}[0,T].$$

In our next theorem we establish a generalized Fubini theorem on function space.

Theorem 3.2. *Let $\{\phi_j\}_{j=1}^{\infty}$ be as in statement (I) above. Let F be a complex-valued Borel measurable functional on $K_{a,b}[0,T]$ such that*

$$\int_{C_{a,b}^2[0,T]} \Big|F(\gamma Z_{h_1}(x,\cdot) + \beta Z_{h_2}(y,\cdot))\Big| d(\mu\times\mu)(x,y) < \infty$$

for all nonzero complex numbers γ and β, where $Z_{h_l}(l=1,2)$ is given by equation (2.1)*. Then for all non-zero complex numbers γ and β,*

$$\begin{aligned}
&\int_{C_{a,b}^2[0,T]} F(\gamma Z_{h_1}(x,\cdot) + \beta Z_{h_2}(y,\cdot)) d(\mu\times\mu)(x,y) \\
&\qquad\qquad = \int_{C_{a,b}[0,T]} F(Z_\nu(w,\cdot) + W_{\gamma,\beta}^{h_1,h_2}(\cdot)) d\mu(w), \quad (3.5)
\end{aligned}$$

where ν and $W_{\gamma,\beta}^{h_1,h_2}$ are given by equations (3.3) *and* (3.4)*, respectively.*

Proof. First note that for each $h \in L_{a,b}^2[0,T]$ and non-zero complex number γ,

$$\gamma Z_h(x,t) = \sum_{j=1}^{\infty} \int_0^t \gamma(h,\phi_j)_{a,b}\phi_j(s)dx(s).$$

Hence using the linearity of the PWZ integral, we have

$$\gamma Z_{h_1}(x,t) + \beta Z_{h_2}(y,t) = \sum_{j=1}^{\infty} Z_{\phi_j}(\gamma(h_1,\phi_j)_{a,b}x + \beta(h_2,\phi_j)_{a,b}y, \cdot)$$

and so

$$\int_{C^2_{a,b}[0,T]} F(\gamma Z_{h_1}(x,\cdot)+\beta Z_{h_2}(y,\cdot))d(\mu\times\mu)(x,y)$$
$$= \int_{C^2_{a,b}[0,T]} F(\sum_{j=1}^{\infty} Z_{\phi_j}(\gamma(h_1,\phi_j)_{a,b}x+\beta(h_2,\phi_j)_{a,b}y,\cdot))\, d(\mu\times\mu)(x,y). \quad (3.6)$$

Let $H(x)=F(\sum_{j=1}^{\infty} Z_{\phi_j}(x,\cdot))$. Then it follows that

$$H(\gamma(h_1,\phi_j)_{a,b}x+\beta(h_2,\phi_j)_{a,b}y) = F(\sum_{j=1}^{\infty} Z_{\phi_j}(\gamma(h_1,\phi_j)_{a,b}x+\beta(h_2,\phi_j)_{a,b}y,\cdot)).$$

Thus, applying this formula and equation (3.1) to the last expression in equation (3.6), we have

$$\begin{aligned}
&\int_{C^2_{a,b}[0,T]} F(\gamma Z_{h_1}(x,\cdot)+\beta Z_{h_2}(y,\cdot))d(\mu\times\mu)(x,y)\\
&= \int_{C^2_{a,b}[0,T]} H(\gamma(h_1,\phi_j)_{a,b}x+\beta(h_2,\phi_j)_{a,b}y)d(\mu\times\mu)(x,y)\\
&= \int_{C_{a,b}[0,T]} H((\gamma^2(h_1,\phi_j)^2_{a,b}+\beta^2(h_2,\phi_j)^2_{a,b})^{\frac12}w\\
&\quad + (\gamma(h_1,\phi_j)_{a,b}+\beta(h_2,\phi_j)_{a,b}-(\gamma^2(h_1,\phi_j)^2_{a,b}+\beta^2(h_2,\phi_j)^2_{a,b})^{\frac12})a)d\mu(w)\\
&= \int_{C_{a,b}[0,T]} F\Big(\sum_{j=1}^{\infty}\Big(\int_0^t \phi_j(s)d\Big[(\gamma^2(h_1,\phi_j)^2_{a,b}+\beta^2(h_2,\phi_j)^2_{a,b})^{\frac12}w(s)\\
&\quad + (\gamma(h_1,\phi_j)_{a,b}+\beta(h_2,\phi_j)_{a,b}-(\gamma^2(h_1,\phi_j)^2_{a,b}+\beta^2(h_2,\phi_j)^2_{a,b})^{\frac12})a(s)\Big]\Big)\Big)d\mu(w)\\
&= \int_{C_{a,b}[0,T]} F(Z_\nu(w,\cdot)+W^{h_1,h_2}_{\gamma,\beta}(\cdot))\,d\mu(w),
\end{aligned}$$

which establishes equation (3.5) as desired. □

The following corollary immediately follows from Theorem 3.2 by letting $h_1(t)=h_2(t)=h(t)$ on $[0,T]$.

Corollary 3.3. *Let F be as in Theorem 3.2. Then*

$$\int_{C^2_{a,b}[0,T]} F(\gamma Z_h(x,\cdot)+\beta Z_h(y,\cdot))d(\mu\times\mu)(x,y)$$
$$= \int_{C_{a,b}[0,T]} F(\sqrt{\gamma^2+\beta^2}Z_h(w,\cdot)+(\gamma+\beta-\sqrt{\gamma^2+\beta^2})(h,a'))d\mu(w).$$

Remark 3.4. (1) The main result in [12, Theorem 3.5] follows immediately from Theorem 3.2 above by choosing $h_1(t)=h_2(t)=1$ on $[0,T]$ and $\gamma,\beta\in\mathbb{R}-\{0\}$.

(2) In the setting of one parameter Wiener space $C_0[0,T]$ (i.e., in the case where $a(t)\equiv 0$ and $b(t)=t$ on $[0,T]$ in our research), the function space $C_{a,b}[0,T]$ reduces to the Wiener space $C_0[0,T]$. Thus the result in [8, Theorem 3.5] follows immediately from Theorem 3.2 above.

In our next theorem, we apply the results obtained in Theorem 3.2 to the following double transform.

Theorem 3.5. *Let F be as in Theorem 3.2. Then for all non-zero complex numbers γ_j and β_j for $j=1,2,3$*

$$T^{h_1,h_2}_{\gamma_1,\beta_1}(T^{h_3,h_4}_{\gamma_2,\beta_2}(F))(y)=T^{\nu,h_2h_4}_{1,\beta_1\beta_2}(F_W)(y), \tag{3.7}$$

where $\nu\equiv\nu^{h_3,h_1h_4}_{\gamma_2,\gamma_1\beta_2}$ and $W\equiv W^{h_3,h_1h_4}_{\gamma_2,\gamma_1\beta_2}$ are given by equations (3.3) and (3.4), respectively.

Proof. By using equations (3.2) and (3.5), it follows that

$$\begin{aligned}
&T^{h_1,h_2}_{\gamma_1,\beta_1}(T^{h_3,h_4}_{\gamma_2,\beta_2}(F))(y)\\
&\quad=\int_{C_{a,b}[0,T]}\int_{C_{a,b}[0,T]}F(\beta_1\beta_2Z_{h_2h_4}(y,\cdot)\\
&\qquad+\gamma_2Z_{h_3}(z,\cdot)+\gamma_1\beta_2Z_{h_1h_4}(x,\cdot))d(\mu\times\mu)(x,z)\\
&\quad=\int_{C_{a,b}[0,T]}F(\beta_1\beta_2Z_{h_2h_4}(y,\cdot)+Z_\nu(w,\cdot)+W^{h_3,h_1h_4}_{\gamma_2,\gamma_1\beta_2}(\cdot))d\mu(w)\\
&\quad=T^{\nu,h_2h_4}_{1,\beta_1\beta_2}(F_W)(y).
\end{aligned}$$

From the definition of the transform with respect to the Gaussian process, we obtain the first and third equalities. The second one results from equation (3.5). Thus we have proved the theorem. □

In the following Table 1, we illustrate the usefulness of our Fubini theorem in this paper.

	$T^{\nu,h_2h_4}_{1,\beta_1\beta_2}(F_W)$
$F_1(x)=\langle v,x\rangle$	$\beta_1\beta_2\langle vh_2h_4\rangle+\gamma_2(vh_3,a')+\gamma_1\beta_2(vh_1h_4,a')$
$F_2(x)=\exp\{F_1(x)\}$	$\exp\Big\{\beta_1\beta_2\langle vh_2h_4\rangle+\frac{1}{2}(v^2\nu^2,b')+\gamma_2(vh_3,a')+\gamma_1\beta_2(vh_1h_4,a')\Big\}$
$F_3(x)=F_1(x)F_2(x)$	$(\beta_1\beta_2\langle vh_2h_4\rangle+(v^2\nu^2,b')+\gamma_2(vh_3,a')+\gamma_1\beta_2(vh_1h_4,a'))\cdot\exp\Big\{\beta_1\beta_2\langle vh_2h_4\rangle+\frac{1}{2}(v^2\nu^2,b')+\gamma_2(vh_3,a')+\gamma_1\beta_2(vh_1h_4,a')\Big\}$

TABLE 1.

In Table 1 above, $F_W = F_j(\cdot + W)$ for $j = 1, 2$ and $v \in L^2_{a,b}[0,T]$ and ν is given as in Theorem 3.5.

Now, using Table 1 and Theorem 3.5, we obtain the following formulas:

$$T^{h_1,h_2}_{\gamma_1,\beta_1}(T^{h_3,h_4}_{\gamma_2,\beta_2}(F_1))(y) = \beta_1\beta_2\langle vh_2h_4\rangle + \gamma_2(vh_3, a') + \gamma_1\beta_2(vh_1h_4, a'),$$

$$\begin{aligned} &T^{h_1,h_2}_{\gamma_1,\beta_1}(T^{h_3,h_4}_{\gamma_2,\beta_2}(F_2))(y) \\ &\qquad = \exp\Big\{\beta_1\beta_2\langle vh_2h_4\rangle + \frac{1}{2}(v^2\nu^2, b') + \gamma_2(vh_3, a') + \gamma_1\beta_2(vh_1h_4, a')\Big\}, \end{aligned}$$

and

$$\begin{aligned} &T^{h_1,h_2}_{\gamma_1,\beta_1}(T^{h_3,h_4}_{\gamma_2,\beta_2}(F_3))(y) \\ &\qquad = (\beta_1\beta_2\langle vh_2h_4\rangle + (v^2\nu^2, b') + \gamma_2(vh_3, a') + \gamma_1\beta_2(vh_1h_4, a')) \\ &\qquad\quad \cdot \exp\Big\{\beta_1\beta_2\langle vh_2h_4\rangle + \frac{1}{2}(v^2\nu^2, b') + \gamma_2(vh_3, a') + \gamma_1\beta_2(vh_1h_4, a')\Big\}. \end{aligned}$$

4. Relationships involving two concepts

In this section, we establish the relationships involving exactly two of the three concepts of transform, the $\diamond$-product, and the first variation for functionals on $K_{a,b}[0,T]$.

Now, we show that the transform with respect to the Gaussian process of the $\diamond$-product is a product of their transforms.

Theorem 4.1. *Let F be as in Theorem 3.2 and let G be a complex-valued Borel measurable functional on $K_{a,b}[0,T]$ such that*

$$\int_{C^2_{a,b}[0,T]} \Big|G(\gamma Z_{h_1}(x,\cdot) + \beta Z_{h_2}(y,\cdot))\Big| d(\mu\times\mu)(x,y) < \infty.$$

Assume that $\tau\gamma = \rho$ and $h_1(t)s_2(t) = s_1(t)$ on $[0,T]$. Then for all non-zero complex numbers γ, β, ρ and τ,

$$T^{h_1,h_2}_{\gamma,\beta}((F\diamond G)^{s_1,s_2}_{\rho,\tau})(y) \overset{*}{=} (T^{s_1,h_2s_2}_{\sqrt{2}\rho,\tau\beta}(F_W))(y)(T^{s_1,h_2s_2}_{\sqrt{2}\rho,\tau\beta}(G_B))(y) \tag{4.1}$$

for $y \in K_{a,b}[0,T]$, where $W \equiv (2-\sqrt{2})\rho Z_{s_1}(a,\cdot)$ and $B \equiv -\sqrt{2}\rho Z_{s_1}(a,\cdot)$.

Proof. By using equations (2.2) and (2.3), it follows that

$$\begin{aligned} &T^{h_1,h_2}_{\gamma,\beta}((F\diamond G)^{s_1,s_2}_{\rho,\tau})(y) \\ &\qquad = \int_{C^2_{a,b}[0,T]} F(\tau\beta Z_{h_2s_2}(y,\cdot) + \tau\gamma Z_{h_1s_2}(x,\cdot) + \rho Z_{s_1}(z,\cdot)) \\ &\qquad\qquad \cdot G(\tau\beta Z_{h_2s_2}(y,\cdot) + \tau\gamma Z_{h_1s_2}(x,\cdot) - \rho Z_{s_1}(z,\cdot))\, d(\mu\times\mu)(x,z). \end{aligned} \tag{4.2}$$

Since $\tau\gamma = \rho$ and $h_1(t)s_2(t) = s_1(t)$ on $[0,T]$, the processes $\tau\gamma Z_{h_1s_2}(x,\cdot) + \rho Z_{s_1}(z,\cdot)$ and $\tau\gamma Z_{h_1s_2}(x,\cdot) - \rho Z_{s_1}(z,\cdot)$ are independent processes. Hence the last expression in equation (4.2) above is equal to

$$\int_{C^2_{a,b}[0,T]} F(\tau\beta Z_{h_2s_2}(y,\cdot)+\tau\gamma Z_{h_1s_2}(x,\cdot)+\rho Z_{s_1}(z,\cdot))\,d(\mu\times\mu)(x,z)$$
$$\cdot\int_{C^2_{a,b}[0,T]} G(\tau\beta Z_{h_2s_2}(y,\cdot)+\tau\gamma Z_{h_1s_2}(x,\cdot)-\rho Z_{s_1}(z,\cdot))\,d(\mu\times\mu)(x,z).$$

Now, applying equation (3.5) to the equation above, it follows that

$$\begin{aligned}(T^{h_1,h_2}_{\gamma,\beta}&((F\diamond G)^{s_1,s_2}_{\rho,\tau}))(y)\\ &=\int_{C_{a,b}[0,T]} F(\tau\beta Z_{h_2s_2}(y,\cdot)+\sqrt{2}\rho Z_{s_1}(w,\cdot)+(2-\sqrt{2})\rho Z_{s_1}(a,\cdot))\,d\mu(w)\\ &\quad\cdot\int_{C_{a,b}[0,T]} G(\tau\beta Z_{h_2s_2}(y,\cdot)+\sqrt{2}\rho Z_{s_1}(w,\cdot)-\sqrt{2}\rho Z_{s_1}(a,\cdot))\,d\mu(w)\\ &=(T^{s_1,h_2s_2}_{\sqrt{2}\rho,\tau\beta}(F_W))(y)(T^{s_1,h_2s_2}_{\sqrt{2}\rho,\tau\beta}(G_B))(y).\end{aligned}$$

Thus we have the desired results. □

Theorem 4.1 above tells us that the transform with respect to the Gaussian process of the $\diamond$-product can be calculated from the product of their transforms without the $\diamond$-product.

	$F(x)=\langle v,x\rangle,\quad G(x)=\exp\{\langle v,x\rangle\}$
$T^{s_1,h_2s_2}_{\sqrt{2}\rho,\tau\beta}(F_W)$	$\tau\beta\langle vh_2s_2,y\rangle+2\rho(vs_1,a')$
$T^{s_1,h_2s_2}_{\sqrt{2}\rho,\tau\beta}(G_B)$	$\exp\left\{\tau\beta\langle uh_2s_2,y\rangle+\rho^2(u^2s_1^2,b')\right\}$

Table 2.

In Table 2 above, $F_W=F(\cdot+W)$ and $G_B=G(\cdot+B)$ and $v\in L^2_{a,b}[0,T]$.

Now, using Table 2 and Theorem 4.1, we have

$$\begin{aligned}T^{h_1,h_2}_{\gamma,\beta}&((F\diamond G)^{s_1,s_2}_{\rho,\tau})(y)\\ &=(\tau\beta\langle vh_2s_2,y\rangle+2\rho(vs_1,a'))\exp\left\{\tau\beta\langle uh_2s_2,y\rangle+\rho^2(u^2s_1^2,b')\right\}.\end{aligned}$$

In our next theorem, we show that the transform for the $\diamond$-product with respect to the transform can be calculated from the transform of F and G without the concept of the $\diamond$-product.

Theorem 4.2. *Let F and G be as in Theorem 4.1. Then for all non-zero complex numbers γ, β, ρ and τ:*

$$\begin{aligned}T^{h_1,h_2}_{\gamma,\beta}&((T^{h_1,h_2}_{\gamma,\beta}(F)\diamond T^{h_1,h_2}_{\gamma,\beta}(G))^{s_1,s_2}_{\rho,\tau})(y)\\ &\overset{*}{=}T^{\nu,h_1h_2^2s_2}_{1,\tau\beta^2}(F_{W+\tilde{W}})(y)T^{\nu,h_1h_2^2s_2}_{1,\tau\beta^2}(G_{B+\tilde{W}})(y),\end{aligned}$$

where $W \equiv (2-\sqrt{2})\rho Z_{s_1}(a,\cdot)$, $B \equiv -\sqrt{2}\rho Z_{s_1}(a,\cdot)$, $\nu \equiv \nu_{\gamma,\sqrt{2}\rho\beta}^{h_1,s_1h_2}$, *and* $\tilde{W} \equiv \tilde{W}_{\gamma,\sqrt{2}\rho\beta}^{h_1,s_1h_2}$.

Proof. From equation (4.1), we obtain that

$$T_{\gamma,\beta}^{h_1,h_2}((T_{\gamma,\beta}^{h_1,h_2}(F)\diamond T_{\gamma,\beta}^{h_1,h_2}(G))_{\rho,\tau}^{s_1,s_2})(y) = T_{\sqrt{2}\rho,\tau\beta}^{s_1,h_2s_2}(T_{\gamma,\beta}^{h_1,h_2}(F_W))(y)T_{\sqrt{2}\rho,\tau\beta}^{s_1,h_2s_2}(T_{\gamma,\beta}^{h_1,h_2}(G_B))(y).$$

Applying equation (3.7) to the last expression in the above equation, it follows that

$$T_{\gamma,\beta}^{h_1,h_2}((T_{\gamma,\beta}^{h_1,h_2}(F)\diamond T_{\gamma,\beta}^{h_1,h_2}(G))_{\rho,\tau}^{s_1,s_2})(y) = T_{1,\tau\beta^2}^{\nu,h_1h_2^2s_2}(F_{W+\tilde{W}})(y)T_{1,\tau\beta^2}^{\nu,h_1h_2^2s_2}(G_{B+\tilde{W}})(y).$$

Thus we prove the desired result. □

In our next theorem, we obtain that the transform involving the first variation equals the first variation of the transform with respect to the Gaussian process.

Theorem 4.3. *Let* F *be as in Theorem 4.1. Assume that*

$$\int_{C_{a,b}[0,T]}\Big|\delta F(Z_h(x,\cdot)|Z_s(z,\cdot))\Big|d\mu(x) < \infty.$$

(A) *Let* h, s, $h_j (j=1,2,3,4)$, l *and* m *satisfy the following conditions:*

(1) $h_3(t) = h(t)h_1(t)$
(2) $l(t)h_4(t) = h(t)h_2(t)$
(3) $m(t)h_4(t) = s(t)$ *on* $[0,T]$.

Then for all non-zero complex numbers γ *and* β*:*

$$T_{\gamma,\beta}^{h_1,h_2}(\delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))(y) \overset{*}{=} \delta T_{\gamma,\beta}^{h_3,h_4}(F)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)). \tag{4.3}$$

(B) *Let* h, s, $h_j (j=1,2,3,4)$, l *and* m *satisfy the following conditions:*

(1) $l(t)h_3(t) = h_1(t)$
(2) $l(t)h_4(t) = h(t)h_2(t)$
(3) $m(t) = s(t)h_2(t)$ *on* $[0,T]$.

Then for all non-zero complex numbers γ *and* β*:*

$$\delta T_{\gamma,\beta}^{h_1,h_2}(F)(Z_h(y,\cdot)|Z_s(z,\cdot)) \overset{*}{=} T_{\gamma,\beta}^{h_3,h_4}(\delta F(Z_l(\cdot,\cdot)|\beta Z_m(z,\cdot)))(y). \tag{4.4}$$

Proof. By using equations (2.2) and (2.4), we have

$$\begin{aligned}
&T_{\gamma,\beta}^{h_1,h_2}(\delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))(y)\\
&\quad= \frac{\partial}{\partial k}\bigg[\int_{C_{a,b}[0,T]} F(\gamma Z_{h_1h}(x,\cdot)+\beta Z_{h_2h}(y,\cdot)+kZ_s(z,\cdot))d\mu(x)\bigg]\Big|_{k=0}\\
&\quad= \frac{\partial}{\partial k}\bigg[\int_{C_{a,b}[0,T]} F(\gamma Z_{h_3}(x,\cdot)+\beta Z_{lh_4}(y,\cdot)+\frac{\beta k}{\beta}Z_{mh_4}(z,\cdot))d\mu(x)\bigg]\Big|_{k=0}\\
&\quad= \frac{\partial}{\partial k}T_{\gamma,\beta}^{h_3,h_4}(F)(Z_l(y,\cdot)+\frac{k}{\beta}Z_m(z,\cdot))\Big|_{k=0}\\
&\quad= \delta T_{\gamma,\beta}^{h_3,h_4}(F)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)).
\end{aligned}$$

Also, using equations (2.2) and (2.4), and a similar argument as above, we can see that

$$\delta T_{\gamma,\beta}^{h_1,h_2}(F)(Z_h(y,\cdot)|Z_s(z,\cdot)) \overset{*}{=} T_{\gamma,\beta}^{h_3,h_4}(\delta F(Z_l(\cdot,\cdot)|\beta Z_m(z,\cdot)))(y).$$

Thus we have the desired results. □

Combining Theorems 3.5 and 4.3, we have the following corollary.

Corollary 4.4. *Let F be as in Theorem 4.3.*

(A) *Let h, s, $h_j(j=1,2,3,4)$, l, m and ν satisfy the following conditions:*

(1) $h_3(t)=h(t)\nu(t)$

(2) $l(t)h_4(t)=h(t)h_2^2(t)$

(3) $m(t)h_4(t)=s(t)$ *on* $[0,T]$.

Then for all non-zero complex numbers γ and β:

$$T_{\gamma,\beta}^{h_1,h_2}(T_{\gamma,\beta}^{h_1,h_2}(\delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot))))(y) \overset{*}{=} \delta T_{1,\beta^2}^{h_3,h_4}(F_W)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)),$$

where $\nu \equiv \nu_{\gamma,\gamma\beta}^{h_1,h_1h_2}$ and $W \equiv W_{\gamma,\gamma\beta}^{h_1,h_1h_2}$.

(B) *Let h, s, $h_j(j=1,2,3,4)$, l, m and ν satisfy the following conditions:*

(1) $l(t)h_3(t)=\nu(t)$

(2) $l(t)h_4(t)=h(t)h_2^2(t)$

(3) $m(t)=s(t)h_2^2(t)$ *on* $[0,T]$.

Then for all non-zero complex numbers γ and β:

$$\delta T_{\gamma,\beta}^{h_1,h_2}(T_{\gamma,\beta}^{h_1,h_2}(F))(Z_h(y,\cdot)|Z_s(z,\cdot)) \overset{*}{=} T_{1,\beta^2}^{h_3,h_4}(\delta F_W(Z_l(\cdot,\cdot)|\beta Z_m(z,\cdot)))(y).$$

Proof. By using equations (3.7) and (4.3), it follows that

$$\begin{aligned}
T_{\gamma,\beta}^{h_1,h_2}(T_{\gamma,\beta}^{h_1,h_2}(\delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot))))(y) &= T_{1,\beta^2}^{\nu,h_2^2}(\delta F_W(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))(y)\\
&= \delta T_{1,\beta^2}^{h_3,h_4}(F_W)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)).
\end{aligned}$$

On the other hand, using a similar method to that in the proof of Theorem 4.3, we have

$$\delta T_{\gamma,\beta}^{h_1,h_2}(T_{\gamma,\beta}^{h_1,h_2}(F))(Z_h(y,\cdot)|Z_s(z,\cdot)) = T_{\gamma,\beta}^{h_1,h_2}(T_{\gamma,\beta}^{h_1,h_2}\delta F(Z_h(\cdot,\cdot)|\beta^2 Z_{sh_2^2}(z,\cdot)))(y).$$

Applying equation (3.7) to the last expression above, we obtain

$$\delta T_{\gamma,\beta}^{h_1,h_2}(T_{\gamma,\beta}^{h_1,h_2}(F))(Z_h(y,\cdot)|Z_s(z,\cdot)) \overset{*}{=} T_{1,\beta^2}^{h_3,h_4}(\delta F_W(Z_l(\cdot,\cdot)|\beta Z_m(z,\cdot)))(y).$$

Thus we have the desired results. □

5. Relationships involving three concepts

In Section 4, we obtained relationships involving exactly two of the three concepts of transform, the $\diamond$-product and the first variation of functionals on $K_{a,b}[0,T]$. In this section, we establish all possible relationships involving all three of these concepts.

R1. A formula for the transform of the first variation of the $\diamond$-product. Let F and G be as in Theorem 4.3. Let h, s, $h_j (j=1,2,3,4)$, l, m, τ and ρ satisfy the following conditions:

(1) $\tau\gamma h_3(t)s_2(t) = \rho s_1(t)$
(2) $h_3(t) = h(t)h_1(t)$
(3) $l(t)h_4(t) = h(t)h_2(t)$
(4) $m(t)h_4(t) = s(t)$ on $[0,T]$.

Then for all non-zero complex numbers γ, β, ρ and τ:

$$\begin{aligned} &T_{\gamma,\beta}^{h_1,h_2}(\delta(F\diamond G)_{\rho,\tau}^{s_1,s_2}(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))(y) \\ &\quad\overset{*}{=} T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4s_2}(F_W)(Z_l(y,\cdot))\delta T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4s_2}(G_B)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)) \\ &\qquad + \delta T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4s_2}(F_W)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4s_2}(G_B)(Z_l(y,\cdot)), \end{aligned}$$

where $W \equiv (2-\sqrt{2})\rho Z_{s_1}(a,\cdot)$ and $B \equiv -\sqrt{2}\rho Z_{s_1}(a,\cdot)$.

Proof. By using equations (4.1) and (4.3), we have

$$\begin{aligned} &T_{\gamma,\beta}^{h_1,h_2}(\delta(F\diamond G)_{\rho,\tau}^{s_1,s_2}(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))(y) \\ &\quad= \delta T_{\gamma,\beta}^{h_3,h_4}((F\diamond G)_{\rho,\tau}^{s_1,s_2})(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)) \qquad (5.1) \\ &\quad= \delta T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4s_2}(F_W)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4s_2}(G_B)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)). \end{aligned}$$

Now, let $H(Z_l(y,\cdot)) = F(Z_l(y,\cdot))G(Z_l(y,\cdot))$. Then

$$\begin{aligned} \delta H(Z_l(y,\cdot)|Z_m(z,\cdot)) &= F(Z_l(y,\cdot))\delta G(Z_l(y,\cdot)|Z_m(z,\cdot)) \\ &\quad + \delta F(Z_l(y,\cdot)|Z_m(z,\cdot))G(Z_l(y,\cdot)). \end{aligned}$$

Thus, applying this formula to the last expression in the above equation (5.1), we obtain

$$\begin{aligned} &T_{\gamma,\beta}^{h_1,h_2}(\delta(F \diamond G)_{\rho,\tau}^{s_1,s_2}(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))(y) \\ &\quad = T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4 s_2}(F_W)(Z_l(y,\cdot))\delta T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4 s_2}(G_B)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot)) \\ &\qquad + \delta T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4 s_2}(F_W)(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4 s_2}(G_B)(Z_l(y,\cdot)). \end{aligned}$$

Thus we have the desired results. □

R2. A formula for the transform of the $\diamond$-product with respect to the first variation. Let F and G be as in Theorem 4.3. Let h, s, $h_j (j = 1, 2, 3, 4)$, l, m, τ and ρ satisfy the following conditions:

(1) $\tau\gamma h_1(t)s_2(t) = \rho s_1(t)$
(2) $h_3(t) = \sqrt{2}\rho s_1(t)h(t)$
(3) $l(t)h_4(t) = h(t)h_2(t)s_2(t)$
(4) $m(t)h_4(t) = s(t)$ on $[0, T]$.

Then for all non-zero complex numbers γ, β, ρ and τ:

$$\begin{aligned} &T_{\gamma,\beta}^{h_1,h_2}((\delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot)) \diamond \delta G(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))_{\rho,\tau}^{s_1,s_2})(y) \\ &\quad \overset{*}{=} \delta T_{\sqrt{2}\rho,\beta\tau}^{h_3,h_4}(F_W)(Z_l(y,\cdot)|\frac{1}{\beta\tau}Z_m(z,\cdot))\delta T_{\sqrt{2}\rho,\beta\tau}^{h_3,h_4}(G_B)(Z_l(y,\cdot)|\frac{1}{\beta\tau}Z_m(z,\cdot)), \end{aligned}$$

where $W \equiv (2 - \sqrt{2})\rho Z_{s_1}(a, \cdot)$ and $B \equiv -\sqrt{2}\rho Z_{s_1}(a, \cdot)$.

Proof. By using equations (4.1) and (4.3), we have

$$\begin{aligned} &T_{\gamma,\beta}^{h_1,h_2}((\delta F(Z_h(\cdot,\cdot)|Z_s(z,\cdot)) \diamond \delta G(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))_{\rho,\tau}^{s_1,s_2})(y) \\ &\quad = T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_2 s_2}(\delta F_W(Z_h(\cdot,\cdot)|Z_s(\cdot,\cdot)))(y)T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_2 s_2}(\delta G_B(Z_h(\cdot,\cdot)|Z_s(\cdot,\cdot)))(y) \\ &\quad = \delta T_{\sqrt{2}\rho,\beta\tau}^{h_3,h_4}(F_W)(Z_l(y,\cdot)|\frac{1}{\beta\tau}Z_m(z,\cdot))\delta T_{\sqrt{2}\rho,\beta\tau}^{h_3,h_4}(G_B)(Z_l(y,\cdot)|\frac{1}{\beta\tau}Z_m(z,\cdot)), \end{aligned}$$

which completes the proof of **R2**. □

R3. A formula for the transform of the first variation with respect to the $\diamond$-product of transforms. Let F and G be as in Theorem 4.3. Let h, s, $h_j(j = 1, 2, 3, 4)$, l, m, τ, ρ and γ satisfy the following conditions:

(1) $\tau\gamma h_3(t)s_2(t) = \rho s_1(t)$
(2) $h_3(t) = h(t)h_1(t)$
(3) $l(t)h_4(t) = h(t)h_2(t)$
(4) $m(t)h_4(t) = s(t)$ on $[0, T]$.

Then for all non-zero complex numbers γ, β, ρ and τ:

$$
\begin{aligned}
T_{\gamma,\beta}^{h_1,h_2}&(\delta((T_{\gamma,\beta}^{h_1,h_2}(F)\diamond T_{\gamma,\beta}^{h_1,h_2}(G))_{\rho,\tau}^{s_1,s_2})(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))(y)\\
&\overset{*}{=} T_{1,\beta^2\tau}^{\nu,h_2s_2h_4}(F_{W+\tilde{W}})(Z_l(y,\cdot))\delta T_{1,\beta^2\tau}^{\nu,h_2s_2h_4}(G_{B+\tilde{W}})(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))\\
&\quad+\delta T_{1,\beta^2\tau}^{\nu,h_2s_2h_4}(F_{W+\tilde{W}})(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))T_{1,\beta^2\tau}^{\nu,h_2s_2h_4}(G_{B+\tilde{W}})(Z_l(y,\cdot)),
\end{aligned}
$$

where $\nu\equiv\nu_{\gamma,\sqrt{2}\rho\beta}^{h_1,s_1h_2}$, $W\equiv(2-\sqrt{2})\rho Z_{s_1}(a,\cdot)$, $B\equiv-\sqrt{2}\rho Z_{s_1}(a,\cdot)$ and $\tilde{W}\equiv\tilde{W}_{\gamma,\sqrt{2}\rho\beta}^{h_1,s_1h_2}$.

Proof. By using equations (3.7), (4.1) and (4.3), we have

$$
\begin{aligned}
T_{\gamma,\beta}^{h_1,h_2}&(\delta((T_{\gamma,\beta}^{h_1,h_2}(F)\diamond T_{\gamma,\beta}^{h_1,h_2}(G))_{\rho,\tau}^{s_1,s_2})(Z_h(\cdot,\cdot)|Z_s(z,\cdot)))(y)\\
&=\delta T_{\gamma,\beta}^{h_3,h_4}((T_{\gamma,\beta}^{h_1,h_2}(F)\diamond T_{\gamma,\beta}^{h_1,h_2}(G))_{\rho,\tau}^{s_1,s_2})(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))\\
&=\delta T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4s_2}(T_{\gamma,\beta}^{h_1,h_2}(F_W))(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))\\
&\quad\cdot T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_4s_2}(T_{\gamma,\beta}^{h_1,h_2}(G_B))(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))\\
&=\delta T_{1,\beta^2\tau}^{\nu,h_2s_2h_4}(F_{W+\tilde{W}})(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))T_{1,\beta^2\tau}^{\nu,h_2s_2h_4}(G_{B+\tilde{W}})(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))\\
&=T_{1,\beta^2\tau}^{\nu,h_2s_2h_4}(F_{W+\tilde{W}})(Z_l(y,\cdot))\delta T_{1,\beta^2\tau}^{\nu,h_2s_2h_4}(G_{B+\tilde{W}})(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))\\
&\quad+\delta T_{1,\beta^2\tau}^{\nu,h_2s_2h_4}(F_{W+\tilde{W}})(Z_l(y,\cdot)|\frac{1}{\beta}Z_m(z,\cdot))T_{1,\beta^2\tau}^{\nu,h_2s_2h_4}(G_{B+\tilde{W}})(Z_l(y,\cdot)),
\end{aligned}
$$

which completes the proof of **R3**. □

R4. A formula for the first variation of the transform with respect to the $\diamond$-product. Let F and G be as in Theorem 4.3. Let h, s, $h_j (j=1,2,3,4)$, l, m, τ, ρ and γ satisfy the following conditions:

(1) $\tau\gamma h_1(t)s_2(t)=\rho s_1(t)$
(2) $l(t)h_3(t)=\sqrt{2}\rho s_1(t)$
(3) $l(t)h_4(t)=h(t)h_2(t)s_2(t)$
(4) $m(t)=s(t)h_2(t)s_2(t)$ on $[0,T]$.

Then for all non-zero complex numbers γ, β, ρ and τ:

$$
\begin{aligned}
\delta(T_{\gamma,\beta}^{h_1,h_2}&(F\diamond G)_{\rho,\tau}^{s_1,s_2})(Z_h(y,\cdot)|Z_s(z,\cdot))\\
&\overset{*}{=} T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_2s_2}(F_W)(Z_h(y,\cdot))T_{\sqrt{2}\rho,\beta\tau}^{h_3,h_4}(\delta G_B(Z_l(\cdot,\cdot)|\beta\tau Z_m(z,\cdot)))(y)\\
&\quad+T_{\sqrt{2}\rho,\beta\tau}^{h_3,h_4}(\delta F_W(Z_l(\cdot,\cdot)|\beta\tau Z_m(z,\cdot)))(y)T_{\sqrt{2}\rho,\beta\tau}^{s_1,h_2s_2}(G_B)(Z_h(y,\cdot)),
\end{aligned}
$$

where $W\equiv(2-\sqrt{2})\rho Z_{s_1}(a,\cdot)$ and $B\equiv-\sqrt{2}\rho Z_{s_1}(a,\cdot)$.

Proof. By using equations (4.1) and (4.4), we have

$$\begin{aligned}
&\delta(T^{h_1,h_2}_{\gamma,\beta}(F\diamond G)^{s_1,s_2}_{\rho,\tau})(Z_h(y,\cdot)|Z_s(z,\cdot))\\
&\qquad = \delta T^{s_1,h_2s_2}_{\sqrt{2}\rho,\beta\tau}(F_W)(Z_h(y,\cdot)|Z_s(z,\cdot))T^{s_1,h_2s_2}_{\sqrt{2}\rho,\beta\tau}(G_B)(Z_h(y,\cdot)|Z_s(z,\cdot))\\
&\qquad = T^{s_1,h_2s_2}_{\sqrt{2}\rho,\beta\tau}(F_W)(Z_h(y,\cdot))\delta T^{s_1,h_2s_2}_{\sqrt{2}\rho,\beta\tau}(G_B)(Z_h(y,\cdot)|Z_s(z,\cdot))\\
&\qquad\quad + \delta T^{s_1,h_2s_2}_{\sqrt{2}\rho,\beta\tau}(F_W)(Z_h(y,\cdot)|Z_s(z,\cdot))T^{s_1,h_2s_2}_{\sqrt{2}\rho,\beta\tau}(G_B)(Z_h(y,\cdot))\\
&\qquad = T^{s_1,h_2s_2}_{\sqrt{2}\rho,\beta\tau}(F_W)(Z_h(y,\cdot))T^{h_3,h_4}_{\sqrt{2}\rho,\beta\tau}(\delta G_B(Z_l(\cdot,\cdot)|\beta\tau Z_m(z,\cdot)))(y)\\
&\qquad\quad + T^{h_3,h_4}_{\sqrt{2}\rho,\beta\tau}(\delta F_W(Z_l(\cdot,\cdot)|\beta\tau Z_m(z,\cdot)))(y)T^{s_1,h_2s_2}_{\sqrt{2}\rho,\beta\tau}(G_B)(Z_h(y,\cdot)),
\end{aligned}$$

which completes the proof of the **R4**. □

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References

[1] S. J. Chang and J. G. Choi, *Relationships of convolution products, generalized transforms, and the first variation on function spaces*, Int. J. Math. Math. Sci. **29** (2002), 591–608. MR 1900504.

[2] S. J. Chang and D. M. Chung, *Conditional function space integrals with applications*, Rocky Mountain J. Math. **26** (1996), 37–62. MR 1386151.

[3] S. J. Chang and D. Skoug, *Generalized Fourier-Feynman transforms and a first variation on function space*, Integral Transforms Spec. Funct. **14** (2003), 375–393. MR 2005996.

[4] S. J. Chang, H. S. Chung and D. Skoug, *Integral transforms of functionals in $L^2(C_{a,b}[0,T])$*, J. Fourier Anal. Appl. **15** (2009), 441–462. MR 2549938.

[5] S. J. Chang, H. S. Chung and D. Skoug, *Some basic relationships among transforms, convolution products, first variations and inverse transforms*, Cent. Eur. J. Math. **11** (2013), 538–551. MR 3016321.

[6] K. S. Chang, B. S. Kim, T. S. Song and I. Yoo, *Convolution and analytic Fourier-Feynman transforms over paths in abstract Wiener space*, Integral Transforms Spec. Funct. **13** (2002), 345–362. MR 1918957.

[7] K. S. Chang, B. S. Kim and I. Yoo, *Integral transform and convolution of analytic functionals on abstract Wiener space* , Numer. Funct. Anal. Optim. **21** (2000), 97–105. MR 1759990.

[8] J. G. Choi and S. J. Chang, *A rotation on Wiener space with applications*, ISRN Appl. Math. 2012, Art. ID 578174, 13 pp. MR 2957713.

[9] D. M. Chung, C. Park and D. Skoug, *Generalized Feynman integrals via conditional Feynman integrals*, Michigan Math. J. **40** (1993), 377–391. MR 1226837.

[10] H. S. Chung and S. J. Chang, *Inverse integral transforms of functional in $L^2(C_{a,b}(T))$*, Submitted for publication.

[11] H. S. Chung, D. Skoug and S. J. Chang, *Relationships involving transforms and convolutions via the translation theorem*, Stoch. Anal. Appl. **32** (2014), 348–363. MR 3177075.

[12] H. S. Chung, J. G. Choi and S. J. Chang, *A Fubini theorem on a function space and its applications*, Banach J. Math. Anal. **7** (2013), 173–185. MR 3004275.

[13] T. Huffman, D. Skoug and D. Storvick, *A Fubini theorem for analytic Feynman integrals with applications*, J. Korean Math. Soc. **38** (2001), 409–420. MR 1817628.

[14] T. Huffman, D. Skoug and D. Storvick, *Integration formulas involving Fourier-Feynman transforms via a Fubini theorem*, J. Korean Math. Soc. **38** (2001), 421–435. MR 1817629.

[15] B. S. Kim and D. Skoug, *Integral transforms of functionals in $L_2(C_0[0,T])$*, Rocky Mountain J. Math. **33** (2003), 1379–1393. MR 2052494.
[16] Y. J. Lee, *Integral transforms of analytic functions on abstract Wiener spaces*, J. Funct. Anal. **47** (1982), 153–164. MR 0664334.
[17] I. Y. Lee, J. G. Choi and S. J. Chang, *A Fubini theorem for generalized analytic Feynman integral on function space*, Bull. Korean Math. Soc. **50** (2013), 217–231. MR 3029543.
[18] E. Nelson, *Dynamical theories of Brownian motion*, Math. Notes, Princeton University Press, Princeton, 1967. MR 0214150.
[19] J. Yeh, *Singularity of Gaussian measures on function spaces induced by Brownian motion processes with non-stationary increments*, Illinois J. Math. **15** (1971), 37–46. MR 0270427.
[20] J. Yeh, *Stochastic processes and the Wiener integral*, Pure and Applied Mathematics, Vol. 13. Marcel Dekker, New York, 1973. MR 0474528.

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