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PLANAR NORMAL SECTIONS OF FOCAL MANIFOLDS OF ISOPARAMETRIC HYPERSURFACES IN SPHERES

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Abstract. The present paper contains some results about the algebraic sets of planar normal sections associated to the focal manifolds of homogeneous isoparametric hypersurfaces in spheres. With the usual identification of the tangent spaces to the focal manifold with subspaces of the tangent spaces to the isoparametric hypersurface, it is proven that the algebraic set of planar normal sections of the focal manifold is contained in that of the original hypersurface.

1. Introduction

In the present paper we study the real algebraic set of unit tangent vectors generating planar normal sections of the *focal manifolds* of isoparametric hypersurfaces of the spheres. It is well known that isoparametric hypersurfaces of the spheres can be considered isoparametric submanifolds of codimension two in ordinary Euclidean spaces.

By definition, *normal sections* are the curves cut out of a submanifold M^n of $\mathbb{R}^{n+k}$ by the affine subspace generated by a unit tangent vector and the normal space, at a given point p of M^n. A normal section γ at *any* point p of M^n ($p = \gamma(0)$) is called *planar* at p if its first three derivatives $\gamma'(0)$, $\gamma''(0)$, $\gamma'''(0)$ are linearly dependent. The unit tangent vectors defining planar normal sections at $p \in M^n$, form a *real algebraic set* $\widehat{X}_p[M^n]$ (see the next section for a proper definition) which is of interest in the study of the geometry of submanifolds of Euclidean spaces ([2, 11]).

We restrict our considerations to *homogeneous isoparametric hypersurfaces* and their focal manifolds. The collection of homogeneous isoparametric hypersurfaces of the spheres has a finite number of members in opposition to that of non-homogeneous ones ([10, 3]). The reason for this restriction is twofold; on the one hand our algebraic methods are intimately related to the R-spaces (see Section 4), and on the other is the fact that for R-spaces the algebraic set $\widehat{X}_p[M^n]$ can in fact be considered independent of the point $p \in M^n$. Recall that the considered

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submanifold M^n is *extrinsic homogeneous* if for any two points $p, q \in M^n$ there is an isometry g of the ambient space R^{n+k} such that $g(M^n) = M^n$ and $g(p) = q$.

The objective of the present paper is to prove the following result.

Theorem 1. *Let $M^n \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be a full homogenous isoparametric hypersurface and $M_{(+)}$ and $M_{(-)}$ its focal manifolds. If p is a point on M^n and p_1 is a focal point of p in $M_{(+)}$ or $M_{(-)}$ then*

$$\widehat{X}_{p_1}\left[M_{(\pm)}\right] \subset \widehat{X}_p\left[M^n\right].$$

The method to obtain Theorem 1 depends strongly on the fact that the codimension of M^n in $\mathbb{R}^{n+2}$ is 2, and it seems clear that the proof presented here can not be extended to higher codimension.

Theorem 1 indicates an important connection between the sets of planar normal sections of two different manifolds and at the same time shows the presence of interesting algebraic subsets of $\widehat{X}_p[M^n]$.

For an isoparametric submanifold M^n of $\mathbb{R}^{n+k}$ with codimension $k \geq 3$ a similar inclusion can be proved ([4, Corollary 13]) but it requires that the focal manifold be a symmetric R-space.

It is important to remark the fact that not all focal manifolds of homogeneous isoparametric hypersurfaces in spheres are symmetric R-spaces. In fact, interesting examples are the focal manifolds $M_{(+)}$ and $M_{(-)}$ of $M = G_2/T^2$ which is the (essentially unique [7]) isoparametric hypersurface in the sphere of type $(6, 2)$. Here $M_{(+)}$ and $M_{(-)}$ are orbits of G_2 of the form $G_2/U(2) = Q^5$ (the complex quadratic). They are not symmetric since G_2 has only one symmetric space, namely $G_2/SO(4)$, which is an inner symmetric space of quaternionic type, hence it is not an R-space.

The structure of the paper is the following. In the next Section we indicate the basic definitions to be used along the paper and in Section 3 we recall basic results from [11] about the planar normal sections of compact spherical submanifolds and the definition of the algebraic set $\widehat{X}_p[M]$. In Section 4 we recall the definitions of R-spaces and in Section 5 we indicate other notation associated to R-spaces. Particularly in Subsection 5.2 we compute the covariant derivative of the second fundamental form of a general R-space and that of the principal orbits. This basic computation is also contained in [4] but it seems convenient to include it here. Section 6 records basic facts about general isoparametric hypersurfaces in spheres and their focal map. Details could be obtained in [1]; we just present what seems strictly necessary. Section 7 contains the essential steps to get the proof of Theorem 1, which is finally contained in Section 8.

2. Basic definitions

We start recalling some basic definitions. Let M be a compact connected n-dimensional Riemannian manifold and $I : M \to \mathbb{R}^{n+k}$ an isometric embedding into the Euclidean space $\mathbb{R}^{n+k}$. We identify M with its image by I. A submanifold of a Euclidean space $\mathbb{R}^{n+k}$ is usually called *full* if it is not included in any affine hyperplane.

Let $\langle *, * \rangle$ denote the inner product in $\mathbb{R}^{n+k}$. Let ∇^E be the Euclidean covariant derivative in $\mathbb{R}^{n+k}$ and ∇ the Levi-Civita connection in M associated to the induced metric.

We shall say that the submanifold M is *spherical* if it is contained in a sphere of radius r in $\mathbb{R}^{n+k}$ which we may think centered at the origin. Let α denote the second fundamental form of the embedding in $\mathbb{R}^{n+k}$. We denote by $T_p(M)$ and $T_p(M)^{\perp}$ the tangent and normal spaces to M at p, in $\mathbb{R}^{n+k}$, respectively. Let p be a point in M and consider a unit vector Y in the tangent space $T_p(M)$. We may define in $\mathbb{R}^{n+k}$ the affine subspace

$$S(p, Y) = p + \operatorname{Span}\left\{Y, T_p(M)^{\perp}\right\}.$$

If U is a small enough neighborhood of p in M, then the intersection $U \cap S(p, Y)$ can be considered the image of a C^{∞} regular curve $\gamma(s)$, parametrized by arc-length, such that $\gamma(0) = p$, $\gamma'(0) = Y$. This curve is called a *normal section of M at p in the direction of Y*. We say that the normal section γ of M at p in the direction of Y is *planar* at p if its first three derivatives $\gamma'(0)$, $\gamma''(0)$ and $\gamma'''(0)$ are *linearly dependent*. We are interested in studying the set of those unit tangent vectors which generate planar normal sections to M at the point $p \in M$.

3. Compact spherical submanifolds

We recall some known facts and definitions from [11] which are needed here.

Lemma 2. *Let M be spherical. The normal section γ of M at p in the direction of X is planar at p if and only if the covariant derivative of the second fundamental form vanishes on the vector $X = \gamma'(0)$. That is, X satisfies the equation*

$$\left(\overline{\nabla}_X \alpha\right)(X, X) = 0.$$

Given a point p in the submanifold M we shall denote, as in [11],

$$\widehat{X}_p[M] = \left\{Y \in T_p(M) : \|Y\| = 1, \left(\overline{\nabla}_Y \alpha\right)(Y, Y) = 0\right\}.$$

By doing this for each point in M we obtain a subset of the *unit tangent bundle* of M which we shall denote by $\Xi(M)$.

Let M be, as above, a compact connected n-dimensional Riemannian submanifold and also assume that M is spherical. In these conditions, we may take the image $X_p[M]$ of $\widehat{X}_p[M]$ in the real projective space $\mathbb{RP}(T_p(M))$. Then $X_p[M]$ is a real algebraic set of $\mathbb{RP}(T_p(M))$ and its natural complexification $X_p^{\mathbb{C}}[M]$ is a complex algebraic set of $\mathbb{CP}^{n-1}$. The submanifold M is said to be *extrinsically homogeneous* in $\mathbb{R}^{n+k}$ [1, p. 35] if for any two points p and q in M^n there is an isometry g of $\mathbb{R}^{n+k}$ such that $g(M) = M$ and $g(p) = q$. If this is the case, $X_p[M]$ can be considered "independent" of the point $p \in M$. In fact

$$\left(\overline{\nabla}_{\left(g_*|_p X\right)} \alpha\right)\left(g_*|_p X, g_*|_p X\right) = g_*|_p \left(\overline{\nabla}_X \alpha\right)(X, X),$$

and we clearly have that

$$\widehat{X}_q[M] = \widehat{X}_{g(p)}[M] = g_*|_p \left(\widehat{X}_p[M]\right).$$

Then $\widehat{X}_q[M]$ and $\widehat{X}_p[M]$ are isomorphic for each pair p, q and we may "free" ourselves from the point p.

4. R-spaces

Let G/K be a compact simply connected irreducible Riemannian symmetric space where G is a compact connected Lie group and K is a closed subgroup. K is the isotropy subgroup of the basic point $o = [K]$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. The structure of G/K generates an involutive automorphism of θ of $\mathfrak{g}$ such that $\mathfrak{k}$ is the fixed point set of θ and setting $\mathfrak{p} = \{X \in \mathfrak{g} : \theta(X) = -X\}$ we get a direct sum decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

with the properties:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

We can identify $T_o(G/K)$, the tangent space to G/K at $o = [K]$, with the subspace $\mathfrak{p}$. We may assume that the invariant Riemannian metric on G/K is such that at $T_o(G/K)$ the inner product coincides with the restriction to $\mathfrak{p} \times \mathfrak{p}$ of the opposite of the Killing form B of $\mathfrak{g}$. That is, we have for X, Y in $\mathfrak{p}$

$$\langle X, Y \rangle = -B_{\mathfrak{g}}(X, Y). \tag{1}$$

The group $Ad(K)$ acts on $\mathfrak{p}$ effectively, as a compact group of linear isometries. The orbit of a non-zero element of $\mathfrak{p}$ by this group is a submanifold of $\mathfrak{p}$ called an *R-space*. Let $\mathbb{S} \subset \mathfrak{p}$ be the unit sphere. Since the orbit of an element λA ($\lambda \neq 0$) is homothetic to that of A it is customary to consider only the orbits of elements in $\mathbb{S}$. Such orbits are, of course, contained in $\mathbb{S}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$ which we shall maintain fixed. It is furthermore known that the orbit of any $A \in \mathbb{S}$ meets $\mathfrak{a}$ orthogonally (polarity, [1, p. 41-46]), so the relevant orbits are those of vectors in the unit sphere of $\mathfrak{a}$, that is $\mathfrak{a} \cap \mathbb{S}$. We refer the reader to [1, p. 37, p. 46 (3.2.10)] for a description of "orbit types". The relevant comment to make here is that, in our setting, for the action of $Ad(K)$ on $\mathfrak{p}$ we have two classes of orbits, namely the *principal orbits* (generated by the points in an *open dense* subset U of $\mathfrak{a} \cap \mathbb{S}$) and the singular ones (generated by the points in $((\mathfrak{a} \cap \mathbb{S}) - U)$). In the following lines we shall call *regular elements* to those in the set U.

The objective of the present part is to study the set of planar normal sections of the orbit of a general element in $\mathfrak{a} \cap \mathbb{S}$.

5. The geometry of an R-space

We are going to study the geometry of the orbit for any $F \in \mathfrak{a} \cap \mathbb{S}$ (*which may not be regular*) by the group $Ad(K)$. We start by fixing some notation that will simplify things.

5.1. **Notation.** We have

$$M = Ad\,(K)\,F \subset \mathfrak{p}.$$

It is convenient to define for $\lambda \in \mathfrak{a}^*$ (the dual space of $\mathfrak{a}$) the subspaces

$$\mathfrak{k}_\lambda = \{X \in \mathfrak{k} : (ad\,(H))^2\,X = \lambda^2\,(H)\,X, \quad \forall H \in \mathfrak{a}\},$$
$$\mathfrak{p}_\lambda = \{X \in \mathfrak{p} : (ad\,(H))^2\,X = \lambda^2\,(H)\,X, \quad \forall H \in \mathfrak{a}\},$$

and observe that obviously

$$\mathfrak{k}_\lambda = \mathfrak{k}_{-\lambda}, \qquad \mathfrak{p}_\lambda = \mathfrak{p}_{-\lambda}, \qquad \mathfrak{p}_0 = \mathfrak{a},$$

and $\mathfrak{k}_0$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. If $\lambda \in \mathfrak{a}^*$ is $\lambda \neq 0$ and $\mathfrak{p}_\lambda \neq \{0\}$ then λ is called a *restricted root* of $\mathfrak{g}$ *with respect to* $\mathfrak{a}$. We denote by

$$\Delta_R = \Delta_R\,(\mathfrak{g}, \mathfrak{a})$$

the set of restricted roots of $\mathfrak{g}$ with respect to $\mathfrak{a}$. It is important to notice that the set of restricted roots $\Delta_R \subset (\mathfrak{a}^* - \{0\})$ may be *non-reduced*.

Now consider the subsets of restricted roots associated to F:

$$\Delta_{R0}\,(F) = \{\lambda \in \Delta_R : \lambda\,(F) = 0\} \subset \Delta_R,$$
$$\Delta_{R+}\,(F) = \{\lambda \in \Delta_R : \lambda\,(F) > 0\} \subset \Delta_R.$$

With them we may define the following four subspaces:

$$\mathfrak{k}_F = \mathfrak{k}_0 \oplus \sum_{\lambda \in \Delta_{R0}} \mathfrak{k}_\lambda, \qquad \mathfrak{p}_F = \mathfrak{a} \oplus \sum_{\lambda \in \Delta_{R0}} \mathfrak{p}_\lambda,$$
$$\mathfrak{k}_+\,(F) = \sum_{\lambda \in \Delta_{R+}} \mathfrak{k}_\lambda, \qquad \mathfrak{p}_+\,(F) = \sum_{\lambda \in \Delta_{R+}} \mathfrak{p}_\lambda,$$

where all sums are direct, and of course

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = (\mathfrak{k}_F \oplus \mathfrak{k}_+) \oplus (\mathfrak{p}_F \oplus \mathfrak{p}_+)\,.$$

A more complete notation may be required to indicate the dependence of $\mathfrak{k}_+$ and $\mathfrak{p}_+$ from F; in that case we shall use $\mathfrak{k}_+\,(F)$ and $\mathfrak{p}_+\,(F)$, but if there is no risk of confusion we shall keep the simpler notation.

If $X \in \mathfrak{k}_\lambda$ ($\lambda \in \Delta_{R+}$) we have $[X, F] \in \mathfrak{p}$ and furthermore $[\mathfrak{k}_\lambda, F] \subset \mathfrak{p}_\lambda$, and similarly taking $X \in \mathfrak{p}_\lambda$ we obtain $[\mathfrak{p}_\lambda, F] \subset \mathfrak{k}_\lambda$. But in fact, the function $ad\,(F)$ which is

$$ad\,(F) : \mathfrak{k}_\lambda \longrightarrow \mathfrak{p}_\lambda, \qquad ad\,(F) : \mathfrak{p}_\lambda \longrightarrow \mathfrak{k}_\lambda$$

is an isomorphism for each $\lambda \in \Delta_{R+}$ because in any of these two spaces $ad\,(F)^2 = \lambda\,(F)^2\,Id$.

Then we have

$$T_F\,(M) = [\mathfrak{k}, F] = [\mathfrak{k}_+, F] = \mathfrak{p}_+.$$

Now we notice that $\mathfrak{p}_+$ and $\mathfrak{p}_F$ are orthogonal, $\langle \mathfrak{p}_+, \mathfrak{p}_F \rangle = 0$; in fact, since any $Y \in \mathfrak{p}_+$ may be written as $Y = [X, F]$ for some $X \in \mathfrak{k}_+$, by taking $U \in \mathfrak{p}_F$ we see that

$$\langle U, Y \rangle = \langle U, [X, F] \rangle = \langle [X, F]\,, U \rangle = \langle X, [F, U] \rangle = 0$$

because $[U,F]=0$. Then we have

$$T_F(M)^{\perp}=\mathfrak{p}_F;$$

we must observe that if we take (instead of F) an element E in $\mathfrak{a}\cap\mathbb{S}$ which is *regular* then

$$T_E(M)^{\perp}=\mathfrak{a}.$$

5.2. **Computation of $\overline{\nabla}\alpha$.** We have the Euclidean covariant derivative ∇^E in $(\mathfrak{p},\langle *,*\rangle)$ and the Levi-Civita connection ∇ associated to the induced metric on M. Also on M we are going to consider the *canonical connection* determined by the decomposition $\mathfrak{k}=\mathfrak{k}_F\oplus\mathfrak{k}_+$, which we shall denote by ∇^c [1, p. 203].

We have the second fundamental form of M on $\mathfrak{p}$ and we have the Gauss formula

$$\nabla^E_U W=\nabla_U W+\alpha(U,W).$$

We need to compute $\nabla^E_U W$. Let us take, for some $X\in\mathfrak{k}$, the curve in M of the form

$$\gamma(t)=(Ad(\exp(tX))F). \tag{2}$$

Its tangent vector at F is $\gamma'(0)=[X,F]$ and if we take $t_1>0$ then we may compute the derivative $\gamma'(t_1)$ by

$$\begin{aligned}\gamma'(t_1)&=\tfrac{d}{dt}\Big|_{t=0}(Ad(\exp((t_1+t)X))F)\\&=\tfrac{d}{dt}\Big|_{t=0}(Ad(\exp((t_1)X))Ad(\exp((t)X))F)\\&=Ad(\exp((t_1)X))\tfrac{d}{dt}\Big|_{t=0}Ad(\exp((t)X))F\\&=Ad(\exp((t_1)X))[X,F],\end{aligned}\tag{3}$$

and so this gives the tangent field along $\gamma(t)$.

It is well known that the curves like γ in (2) are ∇^c-*geodesics* and that the ∇^c-*parallel* translation along these geodesics is precisely given by (3) [1, p. 203].

Let us take a tangent vector at F

$$[Y,F]\in T_F(M)=[\mathfrak{k},F]=[\mathfrak{k}_+,F]$$

and extend it to a field along γ by

$$[Y,F]^*=Ad(\exp(tX))[Y,F]. \tag{4}$$

Then compute, for $X,Y\in\mathfrak{k}$,

$$\nabla^E_{[X,F]}[Y,F]^*=\frac{d}{dt}\bigg|_{t=0}(Ad(\exp(tX))[Y,F])=[X,[Y,F]]\in\mathfrak{p},$$

noticing that, in fact, we may take $X,Y\in\mathfrak{k}_+$.

Now writing the Gauss formula for $[X,F]$ and $[Y,F]^*$ we have

$$\nabla^E_{[X,F]}[Y,F]^*=\nabla_{[X,F]}[Y,F]^*+\alpha([X,F],[Y,F]);$$

then clearly

$$\begin{aligned}\nabla_{[X,F]}[Y,F]^*&=([X,[Y,F]])_{\mathfrak{p}_+}\\\alpha([X,F],[Y,F])&=([X,[Y,F]])_{\mathfrak{p}_F}.\end{aligned}\tag{5}$$

Let us now compute the covariant derivative of α. By definition this is

$$\begin{aligned}\left(\overline{\nabla}_{[X,F]}\alpha\right)([Y,F],[Z,F]) = {} & \nabla^{\perp}_{[X,F]}\alpha\left([Y,F],[Z,F]\right)\\ & - \alpha\left(\nabla_{[X,F]}[Y,F],[Z,F]\right)\\ & - \alpha\left([Y,F],\nabla_{[X,F]}[Z,F]\right).\end{aligned} \tag{6}$$

Now in [8] (see also [1, p. 212]) the canonical covariant derivative of the second fundamental form α was introduced as

$$\begin{aligned}\left(\nabla^{c}_{[X,F]}\alpha\right)([Y,F],[Z,F]) = {} & \nabla^{\perp}_{[X,F]}\alpha\left([Y,F],[Z,F]\right)\\ & - \alpha(\nabla^{c}_{[X,F]}[Y,F],[Z,F])\\ & - \alpha([Y,F],\nabla^{c}_{[X,F]}[Z,F]),\end{aligned}$$

and an important result of [8] is that the condition

$$\left(\nabla^{c}_{[X,F]}\alpha\right)([Y,F],[Z,F]) = 0$$

in fact *characterizes* R-spaces.

Since M is a R-space we have:

$$\begin{aligned}0 = {} & \nabla^{\perp}_{[X,F]}\alpha\left([Y,F],[Z,F]\right) - \alpha\left(\nabla^{c}_{[X,F]}[Y,F],[Z,F]\right)\\ & - \alpha\left([Y,F],\nabla^{c}_{[X,F]}[Z,F]\right),\end{aligned} \tag{7}$$

and subtracting (7) from (6) we get

$$\begin{aligned}\left(\overline{\nabla}_{[X,F]}\alpha\right)([Y,F],[Z,F]) = {} & -\alpha\left(D\left([X,F],[Y,F]\right),[Z,F]\right)\\ & - \alpha\left([Y,F],D\left([X,F],[Z,F]\right)\right),\end{aligned}$$

where

$$D = \nabla - \nabla^{c}$$

is the *difference tensor of the two connections*, and so we have an important equivalence:

$$\left(\overline{\nabla}_{[X,F]}\alpha\right)([X,F],[X,F]) = 0 \iff \alpha\left(D\left([X,F],[X,F]\right),[X,F]\right) = 0. \tag{8}$$

So we need to compute the difference tensor $D\left([X,F],[Y,F]\right)$. To that end we use the fact that the tangent field $[Y,F]^{*}$ (4) is *parallel with respect to the canonical connection* ∇^{c} *along* γ. We have

$$D\left([X,F],[Y,F]\right) = \nabla_{[X,F]}[Y,F]^{*} - \nabla^{c}_{[X,F]}[Y,F]^{*} = \nabla_{[X,F]}[Y,F]^{*},$$

and going back to (5) we have

$$D\left([X,F],[Y,F]\right) = \nabla_{[X,F]}[Y,F]^{*} = Ta\left([X,[Y,F]]\right) = \left([X,[Y,F]]\right)_{\mathfrak{p}_{+}}.$$

Then we have that the indicated equivalence (8) becomes here:

$$\left(\overline{\nabla}_{[X,F]}\alpha\right)([X,F],[X,F]) = 0 \iff \left(\left[X,\left([X,[X,F]]\right)_{\mathfrak{p}_{+}(F)}\right]\right)_{\mathfrak{p}_{F}} = 0. \tag{9}$$

We want to mention the important case in which the orbit M is *principal*. To distinguish this case we use E instead of F, so here E is a *regular element* in $(\mathfrak{a} \cap \mathbb{S})$. Formula (9) becomes then

$$\left(\overline{\nabla}_{[X,E]}\alpha\right)\left([X,E],[X,E]\right) = 0 \iff \left(\left[X, \left([X,[X,E]]\right)_{\mathfrak{p}_+(E)}\right]\right)_{\mathfrak{a}} = 0.$$

Observe that when E is regular, $\mathfrak{p} = \mathfrak{a} \oplus \mathfrak{p}_+(E)$, and in all these formulae $X \in \mathfrak{k} = \mathfrak{k}_F \oplus \mathfrak{k}_+(F)$, but we may take in fact $X \in \mathfrak{k}_+(F)$.

6. Isoparametric hypersurfaces in spheres

We need to indicate in this section some notation for the case of isoparametric hypersurfaces in spheres or equivalently isoparametric submanifolds of rank two in Euclidean spaces. The literature about this type of hypersurfaces and their many interesting properties is quite vast, and we refer the reader to the book [1] and its references for a rather complete treatment of this particular subject within the framework of submanifold geometry. In fact we shall only indicate here the things we need to develop the present work and refer to this source whenever it is necessary.

Let us recall that a compact connected hypersurface $M \subset \mathbb{S}^{n+1}$ is called *isoparametric* if its principal curvatures $\lambda_1(p), \ldots, \lambda_g(p)$ are constant functions of $p \in M$.

It is usual ([1, p. 152]), for the constants $\lambda_1, \ldots, \lambda_g$, to adopt the notation

$$\lambda_j = \cot(\theta_j), \qquad 0 < \theta_1 < \cdots < \theta_g < \pi,$$

and one has the result of Münzner [1, Th. 5.2.16, p. 152]:

$$\theta_k = \theta_1 + \frac{k-1}{g}\pi, \qquad k = 1, \ldots, g.$$

The *multiplicity* of the principal curvature λ_j is denoted by m_j. These satisfy

$$m_j = m_{j+2} \quad \text{(modulo } g \text{ indexing)}.$$

Any pair of shape operators corresponding to a pair of normal vectors (in $\mathbb{R}^{n+2}$) commute:

$$[A_\xi, A_\gamma] = 0.$$

It follows from the Ricci equation [1, p. 11] that the *normal curvature tensor* $R^\perp$ vanishes on M and all the shape operators can be simultaneously diagonalized on each tangent space. We have then in $T_p(M)$ *common eigendistributions* $D_j(p)$ $(j = 1, \ldots, g)$ called the *curvature distributions* of M, and we may write $T_p(M)$ as a direct sum

$$T_p(M) = \sum_{j=1}^{g} D_j(p).$$

6.1. **The focal map.** For each point $p \in M$ the normal space in $\mathbb{R}^{n+2}$ is the two dimensional real vector space $T_p(M)^{\perp}$. We may take the normal vector $H(p)$ such that $\{p, H(p)\}$ is an *orthonormal basis* of $T_p(M)^{\perp}$ (notice that $H(p)$ is tangent to the sphere $\mathbb{S}^{n+1}$ at each $p \in M$).

Let us consider the unit speed geodesic in $\mathbb{S}^{n+1}$ normal to M starting at p, moving in the direction of $H(p)$. We may write it as

$$\Phi_s(p) = \cos(s)\, p + \sin(s)\, H(p), \quad s \in [0, \pi]. \tag{10}$$

We have then (for each fixed s) a function

$$\Phi_s : M \longrightarrow \mathbb{S}^{n+1}.$$

The derivative of Φ_s at $p \in M$ is [1, p. 152]:

$$(\Phi_s)_*|_p = \cos(s)\, Id - (\sin t)\, A_{H(p)}, \tag{11}$$

and we observe that for

$$s \in [0, \pi], \quad s \neq \theta_j \ (j = 1, \ldots, g) \tag{12}$$

the derivative (11), at each $p \in M$, is a monomorphism in each $D_j(p)$. In fact, if $X \in D_j(p)$,

$$(\Phi_s)_*|_p X = \frac{\sin(\theta_j - s)}{\sin(\theta_j)} X$$

and, by our condition on s, $(\Phi_s)_*|_p$ is a monomorphism on $T_p(M)$ [1, p. 152].

Remark 3. *It is important to notice that the subspace $D_j(p)$ is the eigenspace of eigenvalue $\lambda_j = \cot(\theta_j)$ for the operator $A_{H(p)}$.*

Each of the points $\Phi_s(p)$, for s satisfying (12), belongs to another isoparametric hypersurface of the family. On the other hand if $s = \theta_j$ we see that $(\Phi_s)_*|_p$ sends $D_j(p)$ to zero and therefore

$$\widehat{p} = \Phi_{\theta_j}(p)$$

is a *focal point* of p along the normal geodesic, and so $\widehat{p}$ is in one of the two focal manifolds.

Let us call M_{θ_j} the focal manifold reached at $s = \theta_j$ (of course this is either $M_{(+)}$ or $M_{(-)}$ but this is not relevant for our purposes). The tangent space to the corresponding focal manifold M_{θ_j} at $\widehat{p}$ can be identified with

$$T_{\widehat{p}}\left(M_{\theta_j}\right) = \sum_{k=1, k\neq j}^{g} D_k(p).$$

Let us notice that the two dimensional plane generated by $\{p, H(p)\}$ which coincides with $T_p(M)^{\perp}$ is orthogonal to M_{θ_j} at $\widehat{p}$. But also the component $D_j(p)$ is now contained in the normal space of this focal manifold at $\widehat{p}$. Furthermore we may identify

$$T_{\widehat{p}}\left(M_{\theta_j}\right)^{\perp} = \mathrm{Span}_{\mathbb{R}}\{p, H(p)\} \oplus D_j(p).$$

7. Planar normal sections of M_{θ_j}

The fundamental fact here is that the manifold M_{θ_j} is a *spherical* submanifold of $\mathbb{R}^{n+2}$ and so we may study its set of planar normal sections at the point $\widehat{p}$ as in Section 3. However we shall restrict our considerations to the case in which M is a *homogeneous* isoparametric hypersurface in the sphere $\mathbb{S}^{n+1}$. This shall allows us to use the above facts for R-spaces which apply both to the cases of M and M_{θ_j}.

We need to make compatible the notations used above. We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

We have $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace ($\dim \mathfrak{a} = 2$), $E \in (\mathfrak{a} \cap \mathbb{S})$ a *regular element*, and $M = Ad\,(K)\,E$ is a *principal orbit* in $\mathfrak{p}$. This is the isoparametric hypersurface of the unit sphere $\mathbb{S}\,(\mathfrak{p}) \subset \mathfrak{p}$ under consideration. We have the orthonormal basis $\{E, H\,(E)\}$ of $\mathfrak{a} = T_E\,(M)^{\perp}$ and consider the focal point (in M_{θ_j})

$$E_1 = \Phi_{\theta_j}\,(E) = \cos\,(\theta_j)\,E + \sin\,(\theta_j)\,H\,(E)$$

given by (10) at $s = \theta_j$. We want to apply (9) to the spherical submanifold M_{θ_j} hence here we have $F = E_1$, and we need to recover some of the notation from Section 5 in this special case. In particular, we have to identify the subspace $D_j\,(E)$ in the notation of the R-spaces.

By definition (Remark 3),

$$D_j\,(E) = \text{eigenspace of } \cot\,(\theta_j) \text{ of } A_{H(E)}.$$

It is on the other hand known [1, p. 63] that (for the principal orbit $M = Ad\,(K)\,E$) the eigenvalues of any shape operator can be computed in terms of the restricted roots $\lambda \in \Delta_R$ as

$$A_\xi\,(X) = \left(-\frac{\lambda\,(\xi)}{\lambda\,(E)}\right) X, \qquad X \in \mathfrak{p}_\lambda,\ \xi \in \mathfrak{a}, \tag{13}$$

so the common eigenspaces of the shape operators of M are

$$\Omega_\lambda = \mathfrak{p}_\lambda + \mathfrak{p}_{2\lambda},$$

where $\mathfrak{p}_{2\lambda} = \{0\}$ if 2λ is not a restricted root.

By Remark 3 we have that, on $X \in D_j$,

$$A_{H(E)}\,(X) = \cot\,(\theta_j)\,X \quad \Longrightarrow \quad \left(-\frac{\lambda\,(H\,(E))}{\lambda\,(E)}\right) = \cot\,(\theta_j)\,.$$

We need to identify which is the $\lambda \in \Delta_R$ such that $D_j = \Omega_\lambda$. To that end, since $E_1 \in \mathfrak{a}$, we may use formula (13) to compute (at the point $E \in M$) the shape operator $A_{E_1}\,(X)$ for $X \in \mathfrak{p}_\lambda$. We get

$$A_{E_1}\,(X) = \left(-\frac{\lambda\,(E_1)}{\lambda\,(E)}\right) X, \qquad X \in \mathfrak{p}_\lambda.$$

On the other hand, by the definition of E_1, we may compute (again at the point $E \in M$) on $X \in D_j$,

$$\begin{aligned} A_{E_1}(X) &= \cos(\theta_j)\, A_E(X) + \sin(\theta_j)\, A_{H(E)}X \\ &= \cos(\theta_j)(-X) + \sin(\theta_j)\cot(\theta_j)\, X = 0. \end{aligned}$$

Then we see that $X \in D_j$ implies $\lambda(E_1) = 0$. We conclude that

$$D_j \subset \sum_{\lambda} \Omega_\lambda \quad \text{such that } \lambda(E_1) = 0. \tag{14}$$

Then in our notation for the R-spaces we have the two sets of restricted roots associated to the point E_1, that is

$$\begin{aligned} \Delta_{R0}(E_1) &= \{\lambda \in \Delta_R : \lambda(E_1) = 0\}, \\ \Delta_{R+}(E_1) &= \{\lambda \in \Delta_R : \lambda(E_1) > 0\}. \end{aligned}$$

and (14) yields

$$D_j = \sum_{\lambda \in \Delta_{R0}(E_1)} \mathfrak{p}_\lambda. \tag{15}$$

We also have

$$\begin{array}{ll} \mathfrak{k} = \mathfrak{k}_E \oplus \mathfrak{k}_+(E) & \mathfrak{k} = \mathfrak{k}_{E_1} \oplus \mathfrak{k}_+(E_1) \\ \mathfrak{k}_{E_1} = \mathfrak{k}_0 \oplus \sum_{\lambda \in \Delta_{R0}} \mathfrak{k}_\lambda & \mathfrak{p}_{E_1} = \mathfrak{a} \oplus \sum_{\lambda \in \Delta_{R0}} \mathfrak{p}_\lambda \\ \mathfrak{k}_+(E_1) = \sum_{\lambda \in \Delta_{R+}} \mathfrak{k}_\lambda & \mathfrak{p}_+(E_1) = \sum_{\lambda \in \Delta_{R+}} \mathfrak{p}_\lambda. \end{array}$$

To determine the planar normal sections of M_{θ_j} we consider now (9). Since we have the orthogonal direct sum $\mathfrak{p}_{E_1} = \mathfrak{a} \overset{\perp}{\oplus} D_j$, we see that

$$\begin{aligned} &\Big[X, ([X,[X,E_1]])_{\mathfrak{p}_+(E_1)}\Big]_{\mathfrak{p}_{E_1}} \\ &\qquad = \Big[X, ([X,[X,E_1]])_{\mathfrak{p}_+(E_1)}\Big]_{\mathfrak{a}} + \Big[X, ([X,[X,E_1]])_{\mathfrak{p}_+(E_1)}\Big]_{D_j}. \end{aligned} \tag{16}$$

Therefore, if an X vanishes the left hand side (i.e. if the vector $[X, E_1]$ defines a planar normal section to M_{θ_j} at E_1) then it has to vanish each term on the right hand side (because they belong to different orthogonal subspaces). In these formulas $X \in \mathfrak{k} = \mathfrak{k}_{E_1} \oplus \mathfrak{k}_+(E_1)$, but we may take in fact $X \in \mathfrak{k}_+(E_1)$ and $\mathfrak{k}_+(E_1) \subset \mathfrak{k}_+(E)$.

So, we study the condition

$$\Big[X, ([X,[X,E_1]])_{\mathfrak{p}_+(E_1)}\Big]_{\mathfrak{a}} = 0.$$

It is also clear, by definition of E_1, that we have (writing H instead of $H(E)$ to simplify notation):

$$\begin{aligned}&\left[X,\left([X,[X,E_1]]\right)_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}\\&\quad=(\cos\theta_j)\left[X,\left([X,[X,E]]\right)_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}+(\sin\theta_j)\left[X,\left([X,[X,H]]\right)_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}.\end{aligned}\tag{17}$$

Now we study the two terms in (17) separately,

$$\left[X,\left([X,[X,E]]\right)_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}\quad\text{and}\quad\left[X,\left([X,[X,H]]\right)_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}.$$

7.1. **First term** $\left[X,\left([X,[X,E]]\right)_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}$**.** Since $\mathfrak{p}_+(E)=\mathfrak{p}_+(E_1)\oplus D_j$, it is clear that we have

$$\left[X,\left([X,[X,E]]\right)_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}}=\left[X,\left([X,[X,E]]\right)_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}+\left[X,\left([X,[X,E]]\right)_{D_j}\right]_{\mathfrak{a}}.$$

Now we study

$$\left[X,\left([X,[X,E]]\right)_{D_j}\right]_{\mathfrak{a}}.$$

Since, by definition, $\left([X,[X,E]]\right)_{D_j}\in D_j$ and $X\in\mathfrak{k}_+(E_1)\subset\mathfrak{k}_+(E)$ we have

$$\left[X,\left([X,[X,E]]\right)_{D_j}\right]_{\mathfrak{a}}\subset\left[\mathfrak{k}_+(E_1),D_j\right]_{\mathfrak{a}},$$

so we study $\left[\mathfrak{k}_+(E_1),D_j\right]_{\mathfrak{a}}$ in the following lemma.

Lemma 4. $\left[\mathfrak{k}_+(E_1),D_j\right]_{\mathfrak{a}}=0$.

Proof. By (15) we have to consider here

$$\left[\mathfrak{k}_+(E_1),D_j\right]=\left[\sum_{\lambda\in\Delta_{R+}(E_1)}\mathfrak{k}_\lambda,\sum_{\mu\in\Delta_{R0}(E_1)}\mathfrak{p}_\mu\right]=\sum_{\lambda\in\Delta_{R+}(E_1)}\sum_{\mu\in\Delta_{R0}(E_1)}\left[\mathfrak{k}_\lambda,\mathfrak{p}_\mu\right].$$

It is well known [5] that

$$\left[\mathfrak{k}_\lambda,\mathfrak{p}_\mu\right]\subset\mathfrak{p}_{\lambda+\mu}+\mathfrak{p}_{\lambda-\mu},$$

and since $\lambda\in\Delta_{R+}(E_1)$ and $\mu\in\Delta_{R0}(E_1)$ we see that $\lambda+\mu$ and $\lambda-\mu$ belong to $\Delta_{R+}(E_1)$. Then we have

$$\left[\mathfrak{k}_+(E_1),D_j\right]\subset\mathfrak{p}_+(E_1)\subset\mathfrak{p}_+(E).$$

But $\mathfrak{p}_+(E)$ is orthogonal to $\mathfrak{a}$ and therefore the lemma follows. □

We see then that

$$\left[X,\left([X,[X,E]]\right)_{D_j}\right]_{\mathfrak{a}}=0$$

and hence we have the equality

$$\left[X,\left([X,[X,E]]\right)_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}}=\left[X,\left([X,[X,E]]\right)_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}.\tag{18}$$

7.2. **Second term** $\left[X,([X,[X,H]])_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}$. Let us consider now the second term in (17), that is

$$\left[X,([X,[X,H]])_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}. \tag{19}$$

Since $H=\Phi_s(E)$ (for $s=\frac{\pi}{2}$) we have two possibilities: either H is also regular (if $\theta_k\neq\frac{\pi}{2}$, $\forall k=1,\dots,g$) or it is a focal point of E (which requires $\theta_k=\frac{\pi}{2}$ for some $k\in\{1,\dots,g\}$). In any case we study (19) in the same way as before. We have again the same decomposition

$$\left[X,([X,[X,H]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}}=\left[X,([X,[X,H]])_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}+\left[X,([X,[X,H]])_{D_j}\right]_{\mathfrak{a}},$$

and since $X\in\mathfrak{k}_+(E_1)\subset\mathfrak{k}_+(E)$ we have here

$$\left[X,([X,[X,H]])_{D_j}\right]_{\mathfrak{a}}\subset[\mathfrak{k}_+(E_1),D_j]_{\mathfrak{a}},$$

but also in this case Lemma 4 yields

$$\left[X,([X,[X,H]])_{D_j}\right]_{\mathfrak{a}}=0,$$

so we have again a version of (18) but here with H *instead of* E. That is

$$\left[X,([X,[X,H]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}}=\left[X,([X,[X,H]])_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}. \tag{20}$$

Then in view of (18) and (20) we may change the two terms in (17) and write

$$\begin{aligned}&\left[X,([X,[X,E_1]])_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}}\\&\quad=(\cos\theta_1)\left[X,([X,[X,E]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}}+(\sin\theta_1)\left[X,([X,[X,H]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}}.\end{aligned} \tag{21}$$

Now we have the following proposition.

Proposition 5. *For any $X\in\mathfrak{k}_+(E_1)$ we have that $\left[X,([X,[X,E]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}}$ is orthogonal to E and $\left[X,([X,[X,H]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}}$ is orthogonal to H.*

Proof. In fact we prove that for *any nonzero vector* V in $\mathfrak{a}$ and $E\in\mathfrak{a}$ *regular* we have:

$$\left[X,([X,[X,V]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}}\text{ is orthogonal to }V\text{ for any }X\in\mathfrak{k}.$$

We have to compute

$$\left\langle\left[X,([X,[X,V]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}},V\right\rangle=\left\langle\left[X,([X,[X,V]])_{\mathfrak{p}_+(E)}\right],V\right\rangle.$$

Recall that the inner product that we have in $\mathfrak{p}=\mathfrak{a}\overset{\perp}{\oplus}\mathfrak{p}_+(E)$ is (1). Then we have to compute:

$$\begin{aligned}-B_{\mathfrak{g}}\left(\left[X,([X,[X,V]])_{\mathfrak{p}_+(E)}\right],V\right)&=B_{\mathfrak{g}}\left(\left[([X,[X,V]])_{\mathfrak{p}_+(E)},X\right],V\right)\\&=B_{\mathfrak{g}}\left(([X,[X,V]])_{\mathfrak{p}_+(E)},[X,V]\right)\\&=B_{\mathfrak{g}}([X,[X,V]],[X,V])\end{aligned}$$

because $[X,V] \in \mathfrak{p}_+(E)$. Now

$$\begin{aligned} B_{\mathfrak{g}}([X,[X,V]],[X,V]) &= -B_{\mathfrak{g}}([[X,V],X],[X,V]) \\ &= -B_{\mathfrak{g}}([X,V],[X,[X,V]]) \\ &= -B_{\mathfrak{g}}([X,[X,V]],[X,V]) \text{ (by symmetry of } B_{\mathfrak{g}}). \end{aligned}$$

But the equality

$$B_{\mathfrak{g}}([X,[X,V]],[X,V]) = -B_{\mathfrak{g}}([X,[X,V]],[X,V])$$

clearly implies

$$B_{\mathfrak{g}}([X,[X,V]],[X,V]) = 0,$$

and in turn

$$B_{\mathfrak{g}}\left(\left[X,([X,[X,V]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}},V\right) = 0.$$

Now setting $V = E$ and $V = H$ we have both assertions in Proposition 5. □

Corollary 6. *For any* $X \in \mathfrak{k}_+(E_1)$*, if* $\left[X,([X,[X,E]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}}$ *and* $\left[X,([X,[X,H]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}}$ *are both non-zero then they are linearly independent in* $\mathfrak{a}$.

8. Proof of Theorem 1

Proof. As we have observed above, if an X vanishes the left hand side of (16) then it has to vanish each term on the right hand side. Therefore

$$\left[X,([X,[X,E_1]])_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{p}_F} = 0 \quad\Longrightarrow\quad \left[X,([X,[X,E_1]])_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}} = 0;$$

on the other hand (21) and Corollary 6 show that

$$\begin{aligned} 0 &= \left[X,([X,[X,E_1]])_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{a}} \\ &\Longleftrightarrow \left[X,([X,[X,E]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}} = 0 \quad\text{and}\quad \left[X,([X,[X,H]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}} = 0. \end{aligned}$$

In fact, if one of the two terms on the right hand side of (21) vanishes then so does the other one. On the other hand *if both terms are non-zero* then by Corollary 6 they must be linearly independent and this leads to a contradiction because $\cos\theta_1$ and $\sin\theta_1$ are not zero.

Since by definition

$$\widehat{X}[M] = \left\{X \in \mathfrak{k} : \left[X,([X,[X,E]])_{\mathfrak{p}_+(E)}\right]_{\mathfrak{a}} = 0\right\}$$

and

$$\widehat{X}\left[M_{\theta_j}\right] = \left\{X \in \mathfrak{k}_+ : \left[X,([X,[X,E_1]])_{\mathfrak{p}_+(E_1)}\right]_{\mathfrak{p}_F} = 0\right\},$$

we get the inclusion

$$\widehat{X}\left[M_{\theta_j}\right] \subset \widehat{X}[M].$$ □

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