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A NOTE ON THE CLASSIFICATION OF GAMMA FACTORS

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Abstract. One of the earliest invariants introduced in the study of finite von Neumann algebras is the property Gamma of Murray and von Neumann. The set of separable II_1 factors can be split in two disjoint subsets: those that have the property Gamma and those that do not have it, called full factors by Connes. In this note we prove that it is not possible to classify separable II_1 factors satisfying the property Gamma up to isomorphism by a Borel measurable assignment of countable structures as invariants. We also show that the same holds true for the full II_1 factors.

1. Introduction

In this note we continue with the line of research initiated by the author in collaboration with A. Törnquist in [19], [20] and [21], where we applied the notion of *Borel reducibility* from descriptive set theory to study the complexity of the classification problem of several different classes of separable von Neumann algebras.

Recall that if E and F are equivalence relations on standard Borel spaces X and Y, respectively, we say that E is *Borel reducible* to F if there is a Borel function $f : X \to Y$ such that

$$(\forall x, x' \in X) x E x' \iff f(x) F f(x'),$$

and if this is the case we write $E \leq_B F$. Thus if $E \leq_B F$ then the points of X can be classified up to E equivalence by a Borel assignment of invariants that we may think of as F-equivalence classes. E is *smooth* if it is Borel reducible to the equality relation on $\mathbb{R}$. While smoothness is desirable, it is most often too much to ask for. A more generous class of invariants which seems natural to consider are countable groups, graphs, fields, or other countable structures, considered up to isomorphism. Thus, following [14], we will say that an equivalence relation E is *classifiable by countable structures* if there is a countable language $\mathcal{L}$ such that $E \leq_B \simeq^{\mathrm{Mod}(\mathcal{L})}$, where $\simeq^{\mathrm{Mod}(\mathcal{L})}$ denotes isomorphism in $\mathrm{Mod}(\mathcal{L})$, the Polish space of countable models of $\mathcal{L}$ with universe $\mathbb{N}$.

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In [20] it was proved that the isomorphism relation in the set of finite von Neumann algebras is not classifiable by countable structures. Nonetheless, it can certainly be the case that some subclasses of finite factors are possible to classify by countable structures. For instance, Connes' celebrated theorem [3], says that the set of infinite dimensional injective finite factors has only one element on its isomorphism class, namely the hyperfinite II_1 factor R. In contrast with the injective case, in this note we show that it is not possible to obtain a reasonable classification up to isomorphisms for a well studied family of finite factors that includes R. In order to state our results we observe first that the set of finite factors can be split in two disjoint subsets: those who satisfy the property Γ of Murray and von Neumann and those who are *full*. The first set contains the hyperfinite II_1 factor R and more generally, the class of McDuff factors, i.e. those factors of the form $M \otimes R$ for M a II_1 factor. On the other hand the set of full factors contains the free group factors $L(\mathbb{F}_n)$. In this article we show that the II_1 factors constructed in [20] are full. As a consequence Theorem 7 in [20] strengthens to prove:

Theorem 1.1. *The isomorphism relation for full type* II_1 *factors is not classifiable by countable structures.*

It remained then to analyze the complexity of the classification of II_1 factors with the property Γ. In this note we address this problem by showing that:

Theorem 1.2. *The isomorphism relation for McDuff factors is not classifiable by countable structures.*

An immediate consequence is:

Corollary 1.3. *The isomorphism relation for type* II_1 *factors satisfying the property* Γ *of Murray and von Neumann is not classifiable by countable structures.*

We end this introduction by mentioning that the study of the connections between logic and operator algebras has recently attracted many researchers from both fields. As a consequence, in the past five years there has been a burst of activity in proving results along the lines of the ones presented in this note and first unveiled in [19], [20] and [21]. We refer the reader who wants to learn more on these exciting new developments to the recent survey of I. Farah [9].

2. Gamma factors

We start by recalling the definitions of the objects we study in this article. Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and denote by $\mathcal{B}(\mathcal{H})$ the space of bounded operators on $\mathcal{H}$, which we give the weak topology. A separable von Neumann algebra is a weakly closed self-adjoint subalgebra of $\mathcal{B}(\mathcal{H})$. The set of von Neumann algebras acting on $\mathcal{H}$ is denoted $\mathrm{vN}(\mathcal{H})$. A von Neumann algebra M is said to be *finite* if it admits a finite faithful normal tracial state, i.e. a linear functional $\tau : M \to \mathbb{C}$ such that: $\tau(x^*x) \geq 0$, $\tau(x^*x) = 0$ iff $x = 0$, $\tau(1) = 1$, $\tau(xy) = \tau(yx)$ and the unit ball of M is complete with respect to the norm given by the trace $\|x\|_2 = \tau(x^*x)$. If a finite von Neumann algebra is also a *factor*, i.e. its center is trivial, then it has a unique such trace. A finite von Neumann factor that is not a matrix algebra is called a type II_1 factor. This terminology is due to the general classification of von Neumann algebras according to types (see [4, Chapter 5.1] for an historical account of the theory of types).

In this note we will be interested in II_1 factors arising from the so called *group-measure space construction*, that we proceed to describe. For that, let G be a countably infinite discrete group which acts in a measure preserving way on a Borel probability space (X,μ). For each $g \in G$ and $\zeta \in L^2(X,\mu)$ the formula

$$\sigma_g(\zeta)(x) = \zeta(g^{-1}\cdot x)$$

defines a unitary operator on $L^2(X,\mu)$.

We identify the Hilbert space $\mathcal{H} = L^2(G, L^2(X,\mu))$ with the Hilbert space of formal sums $\sum_{g\in G}\zeta_g\xi_g$, where the coefficients ζ_g are in $L^2(X,\mu)$ and satisfy $\sum_g \|\zeta_g\|^2_{L^2(X,\mu)} < \infty$, and ξ_g are indeterminates indexed by the elements of G. The inner product on $\mathcal{H}$ is given by

$$\left\langle \sum_{g\in G}\zeta_g(x)\xi_g, \sum_{g\in G}\zeta'_g(x)\xi_g \right\rangle = \sum_{g\in G}\langle \zeta_g, \zeta'_g\rangle_{L^2(X,\mu)}.$$

Both $L^\infty(X,\mu)$ and G act by left multiplication on $\mathcal{H}$ by the formulas

$$f(\zeta_g(x)\xi_g) = ((f(x)\zeta_g(x))\xi_g,$$
$$u_h(\zeta_g(x)\xi_g) = \sigma_h(\zeta_g)(x)\xi_{hg},$$

where $f \in L^\infty(X,\mu)$, $\zeta_g(x) \in L^2(X,\mu)$ and $g,h \in G$. Thus if we denote by $\mathcal{FS}$ the set of finite sums,

$$\mathcal{FS} = \left\{ \sum_{g\in G} f_g u_g : f_g \in L^\infty(X,\mu),\ f_g = 0, \text{ except for finitely many } g \right\},$$

then each element in $\mathcal{FS}$ defines a bounded operator on $\mathcal{H}$. Moreover, multiplication and involution in $\mathcal{FS}$ satisfy the formulas

$$(f_g u_g)(f_h u_h) = f_g\sigma_g(f_h)u_{gh}$$

and

$$(fu_g)^* = \sigma_{g^{-1}}(f^*)u_{g^{-1}},$$

and so $\mathcal{FS}$ is a $*$-algebra. By definition, the *group-measure space von Neumann algebra* is the weak operator closure of $\mathcal{FS}$ on $\mathcal{B}(\mathcal{H})$ and it is denoted by $L^\infty(X,\mu)\rtimes_\sigma G$. The trace on $\mathcal{FS}$, defined by

$$\tau(\sum_{g\in G} f_g u_g) = \int_X f_e\, d\mu,$$

extends to a faithful normal tracial state in $L^\infty(X)\rtimes_\sigma G$ by the formula $\tau(T) = \langle T(\xi_e), \xi_e\rangle$, where e represents the identity of G.

Definition 2.1 (Murray-von Neumann [15])**.** A finite von Neumann algebra M has the property Γ if given $x_1,\dots,x_n \in M$, and $\varepsilon > 0$ there exists $u \in \mathcal{U}(M)$, $\tau(u) = 0$ such that

$$\|x_i u - u x_i\|_2 < \varepsilon, \quad \text{for all } 1 \le i \le n.$$

It follows immediately from its definition that the hyperfinite II_1 factor R is a Γ-factor. Moreover, it is clear that any finite factor of the form $M \otimes N$ with N a Γ-factor is also a Γ-factor. In particular, *McDuff factors*, i.e. factors of the form $M \otimes R$, are Γ-factors. The paradoxical decomposition of the free groups $\mathbb{F}_n$, $n \geq 2$, is the key ingredient [15, Lemmas 6.2.1, 6.2.2] to show that the corresponding group von Neumann factors $L(\mathbb{F}_n)$, $n \geq 2$, do not have the property Γ. More generally, Effros showed in [8] that if G is a discrete ICC[1] group and $L(G)$ has the property Γ, then G is inner amenable (so, in particular, free groups are not inner amenable). That the converse of Effros' theorem is false is a recent result of Vaes [24].

If M is a finite von Neumann algebra with trace τ, $\mathrm{Aut}(M,\tau)$, the set of τ-preserving automorphisms of M, is a Polish group. A basis for that topology is given by the sets $\mathcal{V}_{T,a_1,\dots,a_n,\varepsilon} = \{S \in \mathrm{Aut}(M,\tau) : ||S(a_i) - T(a_i)||_2 \leq \varepsilon, \forall 1 \leq i \leq n,\ a_i \in M\}$. $\mathrm{Inn}(M)$ denotes the set of inner automorphisms of M, i.e. those of the form $Ad(u)$, $u \in \mathcal{U}(M)$.

Definition 2.2 (Connes [2])**.** A finite von Neumann algebra M is full if $\mathrm{Inn}(M)$ is closed in $\mathrm{Aut}(M)$.

In [2, Corollary 3.8], Connes showed that a II_1 factor M is full if and only if M does not have the property Γ. It follows that for each $n \geq 2$, $L(\mathbb{F}_n)$ is a full factor. In order to discern when group measure space von Neumann algebras are full we need the following:

Definition 2.3 (Schmidt [22])**.** Let G be a discrete group and σ an ergodic measure preserving action of G on a probability space (X,μ). A sequence $(B_n)_{n\in\mathbb{N}}$ of measurable subsets of X is *asymptotically invariant* if

$$\mu(B_n \triangle \sigma_g(B_n)) \to 0, \quad \text{for all } g \in G.$$

The sequence is *trivial* if

$$\mu(B_n)(1-\mu(B_n)) \to 0.$$

The action σ is *strongly ergodic* if every asymptotically invariant sequence is trivial.

The relation between strong ergodicity and fullness has been studied by several authors. For the purpose of this note, it is enough to mention the following theorem of Choda [1]:

Theorem 2.4. *Let G be a discrete group that is not inner amenable, and σ a strongly ergodic measure preserving action of G on a probability space (X,μ). Then $L^\infty(X,\mu) \rtimes_\sigma G$ is a full factor.*

Remark 2.5. The condition that is really used in the proof of Theorem 2.4 is that $L(G)$ is full.

It is known that a group is amenable if and only if it does not admit strongly ergodic actions [22], while a group has the property (T) of Kazdhan if and only if every m.p. ergodic action of it is strongly ergodic [6]. We describe now a concrete example of a strongly ergodic action of $\mathbb{F}_2$ that we will use in this work. Since

[1] ICC stands for infinite conjugacy classes. G is ICC if and only if $L(G)$ is a factor.

$\mathbb{F}_2$ can be identified with the finite index subgroup of $SL(2,\mathbb{Z})$ generated by the matrices $\left\{\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\\ 0 & 1\end{bmatrix}\right\}$ (see [7, II.B.25]), it follows that $\mathbb{F}_2$ naturally acts on $\mathbb{T}^2$. Let us denote such action by σ and by T_a, T_b the automorphisms corresponding to the generators a, b of $\mathbb{F}_2$. This action is clearly measure preserving, and one of the main results in [22] is that σ is strongly ergodic. Inspired by earlier work of Gaboriau and Popa and Törnquist ([12], [23]), in [20] we used this action as the starting point for showing that II_1 factors are not classifiable by countable structures. More precisely, the set

$$\mathrm{Ext}(\sigma) = \{S \in \mathrm{Aut}(\mathbb{T}^2,\mu) : T_a, T_b, S \text{ generates a free action of } \mathbb{F}_3\}$$

was shown in [23, §3] to be a dense G_δ subset of $\mathrm{Aut}(\mathbb{T}^2,\mu)$. Thus $\mathrm{Ext}(\sigma)$ is a standard Borel space. For each $S \in \mathrm{Ext}(\sigma)$ denote by σ_S the corresponding $\mathbb{F}_3$ action and $M_S \in \mathrm{vN}(L^2(\mathbb{F}_3, L^2(\mathbb{T}^2,\mu))$ the corresponding group-measure space von Neumann algebra

$$M_S = L^\infty(\mathbb{T}^2) \rtimes_{\sigma_S} \mathbb{F}_3.$$

In [20] the author and Törnquist showed:

Theorem 2.6. *The equivalence relation on* $\mathrm{Ext}(\sigma)$ *given by* $S \simeq^{\mathscr{F}_{\mathrm{II}_1}} S'$ *if* M_S *is isomorphic to* $M_{S'}$ *is not classifiable by countable structures.*

Also in [20] it was shown that $S \to M_S$ is a Borel map from $\mathrm{Ext}(\sigma)$ to $\mathrm{vN}(L^2(\mathbb{F}_3, L^2(\mathbb{T}^2,\mu)))$. Thus Theorem 1.1 is an immediate consequence of the previous theorem and the next lemma.

Lemma 2.7. *For each* $S \in \mathrm{Ext}(\sigma)$*,* M_S *is a full factor.*

Proof. Let $(B_n)_{n\in\mathbb{N}}$ be an asymptotically invariant sequence for the $\mathbb{F}_3$-action σ_S. Then $(B_n)_{n\in\mathbb{N}}$ is an asymptotically invariant sequence for the action restricted to the subgroup generated by $\{T_a, T_b\}$. By construction, this is the $\mathbb{F}_2$-action σ described above, thus it is strongly ergodic by [22, §4]. It follows that $(B_n)_{n\in\mathbb{N}}$ is trivial and then σ_S is strongly ergodic. Since $\mathbb{F}_3$ is not inner amenable, the result now follows from Theorem 2.4. □

In order to prove Theorem 1.2 we require the following theorem of Popa ([18, Theorem 5.1]):

Theorem 2.8. *If* M_1 *and* M_2 *are full type* II_1 *factors such that* $M_1 \otimes R$ *is isomorphic to* $M_2 \otimes R$ *then there exists* $t \in \mathbb{R}_{>0}$ *such that* M_1 *is isomorphic to* M_2^t*.*

Remark 2.9. By interchanging the roles of M_1 and M_2 one can assume that $t \in (0,1]$, in which case M_2^t is by definition the type II_1 factor pM_2p, where $p \in \mathcal{P}(M_2)$ is any projection of trace equal to t in M_2.

Theorem 2.10. *The assignment* $M_S \to M_S \otimes R$ *is a Borel reduction of* $\simeq^{\mathscr{F}_{\mathrm{II}_1}}$ *to isomorphism of McDuff factors.*

Proof. It is fairly straightforward to prove that the map $M_S \to M_S \otimes R$ is a Borel assignment (see for instance [13, Corollary 3.8]). We are left to show that if $M_S \otimes R$ is isomorphic to $M_{S'} \otimes R$, then M_S is isomorphic to $M_{S'}$.

Let us fix $S, S' \in \mathrm{Ext}(\sigma)$. Lemma 2.7 shows that M_S and $M_{S'}$ are full factors. By Theorem 2.8, $M_S \otimes R$ is isomorphic to $M_{S'} \otimes R$ if and only if there exists $t > 0$ such that $M_{S'}$ is isomorphic to $(M_S)^t$. The proof is over once we show that $t = 1$. For this we make use of the celebrated theorem of Popa on II_1 factors with trivial fundamental group [16], [17] (see also Connes's account in the Bourbaki Séminaire [5]). Indeed by [17, Proposition], there exists a projection $p \in \mathcal{P}(L^\infty(\mathbb{T}^2))$, $\tau(p) = t$, such that the inclusion of von Neumann algebras $(L^\infty(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2) \rtimes_{\sigma_{S'}} \mathbb{F}_3)$ is isomorphic to the inclusion of von Neumann algebras $pL^\infty(\mathbb{T}^2) \subset p\left(L^\infty(\mathbb{T}^2) \rtimes_{\sigma_S} \mathbb{F}_3\right) p$. Feldman–Moore's theorem [10] applies to conclude that the action σ_S is stable orbit equivalent to the action $\sigma_{S'}$, with compression constant $c = t$. Since $\mathbb{F}_3$ has non trivial Atiyah's ℓ^2-Betti numbers, Gaboriau's theorem on ℓ^2-Betti numbers for orbit equivalence relations [11, Theorem 3.12] then implies that $t = 1$. □

Proof of Theorem 1.2. Since $M_S \to M_S \otimes R$ is a Borel reduction of $\simeq^{\mathscr{F}_{\mathrm{II}_1}}$ to isomorphism of McDuff factors and the equivalence relation $\simeq^{\mathscr{F}_{\mathrm{II}_1}}$ is not classifiable by countable structures, it follows that the equivalence relation of isomorphism of McDuff factors is not classifiable by countable structures. □

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