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CONJUGACY CLASSES OF EXTENDED GENERALIZED HECKE GROUPS

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Abstract. Generalized Hecke groups $H_{p,q}$ are generated by $X(z) = -(z - \lambda_p)^{-1}$ and $Y(z) = -(z + \lambda_q)^{-1}$, where $\lambda_p = 2\cos\frac{\pi}{p}$, $\lambda_q = 2\cos\frac{\pi}{q}$, p, q are integers such that $2 \leq p \leq q$, $p + q > 4$. Extended generalized Hecke groups $\overline{H}_{p,q}$ are obtained by adding the reflection $R(z) = 1/\overline{z}$ to the generators of generalized Hecke groups $H_{p,q}$. We determine the conjugacy classes of the torsion elements in extended generalized Hecke groups $\overline{H}_{p,q}$.

1. Introduction

Hecke introduced in [6] the Hecke groups $H(\lambda)$ generated by two linear fractional transformations

$$T(z) = -\frac{1}{z} \quad \text{and} \quad U(z) = z + \lambda,$$

where λ is a fixed positive real number. Let $S = TU$, i.e.,

$$S(z) = -\frac{1}{z + \lambda}.$$

Hecke showed that $H(\lambda)$ is discrete if and only if $\lambda = \lambda_q = 2\cos(\frac{\pi}{q})$, $q \geq 3$ integer, or $\lambda \geq 2$. We consider the former case $q \geq 3$ integer and we denote it by $H_q = H(\lambda_q)$. The Hecke group H_q is isomorphic to the free product of two finite cyclic groups of orders 2 and q,

$$H_q = \langle T, S : T^2 = S^q = I\rangle \simeq \mathbb{Z}_2 * \mathbb{Z}_q.$$

The first few Hecke groups H_q are $H_3 = \Gamma = PSL(2, \mathbb{Z})$ (the modular group), $H_4 = H(\sqrt{2})$, $H_5 = H(\frac{1+\sqrt{5}}{2})$, and $H_6 = H(\sqrt{3})$. It is clear from the above that $H_q \subsetneq PSL(2, \mathbb{Z}[\lambda_q])$ for $q > 3$. These groups and their subgroups have been studied extensively for many aspects in the literature, see [3, 4, 5, 9, 16].

The extended Hecke groups have been defined in [13, 14] by adding the reflection $R(z) = 1/\overline{z}$ to the generators of Hecke groups H_q. They studied even subgroups, commutator subgroups, and principal subgroups of the extended Hecke groups $\overline{H}_q$.

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In [11], Lehner studied a more general class $H_{p,q}$ of Hecke groups H_q, by taking

$$X = \frac{-1}{z-\lambda_p} \quad \text{and} \quad V = z + \lambda_p + \lambda_q,$$

where $2 \leq p \leq q$, $p+q > 4$. Here if we take $Y = XV = -\frac{1}{z+\lambda_q}$, then we have the presentation

$$H_{p,q} = \langle X, Y : X^p = Y^q = I \rangle \simeq \mathbb{Z}_p * \mathbb{Z}_q.$$

Also, $H_{p,q}$ has the signature $(0; p, q, \infty)$. We call these groups *generalized Hecke groups* $H_{p,q}$. We know from [11] that $H_{2,q} = H_q$, $[H_q : H_{q,q}] = 2$, and there is no group $H_{2,2}$. Also, all Hecke groups H_q are included in generalized Hecke groups $H_{p,q}$. Generalized Hecke groups $H_{p,q}$ have been also studied by Calta and Schmidt in [1, 2].

Now we define extended generalized Hecke groups $\overline{H}_{p,q}$, similar to extended Hecke groups $\overline{H}_q$, by adding the reflection $R(z) = 1/\overline{z}$ to the generators of generalized Hecke groups $H_{p,q}$. Then, extended generalized Hecke groups $\overline{H}_{p,q}$ have a presentation

$$\overline{H}_{p,q} = \langle X, Y, R : X^p = Y^q = R^2 = I,\ RX = X^{-1}R, RY = Y^{-1}R \rangle.$$

It is clear that $\left[\overline{H}_{p,q} : H_{p,q}\right] = 2$.

In this paper, we determine the conjugacy classes of the torsion elements in extended generalized Hecke groups $\overline{H}_{p,q}$. The conjugacy classes of extended modular groups have been studied by Jones and Pinto in [10]. The non-elliptic conjugacy classes of Hecke groups H_q have been studied by Hoang and Ressler in [7]. Also, the conjugacy classes of the torsion elements in Hecke H_q and extended Hecke groups $\overline{H}_q$ have been found by Yılmaz Ozgur and Sahin in [17]. Here, we generalize the results given in [17] to extended generalized Hecke groups $\overline{H}_{p,q}$ by similar methods.

2. Conjugacy classes in $\overline{H}_{p,q}$

Firstly, we give the group structures of extended generalized Hecke groups $\overline{H}_{p,q}$.

Theorem 1. *Extended generalized Hecke groups $\overline{H}_{p,q}$ are given directly as a free product of two groups G_1 and G_2 with amalgamated subgroup $\mathbb{Z}_2$, where G_1 is the dihedral group D_p and G_2 is the dihedral group D_q, that is $\overline{H}_{p,q} \simeq D_p *_{\mathbb{Z}_2} D_q$.*

Proof. In the presentation of extended generalized Hecke groups $\overline{H}_{p,q}$, if we take $G_1 = \langle X, R : X^p = R^2 = (XR)^2 = I \rangle \simeq D_p$ and $G_2 = \langle Y, R : Y^q = R^2 = (YR)^2 = I \rangle \simeq D_q$, then $\overline{H}_{p,q}$ is $G_1 * G_2$ with the identification $R = R$. In the first group G_1, the subgroup generated by R is $\mathbb{Z}_2$ and also this is true for the second group G_2. Therefore the identification induces an isomorphism and $\overline{H}_{p,q}$ is a generalized free product with the subgroup $M \simeq \mathbb{Z}_2$ amalgamated, i.e.,

$$\overline{H}_{p,q} = \langle X, Y, R : X^p = Y^q = R^2 = (XR)^2 = (YR)^2 = I \rangle \cong D_p *_{\mathbb{Z}_2} D_q. \qquad \square$$

Now, we obtain the conjugacy classes of torsion elements in the group $\overline{H}_{p,q}$. We need the following two lemmas.

Lemma 1. *Let p and q be integers satisfying $2 \leq p \leq q$, $p+q>4$. Then in $\overline{H}_{p,q}$ we have*

$$X^t R = RX^{p-t},$$
$$Y^m R = RY^{q-m},$$

$1 \leq t \leq p-1$, $1 \leq m \leq q-1$.

Lemma 2. *Let p and q be integers satisfying $2 \leq p \leq q$, $p+q>4$. Then in $\overline{H}_{p,q}$ we have:*

1) $X^t R$, $1 \leq t \leq p-1$, is conjugate to R by $X^w R$, where $w = \frac{pk+t}{2}$ for some $k \in \mathbb{Z}$ satisfying the condition $w \in \mathbb{Z}$ unless p is even and t is odd. If so, $X^t R$, $1 \leq t \leq p-1$, is conjugate to XR by $X^w R$, where $w = \frac{pk+t+1}{2}$ for some $k \in \mathbb{Z}$ satisfying the condition $w \in \mathbb{Z}$.

2) X^u, $1 \leq u \leq \frac{p-1}{2}$, is conjugate to X^{p-u}.

3) $Y^m R$, $1 \leq m \leq q-1$, is conjugate to R by $Y^v R$, where $v = \frac{qk+m}{2}$ for some $k \in \mathbb{Z}$ satisfying the condition $v \in \mathbb{Z}$ unless q is even and m is odd. If so, $Y^m R$, $1 \leq m \leq q-1$, is conjugate to YR by $Y^v R$, where $v = \frac{qk+m+1}{2}$ for some $k \in \mathbb{Z}$ satisfying the condition $v \in \mathbb{Z}$.

4) $Y^n, 1 \leq n \leq \frac{q-1}{2}$, is conjugate to Y^{q-n}.

Proof. 1) Let p be even and t odd. Then there is some $k \in \mathbb{Z}$ such that $w = \frac{pk+t+1}{2} \in \mathbb{Z}$. Thus $X^t R$ is conjugate to $X^w R.X^t R.\left(X^w R\right)^{-1} = XR$. The other case can be obtained similarly.

2) From the presentation of $\overline{H}_{p,q}$ we have X^u is conjugate to $R.X^u.R^{-1} = X^{p-u}$. The proofs of 3 and 4 are similar. □

Now we can give the following theorem for $\overline{H}_{p,q}$.

Theorem 2. *If p and q are prime numbers satisfying $2 \leq p \leq q$, $p+q>4$, then the conjugacy classes of torsion elements in group $\overline{H}_{p,q}$ are given in the following table:*

Condition	Type	Order	Classes of elliptic elements
p, q primes	Elliptic	p	$X^1, X^2, X^3, \ldots, X^{\frac{p-1}{2}}$
	Elliptic	q	$Y^1, Y^2, Y^3, \ldots, Y^{\frac{q-1}{2}}$
	Reflection	2	$R, X^{(p,2)-1}R$

Proof. We have $\overline{H}_{p,q} \simeq D_p *_{\mathbb{Z}_2} D_q$. From a theorem of Kurosh [12], we know that any element of finite order in an amalgamated free product $A *_H B$ is conjugate to an element in one of the factors. So every finite order element $g \in \overline{H}_{p,q}$ is conjugate to an element in G_1 or G_2. We know that

$$G_1 = \langle X, R : X^p = R^2 = (XR)^2 = I \rangle,$$
$$G_2 = \langle Y, R : Y^q = R^2 = (YR)^2 = I \rangle.$$

In G_1 the possible conjugacy classes are R, X^1, X^2,..., $X^{\frac{p-1}{2}}$, X^1R, X^2R,..., $X^{\frac{p-1}{2}}R$, and in G_2 the conjugacy classes are Y^1, Y^2,..., $Y^{\frac{q-1}{2}}$, Y^1R, Y^2R,..., $Y^{\frac{q-1}{2}}R$.

From Lemma 2, if $p \neq 2$, then $X^tR \sim R$ and $Y^mR \sim R$, and so G_1 has $\frac{p-1}{2}+1$ conjugacy classes with representatives $R, X^1, X^2, \ldots, X^{\frac{p-1}{2}}$, and G_2 has $\frac{q-1}{2}$ conjugacy classes with representatives $Y, Y^2, Y^3, \ldots, Y^{\frac{q-1}{2}}$. Of course, if $p = 2$ we have one extra conjugacy class with representative XR. □

Example 1. *In $\overline{H}_{3,5}$ we have four conjugacy classes of finite order elements with representatives R, X, Y, Y^2.*

Now let us examine the conjugacy classes of finite order elements in the group $\overline{H}_{p,q}$, where p and q are integers satisfying $2 \leq p \leq q$, $p+q > 4$.

Case (i): p and q are odd.

From Lemma 1 and Lemma 2, the conjugacy classes of elliptic elements of order p are $X^{r_1}, X^{r_2}, \ldots, X^{r_{\frac{\phi(p)}{2}}}$; $1 \leq i \leq \frac{\phi(p)}{2}$, $(r_i, p) = 1$. Similarly, we have the q ordered conjugacy classes as $Y^{s_1}, Y^{s_2}, \ldots, Y^{s_{\frac{\phi(q)}{2}}}$; $1 \leq j \leq \frac{\phi(q)}{2}$, $(s_j, q) = 1$

One conjugacy class of reflection of order 2 is again R. In this case, we have conjugacy classes of different orders. For every divisor a_i of p, we have conjugacy classes of order a_i with representatives $X^{k\frac{p}{a_i}}$, $k \in \mathbb{Z}$, $k\frac{p}{a_i} < p$. From Lemma 2, the number of these classes reduce by half, and so we have $\frac{p-1-\phi(p)}{2}$ classes. Also, for every divisor b_i of q there is a conjugacy class of order b_i with representative $Y^{k\frac{q}{b_i}}$, $k \in \mathbb{Z}$, $k\frac{q}{b_i} < q$. The number of these classes is $\frac{q-1-\phi(q)}{2}$. Consequently, in total we have $\frac{p+q}{2}$ conjugacy classes of torsion element in the group $\overline{H}_{p,q}$.

Case (ii): p and q are even.

The number of conjugacy classes of elliptic elements of order p and q is the same as in case (i). Then we have three conjugacy classes of reflection elements R, XR and YR. Differently from case (i), we have now two conjugacy classes of elliptic elements of order two with representatives $X^{\frac{p}{2}}$, $Y^{\frac{q}{2}}$. Also for every divisor a_i of p, $a_i \neq 2$, we have conjugacy classes of order a_i with representatives $X^{k\frac{p}{a_i}}$, $k \in \mathbb{Z}$, $k\frac{p}{a_i} < p$. The number of these classes reduce by half, so we have $\frac{p-2-\phi(p)}{2}$ classes. Also for every divisor b_i of q, $b_i \neq 2$, there are conjugacy classes of order b_i with representative $Y^{k\frac{q}{b_i}}$, $k \in \mathbb{Z}$, $k\frac{q}{b_i} < q$. The number of these classes is $\frac{q-2-\phi(q)}{2}$. In this case, we have $\frac{p+q+6}{2}$ conjugacy classes.

Case (iii): p is even and q is odd.

In this case, we have only one conjugacy class of elliptic elements of order two with representative $X^{\frac{p}{2}}$. Also, differently from case (ii), we have now two conjugacy classes of reflection elements with representatives R, XR. So we have in total $\frac{p+q+3}{2}$ conjugacy classes of torsion elements in the group $\overline{H}_{p,q}$.

Remark 1. *In Theorem 1, if we take $p = 2$ we have $\overline{H}_{2,q} = \overline{H}_q$. Using the same method as in the proof of Theorem 1, the possible conjugacy classes of finite order* *elements are* R, X, XR, Y, Y^2, $Y^3, \ldots, Y^{q-1}$, YR, Y^2R, $Y^3R, \ldots, Y^{q-1}R$. *From Lemma 2, we get* $Y^mR \sim R$ *and* $Y^m \sim Y^{q-m}$. *Hence we have* $\frac{q+5}{2}$ *conjugacy classes with representatives* Y^1, $Y^2, \ldots, Y^{\frac{q-1}{2}}$, R, X, XR. *This result coincides with* [17, Theorem 2.3].

Case (iv): p is odd and q is even.

We obtain results similar to those in case (iii). In this case the conjugacy classes of elliptic elements of order two is represented by $Y^{\frac{q}{2}}$. We have $\frac{p+q+3}{2}$ conjugacy classes of torsion elements in the group $\overline{H}_{p,q}$.

As a result of these four cases, we have the following theorem.

Theorem 3. *If* p *and* q *are integers satisfying* $2 \leq p \leq q$, $p+q > 4$, *then the conjugacy classes of torsion elements in the group* $\overline{H}_{p,q}$ *are given in Table 1.*

Corollary 1. *Let* p *and* q *be integers satisfying* $2 \leq p \leq q$, $p+q > 4$. *There are* $[|p/2|]+[|q/2|]+(2,p)+(2,q)-1$ *conjugacy classes of torsion elements in the group* $\overline{H}_{p,q}$.

In Table 2 we give some examples using these results.

2.1. **An application of conjugacy classes of $\overline{H}_{p,q}$.** In this section, we give an application for normal subgroups of extended generalized Hecke groups $\overline{H}_{p,q}$ which have torsion. If $p = 2$ we have extended Hecke groups $\overline{H}_{2,q} = \overline{H}_q$. In [17] Yılmaz Özgür and Sahin have given the following theorem.

Theorem 4. [17] *If* G *is a normal subgroup of* $\overline{H}_q$, q *prime, and* G *has torsion, then the index* $\left[\overline{H}_q : G\right]$ *is finite.*

So we focus on the condition $2 < p \leq q$.

Theorem 5. *Let* p *and* q *be prime numbers satisfying* $2 < p \leq q$, $p+q > 4$. *If* G *is a normal subgroup of* $\overline{H}_{p,q}$ *such that* G *has torsion, then the index* $\left[\overline{H}_{p,q} : G\right]$ *is finite.*

Proof. Since G has torsion there is at least an element of finite order g in G. Let $N(g)$ denote the normal closure of g in $\overline{H}_{p,q}$. Because of $G \lhd \overline{H}_{p,q}$, we have $N(g) \subseteq G$ implies that $\left[\overline{H}_{p,q} : G\right] \mid \left[\overline{H}_{p,q} : N(g)\right]$.

If g^* is any conjugate of g we know that $\left[\overline{H}_{p,q} : N(g)\right] = \left[\overline{H}_{p,q} : N(g^*)\right]$. We complete the proof by showing that $\left[\overline{H}_{p,q} : N(g^*)\right]$ is finite. Now g^* is any of the conjugacy class representatives of finite order elements listed in Theorem 2. So all the possible representatives are $g^* = X^1, X^2, X^3, \ldots, X^{\frac{p-1}{2}}, Y^1, Y^2, Y^3, \ldots, Y^{\frac{q-1}{2}}$, R. The quotient group $\overline{H}_{p,q}/N(g^*)$ is obtained by adding the reation $g^* = I$ to the relations of $\overline{H}_{p,q}$ [12].

Suppose $g^* = R$. Then

$$\begin{aligned}\overline{H}_{p,q}/N(R) &\simeq \langle X, Y, R : X^p = Y^q = R^2 = (XR)^2 = (YR)^2 = R = I\rangle \\ &\simeq \mathbb{Z}_1.\end{aligned}$$

Condition	Type	Order	Cls. of torsion elements	Total
p, q odd	Elliptic	p	$X^{r_1}, X^{r_2}, \dots, X^{r_{\frac{\phi(p)}{2}}}$	$\frac{\phi(p)}{2}$
	Elliptic	a_i	$X^{k\frac{p}{a_i}}$	$\frac{p-1-\phi(p)}{2}$
	Elliptic	q	$Y^{s_1}, Y^{s_2}, \dots, Y^{s_{\frac{\phi(q)}{2}}}$	$\frac{\phi(q)}{2}$
	Elliptic	b_i	$Y^{k\frac{q}{b_i}}$	$\frac{q-1-\phi(q)}{2}$
	Reflection	2	R	1
p, q even	Elliptic	p	$X^{r_1}, X^{r_2}, \dots, X^{r_{\frac{\phi(p)}{2}}}$	$\frac{\phi(p)}{2}$
	Elliptic	a_i	$X^{k\frac{p}{a_i}}$	$\frac{p-2-\phi(p)}{2}$
	Elliptic	q	$Y^{s_1}, Y^{s_2}, \dots, Y^{s_{\frac{\phi(q)}{2}}}$	$\frac{\phi(q)}{2}$
	Elliptic	b_i	$Y^{k\frac{q}{b_i}}$	$\frac{q-2-\phi(q)}{2}$
	Elliptic	2	$X^{\frac{p}{2}}, Y^{\frac{q}{2}}$	2
	Reflection	2	R, XR, YR	3
p even, q odd	Elliptic	p	$X^{r_1}, X^{r_2}, \dots, X^{r_{\frac{\phi(p)}{2}}}$	$\frac{\phi(p)}{2}$
	Elliptic	a_i	$X^{k\frac{p}{a_i}}$	$\frac{p-2-\phi(p)}{2}$
	Elliptic	q	$Y^{s_1}, Y^{s_2}, \dots, Y^{s_{\frac{\phi(q)}{2}}}$	$\frac{\phi(q)}{2}$
	Elliptic	b_i	$Y^{k\frac{q}{b_i}}$	$\frac{q-1-\phi(q)}{2}$
	Elliptic	2	$X^{\frac{p}{2}}$	1
	Reflection	2	R, XR	2
p odd, q even	Elliptic	p	$X^{r_1}, X^{r_2}, \dots, X^{r_{\frac{\phi(p)}{2}}}$	$\frac{\phi(p)}{2}$
	Elliptic	a_i	$X^{k\frac{p}{a_i}}$	$\frac{p-1-\phi(p)}{2}$
	Elliptic	q	$Y^{s_1}, Y^{s_2}, \dots, Y^{s_{\frac{\phi(q)}{2}}}$	$\frac{\phi(q)}{2}$
	Elliptic	b_i	$Y^{k\frac{q}{b_i}}$	$\frac{q-2-\phi(q)}{2}$
	Elliptic	2	$Y^{\frac{q}{2}}$	1
	Reflection	2	R, YR	2

TABLE 1

Therefore $\left[\overline{H}_{p,q} : N(R)\right] = 1$.

Suppose $g^* = X^a$, $1 \le a \le \frac{p-1}{2}$. Then

$$\begin{aligned}\overline{H}_{p,q}/N(X^a) &\simeq \langle X, Y, R : X^p = Y^q = R^2 = (XR)^2 = (YR)^2 = X^a = I\rangle \\ &\simeq \langle Y, R : Y^q = R^2 = (YR)^2 = I\rangle \simeq D_q.\end{aligned}$$

Therefore $\left[\overline{H}_{p,q} : N(X^a)\right] = 2q$.

Suppose $g^* = Y^b$, $1 \le b \le \frac{q-1}{2}$. Then

$$\begin{aligned}\overline{H}_{p,q}/N(Y^b) &\simeq \langle X, Y, R : X^p = Y^q = R^2 = (XR)^2 = (YR)^2 = Y^b = I\rangle \\ &\simeq \langle X, R : X^p = R^2 = (XR)^2 = I\rangle \simeq D_p.\end{aligned}$$

Groups	Type	Order	Cls. of torsion elements	Total
$\overline{H}_{5,9}$	Elliptic	5	X, X^2	$[\|\frac{5}{2}\|] + [\|\frac{9}{2}\|] + (2,5) + (2,9) - 1 = 7$
	Elliptic	9	Y, Y^2, Y^4	
	Elliptic	3	Y^3	
	Reflection	2	R	
$\overline{H}_{4,6}$	Elliptic	4	X	$[\|\frac{4}{2}\|] + [\|\frac{6}{2}\|] + (2,4) + (2,6) - 1 = 8$
	Elliptic	6	Y	
	Elliptic	3	Y^2	
	Elliptic	2	X^2, Y^3	
	Reflection	2	R, XR, YR	
$\overline{H}_{15,8}$	Elliptic	15	X, X^2, X^4, X^7	$[\|\frac{15}{2}\|] + [\|\frac{8}{2}\|] + (2,15) + (2,8) - 1 = 13$
	Elliptic	3	X^5	
	Elliptic	5	X^3, X^6	
	Elliptic	8	Y, Y^3	
	Elliptic	4	Y^2	
	Elliptic	2	Y^4	
	Reflection	2	R, YR	
$\overline{H}_{2,6}$	Elliptic	2	X	$[\|\frac{2}{2}\|] + [\|\frac{6}{2}\|] + (2,2) + (2,6) - 1 = 7$
	Elliptic	6	Y	
	Elliptic	2	Y^3	
	Elliptic	3	Y^2	
	Reflection	2	R, XR, YR	

TABLE 2

Therefore we have $\left[\overline{H}_{p,q} : N(Y^b)\right] = 2p$.

Thus in all cases the index is finite. □

Corollary 2. *Let p and q be primes satisfying $2 \le p \le q$, $p+q > 4$. If $G \lhd \overline{H}_{p,q}$ and G has an elliptic element or reflection then $\left[\overline{H}_{p,q} : G\right]$ divides $2pq$.*

Corollary 3. *Let p and q be primes satisfying $2 \le p \le q$, $p+q > 4$. If $G \lhd H_{p,q}$ and G has an elliptic element of finite order, then the index $[H_{p,q} : G]$ is finite and divides pq.*

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