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NULL HELIX AND k-TYPE NULL SLANT HELICES IN $\mathbb{E}_1^4$

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Abstract. We study the null helix and k-type null slant helices in 4-dimensional Minkowski space $\mathbb{E}_1^4$ and characterize them in terms of the null curvature, null torsion and structure functions. Some null helices and k-type null slant helices are presented by solving special differential equations.

1. Introduction

In recent years, with the development of the theory of relativity, geometers have extended some topics in classical differential geometry of Riemannian manifolds to that of Lorentzian manifolds. For instance, in the Euclidean space $\mathbb{E}^3$, a curve whose tangent vector makes a constant angle with a fixed direction is called a helix, and the curve with non-vanishing curvature is called a slant helix if the principal normal lines make a constant angle with a fixed direction of the ambient space. Some work has been done extending the definitions of helix and slant helix to those in Minkowski space (see [1, 2, 6, 12]). Many problems in Euclidean space can find their counterparts in Minkowski space. However, due to the causal character of the vectors in Minkowski space, some problems become different and new methods are needed in order to discuss them, especially for the problems related to null curves.

In 2011, the authors of [10] introduced a new method to describe cone curves in which the structure functions of the cone curves on $\mathbb{Q}^3$ are presented, where

$$\mathbb{Q}^3 = \{v \in \mathbb{E}_1^4 \mid \langle v, v\rangle = 0\}.$$

Naturally, a null curve as the integration curve of a cone curve also can be characterized by those structure functions. In [2] the authors defined k-type slant helices and discussed them for the partially null and pseudo null curves in $\mathbb{E}_1^4$. Motivated by these ideas, we introduce the definitions of null helix and k-type null slant helices in $\mathbb{E}_1^4$. Then we discuss their properties by using the Cartan Frenet frame in

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[10] and characterize them in terms of the null curvature, null torsion and structure functions.

The curves under consideration in this paper are regular unless otherwise stated.

2. Preliminaries and definitions

In this section we review some basic facts of null curves in $\mathbb{E}_1^4$ and introduce the definitions of null helix and k-type null slant helices.

2.1. **Minkowski 4-space $\mathbb{E}_1^4$.** Minkowski 4-space $\mathbb{E}_1^4$ is the real 4-dimensional vector space $\mathbb{R}^4$ equipped with the standard flat metric given by

$$g = dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$$

in terms of rectangular coordinates (x_1, x_2, x_3, x_4). Recall that a non-zero vector $v \in \mathbb{E}_1^4$ is space-like if $g(v, v) > 0$, time-like if $g(v, v) < 0$, and null (light-like) if $g(v, v) = 0$. In particular, the zero vector is regarded as a space-like vector. A curve $r(s)$ in $\mathbb{E}_1^4$ is called space-like, time-like or null (light-like) if all of its velocity vectors $r'(s)$ are space-like, time-like or null, respectively.

For null curves, it is not possible to normalize the tangent vector in the usual way because the arc-length vanishes. A method of proceeding is to denote a new parameter called the null arc-length parameter which normalizes the derivative of the tangent vector such that $g(r''(s), r''(s)) = 1$ (see [10]).

Proposition 2.1. *Let $r(s) : I \to \mathbb{E}_1^4$ be a null curve parametrized by null arc-length. Then $r(s)$ can be framed by a unique Cartan Frenet frame $\{x, \alpha, y, \beta\}$ such that*

$$\begin{cases} r'(s) = x(s), \\ x'(s) = \alpha(s), \\ \alpha'(s) = \kappa(s)x(s) - y(s), \\ y'(s) = -\kappa(s)\alpha(s) - \tau(s)\beta(s), \\ \beta'(s) = \tau(s)x(s), \end{cases} \tag{2.1}$$

where

$$g(x, x) = g(y, y) = g(x, \alpha) = g(y, \alpha) = g(x, \beta) = g(y, \beta) = g(\alpha, \beta) = 0,$$
$$g(x, y) = g(\alpha, \alpha) = g(\beta, \beta) = \det(x, \alpha, \beta, y) = 1.$$

In sequence, x, α, y, β is called the tangent, principal normal, first binormal, and second binormal vector field of $r(s)$. The functions $\kappa(s)$ and $\tau(s)$ are called the null curvature and null torsion function, respectively.

2.2. **Representation formulas of null curves in $\mathbb{E}_1^4$.** In [10] the authors described cone curves with the structure functions $f(s)$ and $g(s)$ in $\mathbb{Q}^3 \subset \mathbb{E}_1^4$. A null curve in $\mathbb{E}_1^4$ as the integration curve of a cone curve in $\mathbb{Q}^3$ can be characterized in the sense of [10] as follows:

Proposition 2.2. *Let $r(s) : I \to \mathbb{E}_1^4$ be a null curve with null arc-length parameter s. Then $r(s)$ can be written as*

$$r(s) = \int \rho(2f, 2g, 1 - f^2 - g^2, 1 + f^2 + g^2)\, ds, \tag{2.2}$$

where $f(s)$ and $g(s)$ are called the structure functions of $r(s)$. The null curvature $\kappa(s)$, null torsion $\tau(s)$, structure functions $f(s)$, and $g(s)$ satisfy

$$\begin{aligned}\kappa(s) &= \frac{1}{2}[(\log\rho)_s]^2 + (\log\rho)_{ss} - \frac{1}{2}\theta_s^2,\\ \tau^2(s) &= (\theta_s(\log\rho)_s + \theta_{ss})^2,\end{aligned} \tag{2.3}$$

where

$$4\rho^2(s)[f^2(s)+g^2(s)] = 1, \quad \theta_s = \left(1+\frac{f_s^2}{g_s^2}\right)^{-1}\left(\frac{f_s}{g_s}\right)_s.$$

2.3. **Null helix and k-type null slant helices in $\mathbb{E}_1^4$.**

Definition. Let $r = r(s)$ be a null curve parametrized by null arc-length with the Cartan Frenet frame $\{x,\alpha,y,\beta\}$ in $\mathbb{E}_1^4$. If there exists a non-zero constant vector field V in $\mathbb{E}_1^4$ such that $g(x,V)\neq 0$ is a constant for all $s\in I$, then r is said to be a *null helix* and V is called the *axis* of r.

Definition. Let $r = r(s)$ be a null curve parametrized by null arc-length with the Cartan Frenet frame $\{x,\alpha,y,\beta\}$ in $\mathbb{E}_1^4$. If there exists a non-zero constant vector field V in $\mathbb{E}_1^4$ such that $g(\alpha,V)\neq 0$ (respectively, $g(y,V)\neq 0$, $g(\beta,V)\neq 0$) is a constant for all $s\in I$, then r is said to be a *k-type* ($k=1,2,3$, respectively) *null slant helix* and V is called the *slope axis* of r.

At the end of this section, we make some notations and calculations for later use. Let V be an axis (or a slope axis) of a null helix (or a k-type null slant helix) $r(s)$. Then V can be decomposed as

$$V = v_1x(s) + v_2\alpha(s) + v_3\beta(s) + v_4y(s), \tag{2.4}$$

where $\{x,\alpha,y,\beta\}$ is the Cartan Frenet frame of $r(s)$, and $v_i = v_i(s)$ $(i=1,2,3,4)$ are differentiable functions on null arc-length s. Thus

$$v_1 = g(y,V), \quad v_2 = g(\alpha,V), \quad v_3 = g(\beta,V), \quad v_4 = g(x,V).$$

By taking derivative on both sides of (2.4), we get

$$(v_1' + v_2\kappa + v_3\tau)x + (v_1 + v_2' - v_4\kappa)\alpha + (v_3' - v_4\tau)\beta + (v_4' - v_2)y = 0,$$

which implies

$$\begin{cases} v_1' + v_2\kappa + v_3\tau = 0,\\ v_1 + v_2' - v_4\kappa = 0,\\ v_3' - v_4\tau = 0,\\ v_4' - v_2 = 0.\end{cases} \tag{2.5}$$

3. Null helix

Theorem 3.1. *Let $r = r(s)$ be a null curve in $\mathbb{E}_1^4$. Then r is a null helix if and only if*

$$\kappa' = -\tau\int\tau ds.$$

Proof. According to the definition of null helix, we have

$$v_4 = g(x, V) = C_0,$$

where C_0 is a non-zero constant. Substituting it into (2.5), we get

$$\begin{cases} v_1' + v_3\tau = 0, \\ v_1 = C_0\kappa, \\ v_3' = C_0\tau, \\ v_2 = 0. \end{cases} \tag{3.1}$$

By (3.1) we find that

$$C_0\kappa' = -C_0\tau \int \tau ds.$$

Because $C_0 \neq 0$, we have

$$\kappa' = -\tau \int \tau ds. \tag{3.2}$$

Conversely, suppose (3.2) holds for a null curve $r(s)$. By choosing the vector field V as

$$V = c\kappa x + c\left(\int \tau ds\right)\beta + cy, \quad (c \in \mathbb{R} - \{0\}),$$

we have $V' = 0$ and $g(x, V) = c$. So V is a constant vector field and $r(s)$ is a null helix. □

From Theorem 3.1, we have:

Corollary 3.2. *Let $r = r(s)$ be a null helix in $\mathbb{E}_1^4$. Then the axes of r are obtained by*

$$V = c\kappa x + c\left(\int \tau ds\right)\beta + cy, \quad (c \in \mathbb{R} - \{0\}),$$

and the axes V lie fully in the rectifying space of $r(s)$.

Remark 3.3. The rectifying space of $r(s)$ is the orthogonal complement of its principal normal vector field α in $\mathbb{E}_1^4$.

Corollary 3.4. *Let $r = r(s)$ be a null helix with vanishing null torsion in $\mathbb{E}_1^4$. Then r can be written as follows:*

(1) $r = \left(s, \frac{1}{2}s^2, \frac{1}{6}s^3, \frac{1}{6}s^3 + s\right)$ *for* $\kappa = 0$*;*

(2) $r = \left(as, bs, (a^2+b^2)\cosh(\frac{s}{\sqrt{a^2+b^2}}), (a^2+b^2)\sinh(\frac{s}{\sqrt{a^2+b^2}})\right)$ *for* $\kappa > 0$*;*

(3) $r = \left((b^2-a^2)\cos(\frac{s}{\sqrt{a^2-b^2}}), (a^2-b^2)\sin(\frac{s}{\sqrt{a^2-b^2}}), bs, as\right)$ *for* $\kappa < 0$*,*

where $a, b \in \mathbb{R}$, $a^2 + b^2 \neq 0$ for $\kappa > 0$, and $a^2 - b^2 > 0$ for $\kappa < 0$.

Proof. When the null torsion $\tau \equiv 0$, from Theorem 3.1 the null curvature κ is a constant. From (2.3), the structure functions can be written as follows:

Case 1: If $\kappa = 0$, then we have

$$\begin{cases} f(s) = \frac{1}{s^2+1}, \\ g(s) = \frac{s}{s^2+1}, \\ 2\rho(s) = s^2 + 1. \end{cases} \tag{3.3}$$

Case 2: If $\kappa > 0$, then we get

$$\begin{cases} f(s) = \frac{a}{2\rho}, \\ g(s) = \frac{b}{2\rho}, \\ 2\rho(s) = \sqrt{a^2+b^2}\left(\sinh(\frac{s}{\sqrt{a^2+b^2}}) + \cosh(\frac{s}{\sqrt{a^2+b^2}})\right), \end{cases} \tag{3.4}$$

where $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$.

Case 3: If $\kappa < 0$, then we obtain

$$\begin{cases} f(s) = \frac{\sqrt{a^2-b^2}}{a+b}\left(\sin(\frac{s}{\sqrt{a^2-b^2}})\right), \\ g(s) = \frac{\sqrt{a^2-b^2}}{a+b}\left(\cos(\frac{s}{\sqrt{a^2-b^2}})\right), \\ 2\rho(s) = a + b, \end{cases} \tag{3.5}$$

where $a, b \in \mathbb{R}$ and $a^2 - b^2 > 0$.

From (2.2), (3.3), (3.4) and (3.5), the expression forms of $r(s)$ can be easily achieved (see [10]). □

Theorem 3.5. *Let $r = r(s)$ be a null helix with non-zero constant null curvature and vanishing null torsion in $\mathbb{E}_1^4$. Then*

(1) *r is a Bertrand curve;*
(2) *r is a null osculating curve of the first kind;*
(3) *r is a 2-type and 3-type null slant helix.*

Proof. In $\mathbb{E}_1^4$, a null curve is a Bertrand curve if and only if κ is a non-zero constant and $\tau \equiv 0$ (see [4]), so r is a Bertrand curve. In [8], a null curve is an osculating curve of the first kind if and only if τ is equal to zero, therefore we get the second conclusion. At the same time, from the axes of null helices denoted in Corollary 3.2, we have

$$g(y, V) = c\kappa, \quad g(\beta, V) = c\int \tau ds.$$

Thus, $g(y, V)$ and $g(\beta, V)$ are all non-zero constants when $\tau \equiv 0$ and κ is a non-zero constant, where $\int \tau ds \neq 0$ is always assumed. The proof is thus completed. □

4. k-TYPE NULL SLANT HELICES

In this section we discuss three kinds of null slant helices.

4.1. **1-type null slant helix.**

Theorem 4.1. *Let $r = r(s)$ be a null curve in $\mathbb{E}_1^4$. Then r is a 1-type null slant helix if and only if*

$$2\kappa + \kappa' s = -\tau \int s\tau ds.$$

Proof. Based on the definition of 1-type null slant helix, we have

$$v_2 = g(\alpha, V) = C_0,$$

where C_0 is a non-zero constant. Substituting it into (2.5), we get

$$\begin{cases} v_1' + C_0\kappa + v_3\tau = 0, \\ v_1 - v_4\kappa = 0, \\ v_3' - v_4\tau = 0, \\ v_4' = C_0. \end{cases} \tag{4.1}$$

By the last equation in (4.1), we have $v_4 = C_0 s$ (putting the integration constant to be zero by transformation). Then, the coefficients of V are expressed by

$$\begin{cases} v_1 = C_0 s\kappa, \\ v_2 = C_0, \\ v_3 = -C_0(\frac{2\kappa+\kappa' s}{\tau}) = C_0 \int s\tau ds, \\ v_4 = C_0 s. \end{cases} \tag{4.2}$$

Because $C_0 \neq 0$, from the third equation in (4.2) we get

$$2\kappa + \kappa' s = -\tau \int s\tau ds. \tag{4.3}$$

Conversely, assume (4.3) holds for a null curve $r(s)$. We can define a vector field V as follows:

$$V = cs\kappa x + c\alpha + c\left(\int s\tau ds\right)\beta + csy, \quad (c \in \mathbb{R} - \{0\}).$$

Then, we have $V' = 0$ and $g(\alpha, V) = c$. This completes the proof. □

As a consequence of Theorem 4.1, we have

Corollary 4.2. *Let $r = r(s)$ be a 1-type null slant helix in $\mathbb{E}_1^4$. Then the slope axes of r are written as*

$$V = cs\kappa x + c\alpha + c\left(\int s\tau ds\right)\beta + csy, \quad (c \in \mathbb{R} - \{0\}).$$

Corollary 4.3. *Let $r = r(s)$ be a 1-type null slant helix with vanishing null torsion in $\mathbb{E}_1^4$. Then the null curvature is given by*

$$k = \frac{c}{s^2},$$

where c is a constant, and $r(s)$ can be written as follows:

(1) $r(s) = C_1 s^2 + C_2 s^2 \log s + C_3 s^2 \log^2 s + C_4$, *for* $2c = -1$;

(2) $r(s) = C_1 s^2 + C_2 s^{(2+\sqrt{1+2c})} + C_3 s^{(2-\sqrt{1+2c})} + C_4$, *for* $2c > -1$;

(3) $r(s) = C_1 s^2 + C_2 s^2 \sin[(\sqrt{-1-2c})\log s] + C_3 s^2 \cos[(\sqrt{-1-2c})\log s] + C_4$, *for* $2c < -1$,

where C_i $(i = 1, 2, 3, 4) \in \mathbb{E}_1^4$.

Proof. When the null torsion $\tau = 0$, from Theorem 4.1 we have

$$2\kappa + \kappa' s = 0,$$

from which we get

$$k = \frac{c}{s^2}, \quad (c \in \mathbb{R}).$$

By the Frenet formulas (2.1) we have

$$s^3 x''' - 2csx' + 2cx = 0. \tag{4.4}$$

Solving the above Euler equation (4.4), we get

(1) $x(s) = B_1 s + B_2 s \log s + B_3 s \log^2 s$, for $2c = -1$;

(2) $x(s) = B_1 s + B_2 s^{(1+\sqrt{1+2c})} + B_3 s^{(1-\sqrt{1+2c})}$, for $2c > -1$;

(3) $x(s) = B_1 s + B_2 s \sin[(\sqrt{-1-2c})\log s] + B_3 s \cos[(\sqrt{-1-2c})\log s]$, for $2c < -1$,

where B_i $(i = 1, 2, 3) \in \mathbb{E}_1^4$. Integrating $x(s)$ with respect to s, the expression forms of $r(s)$ can be obtained. □

Theorem 4.4. *Let* $r = r(s)$ *be a* 1*-type null slant helix with vanishing null torsion in* $\mathbb{E}_1^4$. *Then*

(1) *r is a null osculating curve of the first kind;*
(2) *r is a* 3*-type null slant helix.*

Proof. As denoted in the proof of Theorem 3.5, r is a null osculating curve of the first kind. At the same time, from the slope axes of 1-type null slant helices, we have

$$g(V, \beta) = c \int s\tau ds.$$

Thus, $g(V, \beta)$ is a non-zero constant when $\tau = 0$ and $\int s\tau ds \neq 0$. This completes the proof. □

4.2. 2-type null slant helix.

Theorem 4.5. *Let* $r = r(s)$ *be a null curve in* $\mathbb{E}_1^4$. *Then* r *is a* 2*-type null slant helix if and only if*

$$\frac{\tau}{\kappa} = \pm \frac{v_4'}{\sqrt{-2C_0 v_4 - v_4'^2 + a}}, \quad (C_0 \neq 0,\ a \in \mathbb{R}),$$

where v_4 *satisfies* (4.6).

Proof. From the definition of 2-type null slant helix, we have

$$v_1 = g(y, V) = C_0,$$

where C_0 is a non-zero constant. Substituting it into (2.5), we get

$$\begin{cases} v_2\kappa + v_3\tau = 0, \\ C_0 + v_2' - v_4\kappa = 0, \\ v_3' - v_4\tau = 0, \\ v_4' = v_2, \end{cases} \tag{4.5}$$

from which we have

$$v_4'' - \kappa v_4 + C_0 = 0. \tag{4.6}$$

First, in order to solve equation (4.6), we consider

$$v_4'' - \kappa v_4 = 0. \tag{4.7}$$

Case 1: If $\kappa < 0$, then we have

$$v_4 = c \exp\left(\int \sqrt{-\kappa}\, \frac{\cos\theta}{\sin\theta} ds\right),$$

where c is a constant and the newly defined function $\theta(s)$ satisfies

$$\theta' = \sqrt{-\kappa} + \frac{\kappa'}{2\kappa} \sin\theta\cos\theta.$$

Case 2: If $\kappa > 0$, then we get

$$v_4 = c \exp\left(\int \sqrt{\kappa}\, \frac{\cosh\theta}{\sinh\theta} ds\right),$$

where c is a constant and $\theta(s)$ satisfies

$$\theta' = \sqrt{\kappa} + \frac{\kappa'}{2\kappa} \sinh\theta\cosh\theta.$$

Let y_1, y_2 be two non-trivial linearly independent solutions of (4.7). Then the general solution of (4.6) can be obtained in the form

$$v_4 = C_1 y_1 + C_2 y_2 - C_0 y_2 \int y_1 \frac{ds}{W} + C_0 y_1 \int y_2 \frac{ds}{W},$$

where $C_1, C_2 \in \mathbb{R}$ and $W = y_1 y_2' - y_2 y_1'$ (see [11, p. 142]).

At the same time, from (4.5) we have

$$v_3 v_3' + v_4 v_4' \kappa = 0.$$

By (4.6), we have

$$(v_3^2)' = -2C_0 v_4' - 2v_4' v_4''.$$

Therefore,

$$v_3 = \pm\sqrt{-2C_0 v_4 - (v_4')^2 + a}, \quad (C_0 \neq 0,\ a \in \mathbb{R}).$$

Thus, we obtain from (4.5)

$$\frac{\tau}{\kappa} = \pm \frac{v_4'}{\sqrt{-2C_0 v_4 - v_4'^2 + a}}. \tag{4.8}$$

Conversely, assume (4.8) holds for a null curve $r(s)$. By choosing the vector field V as follows:

$$V = cx + v_4'\alpha - \frac{\kappa v_4'}{\tau}\beta + v_4 y, \quad (0 \neq c \in \mathbb{R})$$

we have $V' = 0$ and $g(y, V) = c$. ☐

As a consequence of the above theorem, we have

Corollary 4.6. *Let $r = r(s)$ be a 2-type null slant helix. Then the slope axes of r are obtained by*

$$V = cx + v_4'\alpha - \frac{\kappa v_4'}{\tau}\beta + v_4 y, \quad (0 \neq c \in \mathbb{R}),$$

where v_4 satisfies (4.6).

Theorem 4.7. *Let $r = r(s)$ be a 2-type null slant helix. Suppose the ratio of the null curvature and null torsion is a non-zero constant.*

(1) *If κ is a constant, then we have*

$$r(s) = (\sqrt{a^2+b^2})^{-1}\left(-\frac{1}{a}\cos as, \frac{1}{a}\sin as, \frac{1}{b}\cosh bs, \frac{1}{b}\sinh bs\right).$$

(2) *If κ is not a constant, then the null curvature and null torsion can be expressed by*

$$\kappa = \frac{2c^2 C_0}{-C_0 s^2 + 2a(1+c^2)s + 2b(1+c^2)},$$
$$\tau = \frac{2c C_0}{-C_0 s^2 + 2a(1+c^2)s + 2b(1+c^2)},$$

where $a, b, c, C_0 \in \mathbb{R}$ and $c \neq 0$, $C_0 \neq 0$.

Proof. Assume $\kappa = c\tau$ $(0 \neq c \in \mathbb{R})$. Then we have the following two cases:

Case 1: κ is a constant. This means that τ is also a constant. In such case, the structure functions can be obtained by (2.3) as follows:

$$\begin{cases} f(s) = \frac{\sin as}{\sinh bs + \cosh bs}, \\ g(s) = \frac{\cos as}{\sinh bs + \cosh bs}, \\ 2\rho(s) = \frac{\sinh bs + \cosh bs}{\sqrt{a^2+b^2}}. \end{cases} \tag{4.9}$$

By (2.2) and (4.9), $r(s)$ can be written as

$$r(s) = (\sqrt{a^2+b^2})^{-1}\left(-\frac{1}{a}\cos as, \frac{1}{a}\sin as, \frac{1}{b}\cosh bs, \frac{1}{b}\sinh bs\right),$$

where $a, b \in \mathbb{R}$ and $ab \neq 0$.

Case 2: κ is not a constant. In such case, from $\kappa = c\tau$ and (4.5) we get

$$v_3 = -cv_2.$$

Substituting it into (4.5), we have

$$v_4'' = -\frac{C_0}{1+c^2},$$

from which we have

$$v_4 = -\frac{1}{2}\frac{C_0}{1+c^2}s^2 + as + b, \quad (a, b \in \mathbb{R}).$$

Furthermore, by (4.5) we get

$$\kappa = \frac{2c^2C_0}{-C_0s^2 + 2a(1+c^2)s + 2b(1+c^2)},$$
$$\tau = \frac{2cC_0}{-C_0s^2 + 2a(1+c^2)s + 2b(1+c^2)},$$

where $a, b, c, C_0 \in \mathbb{R}$ and $c \neq 0$, $C_0 \neq 0$. □

4.3. **3-type null slant helix.**

Theorem 4.8. *Let $r = r(s)$ be a null curve in $\mathbb{E}_1^4$. Then r is a 3-type null slant helix if and only if the null torsion vanishes and*

$$v_4''' = 2v_4'\kappa + v_4\kappa',$$

where $v_4 = g(x, V)$.

Proof. From the definition of 3-type null slant helix, we have

$$v_3 = g(\beta, V) = C_0,$$

where C_0 is a non-zero constant. Substituting it into (2.5), we get

$$\begin{cases} v_1' + v_2\kappa + C_0\tau = 0, \\ v_1 + v_2' - v_4\kappa = 0, \\ v_4\tau = 0, \\ v_4' = v_2. \end{cases} \tag{4.10}$$

By the third equation in (4.10), we consider the open subset $\mathcal{O} = \{\tau(s) \neq 0\}$ of r. First, we assume $\mathcal{O} \neq \emptyset$. Then, $v_4 = 0$ on $\mathcal{O}$ by (4.10). However, $v_4 = 0$ implies $v_i = 0$ $(i = 1, 2, 3, 4)$. This is a contradiction. So $\mathcal{O} = \emptyset$ and $\tau \equiv 0$ by continuity.

Thus, (4.10) implies

$$\begin{cases} v_1' + v_2\kappa = 0, \\ v_1 + v_2' - v_4\kappa = 0, \\ v_4' = v_2. \end{cases} \tag{4.11}$$

From (4.11) we have

$$v_4''' = 2v_4'\kappa + v_4\kappa'. \tag{4.12}$$

Solving the above equation (4.12) we get

$$v_4 = c_1\omega_1^2 + c_2\omega_1\omega_2 + c_3\omega_2^2, \tag{4.13}$$

where $c_1, c_2, c_3 \in \mathbb{R}$ and ω_1, ω_2 are linearly independent solutions of the following equation (see [11, p. 141 and p. 538]):

$$2\omega'' - \kappa\omega = 0. \tag{4.14}$$

The solutions of (4.14) are as follows:

Case 1: If $\kappa < 0$, then we get

$$\omega = c \exp\left(\frac{1}{\sqrt{2}} \int \sqrt{-\kappa}\, \frac{\cos\theta}{\sin\theta}\, ds\right),$$

where c is a constant and the newly defined function $\theta(s)$ satisfies

$$\theta' = \sqrt{\frac{-\kappa}{2}} + \frac{\kappa'}{2\kappa} \sin\theta \cos\theta.$$

Case 2: If $\kappa > 0$, then we have

$$\omega = c \exp\left(\frac{1}{\sqrt{2}} \int \sqrt{\kappa}\, \frac{\cosh\theta}{\sinh\theta}\, ds\right),$$

where c is a constant and $\theta(s)$ satisfies

$$\theta' = \sqrt{\frac{\kappa}{2}} + \frac{\kappa'}{2\kappa} \sinh\theta \cosh\theta.$$

Conversely, assume (4.12) holds for a null curve $r(s)$ with vanishing null torsion. We can define the vector field V as

$$V = (v_4\kappa - v_4'')x + v_4'\alpha + c\beta + v_4 y, \quad (0 \neq c \in \mathbb{R});$$

then $V' = 0$ and $g(\beta, V) = c$. □

From Theorem 4.8, we have:

Corollary 4.9. *Let $r = r(s)$ be a 3-type null slant helix. Then the slope axes of r are obtained by*

$$V = (v_4\kappa - v_4'')x + v_4'\alpha + c\beta + v_4 y, \quad (c \in \mathbb{R} - \{0\}),$$

where V_4 satisfies (4.12).

For 3-type null slant helices, because $\tau = 0$, β is a constant vector, $c\beta$ is a constant function. Then, we have:

Corollary 4.10. *The slope axes of 3-type null slant helices lie fully in the first kind osculating space of $r(s)$.*

Remark 4.11. The first kind osculating space of $r(s)$ is the orthogonal complement of its second binormal vector field β in $\mathbb{E}_1^4$.

In [8], a null curve is a null osculating curve of the first kind if and only if the null torsion is equal to zero. Thus, we have:

Theorem 4.12. *Let $r = r(s)$ be a 3-type null slant helix. Then r is congruent to a null osculating curve of the first kind.*

Next, we use an example to show the process to obtain the slope axes of some kind of 3-type null slant helices.

Example. Let $r = r(s)$ be a 3-type null slant helix with vanishing null torsion and the null curvature

$$\kappa = 2as^{-4}, \quad (a \in \mathbb{R} - \{0\}).$$

By solving the equation

$$\omega'' - as^{-4}\omega = 0,$$

we get

(1) $\omega = C_1 s \sin(\frac{\sqrt{-a}}{s}) + C_2 s \cos(\frac{\sqrt{-a}}{s})$, for $a < 0$;

(2) $\omega = C_1 s \sinh(\frac{\sqrt{a}}{s}) + C_2 s \cosh(\frac{\sqrt{a}}{s})$, for $a > 0$,

where $C_1, C_2 \in \mathbb{R}$ (see [11, p. 142 and p. 171]).

Taking the first case ($a < 0$) as an example, we choose

$$\omega_1 = s \sin(\frac{\sqrt{-a}}{s}), \quad \omega_2 = s \cos(\frac{\sqrt{-a}}{s})$$

as two linearly independent solutions. Then, from (4.13), we get

$$v_4 = C_1 s^2 \sin^2(\frac{\sqrt{-a}}{s}) + C_2 s^2 \sin(\frac{\sqrt{-a}}{s}) \cos(\frac{\sqrt{-a}}{s}) + C_3 s^2 \cos^2(\frac{\sqrt{-a}}{s}),$$

where $C_1, C_2, C_3 \in \mathbb{R}$.

From Corollary 4.9, the slope axes of such kind of 3-type null slant helices can be obtained.

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