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SINGULAR INTEGRAL OPERATORS, JOHN–NIRENBERG INEQUALITIES AND TRIEBEL–LIZORKIN TYPE SPACES ON WEIGHTED LEBESGUE SPACES WITH VARIABLE EXPONENTS

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Abstract. We establish the boundedness of singular integral operators, the Fefferman–Stein inequality and the John–Nirenberg inequalities on weighted Lebesgue spaces with variable exponents by using the extrapolation theorem. Moreover, as a consequence of the extrapolation theorem, we have the Fefferman–Stein vector-valued inequalities on weighted Lebesgue spaces, and hence we use it to study the weighted Triebel–Lizorkin spaces with variable exponent.

1. Introduction

In this paper we aim to extend several important results in harmonic analysis, such as the boundedness of singular integral operators, the Fefferman–Stein inequality and the John–Nirenberg inequalities to weighted Lebesgue spaces with variable exponents.

As shown in [3], the method of extrapolation is a powerful tool to extend results from weighted inequality to Lebesgue spaces with variable exponents. Therefore, we use the method of extrapolation to establish the above mentioned results. Roughly speaking, we extend the extrapolation theorem to those weighted Lebesgue spaces with variable exponents $L^{p(\cdot)}_\omega$ for which the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}_\omega$. The reader is referred to Definition 2.4 of the weighted Lebesgue spaces with variable exponents $L^{p(\cdot)}_\omega$.

We apply the extrapolation theorem for the weighted Lebesgue spaces with variable exponents to study the boundedness of singular integral operators, the Fefferman–Stein inequality and the John–Nirenberg inequalities on $L^{p(\cdot)}_\omega$.

We use this John–Nirenberg inequality to establish the characterizations of the function space of bounded mean oscillation BMO via the weighted Lebesgue spaces with variable exponents. Furthermore, the extrapolation theorem also yields the

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Fefferman–Stein vector-valued maximal inequality, which is used to study weighted Triebel–Lizorkin spaces with variable exponents.

This paper is organized as follows. The precise definitions of the classes of weight functions and the corresponding Lebesgue spaces with variable exponents are given in Section 2. The extrapolation theorem for weighted Lebesgue spaces with variable exponents is presented and proved in Section 3.

The boundedness of singular integral operators, the Fefferman–Stein inequality and the John–Nirenberg inequalities for weighted Lebesgue spaces with variable exponents are obtained in Section 4. In addition, we also establish the characterization of BMO in terms of weighted Lebesgue spaces with variable exponents in Section 4. Finally, we study the weighted Triebel–Lizorkin spaces with variable exponents in Section 5.

2. Weighted Lebesgue spaces with variable exponents

Let $B(z,r)=\{x\in\mathbb{R}^n : |x-z|<r\}$ denote the open ball with center $z\in\mathbb{R}^n$ and radius $r>0$. Let $\mathbb{B}=\{B(z,r): z\in\mathbb{R}^n,\ r>0\}$.

We begin with the definition of the well known Muckenhoupt class of weight functions.

Definition 2.1. For $1<p<\infty$, a locally integrable function $\omega:\mathbb{R}^n\to[0,\infty)$ is said to be an A_p weight if

$$[\omega]_{A_p}=\sup_{B\in\mathbb{B}}\left(\frac{1}{|B|}\int_B \omega(x)dx\right)\left(\frac{1}{|B|}\int_B \omega(x)^{-\frac{p'}{p}}dx\right)^{\frac{p}{p'}}<\infty,$$

where $p'=\frac{p}{p-1}$. A locally integrable function $\omega:\mathbb{R}^n\to[0,\infty)$ is said to be an A_1 weight if

$$\frac{1}{|B|}\int_B \omega(y)\,dy\le C\omega(x),\quad \text{a.e. } x\in B$$

for some constant $C>0$. The infimum of all such C is denoted by $[\omega]_{A_1}$. We define $A_\infty=\cup_{p\ge 1}A_p$.

For any Lebesgue measurable set E and $\omega\in A_\infty$, write $\omega(E)=\int_E\omega(x)dx$.

We recall the definition of Lebesgue spaces with variable exponents and some of theirs properties. For any Lebesgue measurable function $p(\cdot):\mathbb{R}^n\to(0,\infty)$, define

$$p_-=\operatorname{ess\,inf}\{p(x):x\in\mathbb{R}^n\}\quad\text{and}\quad p_+=\operatorname{ess\,sup}\{p(x):x\in\mathbb{R}^n\}.$$

Definition 2.2. Let $p(\cdot):\mathbb{R}^n\to(0,\infty)$ be a Lebesgue measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all Lebesgue measurable functions $f:\mathbb{R}^n\to\mathbb{C}$ satisfying

$$\|f\|_{L^{p(\cdot)}}=\inf\{\lambda>0:\rho(|f(x)|/\lambda)\le 1\}<\infty,$$

where

$$\rho(f)=\int_{\mathbb{R}^n}|f(x)|^{p(x)}dx.$$

We call $p(x)$ the exponent function of $L^{p(\cdot)}$.

For any $p : \mathbb{R}^n \to [1, \infty)$, the conjugate function p' is defined by $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

We use the weight functions introduced in [4, Definition 1.4]. This is the class of weight functions for which the Hardy–Littlewood maximal operator is bounded on the corresponding Lebesgue space with variable exponent (see [4, Theorem 1.5]).

Definition 2.3. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a Lebesgue measurable function. A locally integrable function $\omega : \mathbb{R}^n \to (0, \infty)$ belongs to $A_{p(\cdot)}$ if there exists a constant $C > 0$ such that

$$\|\omega\chi_B\|_{L^{p(\cdot)}}\|\omega^{-1}\chi_B\|_{L^{p'(\cdot)}} \leq C|B|, \quad \forall B \in \mathbb{B}.$$

According to the above definition, if $p(\cdot) = p$, $1 < p < \infty$, is a constant function and $\omega \in A_{p(\cdot)}$, then $\omega^p \in A_p$.

Furthermore, whenever $\omega \in A_{p(\cdot)}$, $\omega^{-1} \in A_{p'(\cdot)}$.

Definition 2.4. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a Lebesgue measurable function and $\omega \in A_{p(\cdot)}$. The Lebesgue space with variable exponent $L^{p(\cdot)}_\omega$ consists of all Lebesgue measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\|f\|_{L^{p(\cdot)}_\omega} = \|f\omega\|_{L^{p(\cdot)}} < \infty.$$

The function space $L^{p(\cdot)}_\omega$ is defined in [6, 7] for the study of weighted norm inequalities of the Hardy–Littlewood maximal operator on Lebesgue spaces with variable exponents.

Notice that $L^{p(\cdot)}_\omega$ is not necessarily a Banach function space with respect to the Lebesgue measure. Particularly, when $p(\cdot) = p$, $1 < p < \infty$, for any unbounded Lebesgue measurable E with $|E| < \infty$, $\|\chi_E\|_{L^{p(\cdot)}_\omega}$ is not necessarily finite. For the definition of Banach function spaces, the reader is referred to [7, Section 2.10.3].

Thus, the extrapolation theorem given in [5, Theorem 4.6] for Banach function spaces with respect to Lebesgue measure cannot directly apply to $L^{p(\cdot)}_\omega$. On the other hand, we follow the idea from [5] to obtain our extrapolation for $L^{p(\cdot)}_\omega$ in Theorem 3.4.

The following is the Hölder inequality for the pair $L^{p(\cdot)}_\omega$ and $L^{p'(\cdot)}_{\omega^{-1}}$.

Lemma 2.1. *Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a Lebesgue measurable function and $\omega \in A_{p(\cdot)}$. We have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq C\|f\|_{L^{p(\cdot)}_\omega}\|g\|_{L^{p'(\cdot)}_{\omega^{-1}}}$$

for some $C > 0$.

The proof of the above lemma follows easily from the definition of $L^{p(\cdot)}_\omega$ and the Hölder inequality for $L^{p(\cdot)}$ ([7, Theorem 2.26], [8, Lemma 3.2.20]), hence we leave it to the reader.

Next, we have the norm conjugate formula for $L^{p(\cdot)}_\omega$.

Proposition 2.2. *Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a Lebesgue measurable function and $\omega \in A_{p(\cdot)}$. We have*

$$\|f\|_{L^{p(\cdot)}_\omega} \approx \sup\left\{\int_{\mathbb{R}^n} |f(x)g(x)|dx : g \in L^{p'(\cdot)}_{\omega^{-1}},\ \|g\|_{L^{p'(\cdot)}_{\omega^{-1}}} \leq 1\right\}.$$

The proof of the preceding proposition follows from the proof of the norm conjugate formula for $L^{p(\cdot)}$ ([7, Proposition 2.37], [8, Corollary 3.2.14]), so for brevity we skip it.

3. Extrapolations

We recall the well known log-Hölder continuity condition used in the variable exponent function spaces.

Definition 3.1. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a Lebesgue measurable function. We say that $p(\cdot)$ satisfies the log-Hölder continuity condition if

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|}, \quad \forall x, y \in \mathbb{R}^n,\ |x-y| < 1/2$$

for some $C > 0$. We say that $p(\cdot)$ satisfies the log-Hölder continuity condition at infinity if there exists a $1 \leq p_\infty < \infty$ such that

$$|p(x) - p_\infty| \leq \frac{C}{\log(e + |x|)}, \quad \forall x \in \mathbb{R}^n$$

for some $C > 0$.

We write $p \in LH$ if $p(\cdot)$ satisfies the log-Hölder continuity condition and the log-Hölder continuity condition at infinity.

In view of [6, Theorem 1.5], we have the subsequent boundedness result for the Hardy–Littlewood maximal operator on $L^{p(\cdot)}_\omega$.

Theorem 3.1. *Let $p \in LH$ with $1 < p_- \leq p_+ < \infty$. Then the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}_\omega$ if and only if $\omega \in A_{p(\cdot)}$.*

As stated in Section 1, we present the extrapolation theorem for $L^{p(\cdot)}_\omega$. We now state the classes of weight function corresponding to this extrapolation theorem. The definitions of these classes of weight function are modified from [8, Definition 4.4.6].

We begin with the definition of the averaging operators $T_{\mathcal{Q}}$. For any family of pairwise disjoint open bounded sets $\mathcal{O} = \{O\}$, write

$$(T_{\mathcal{O}} f)(x) = \sum_{O \in \mathcal{O}} \chi_O(x) \frac{1}{|O|} \int_O |f(y)|\, dy.$$

Definition 3.2. Let $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ be a Lebesgue measurable function. For any locally integrable function $\omega : \mathbb{R}^n \to (0, \infty)$, we write $\omega \in \mathcal{A}_{p(\cdot)}$ if there exists a constant $C > 0$ such that for any family of pairwise disjoint open bounded sets $\mathcal{O} = \{O\}$, we have

$$\|T_{\mathcal{O}} f\|_{L^{p(\cdot)}_\omega} \leq C \|f\|_{L^{p(\cdot)}_\omega}. \tag{3.1}$$

We call the smallest C for which (3.1) holds the $\mathcal{A}$-constant of $\varphi(x,t) = (t\omega)^{p(x)}$.

In view of [8, Theorem 4.5.7, Lemma 5.8.2 and (5.8.3)], $\omega \in \mathcal{A}_{p(\cdot)}$ if and only if

$$\|\chi_Q\|_{L^{p(\cdot)}_\omega} \|\chi_Q\|_{L^{p'(\cdot)}_{\omega^{-1}}} \approx |Q| \tag{3.2}$$

uniformly for all cubes $Q \subset \mathbb{R}^n$.

According to [8, Lemma 5.2.2], we have

$$\omega \in \mathcal{A}_{p(\cdot)} \Leftrightarrow \omega^{-1} \in \mathcal{A}_{p'(\cdot)}. \tag{3.3}$$

The set $\mathcal{A}_{p(\cdot)}$ also gives a characterization for the boundedness of the Hardy–Littlewood maximal operator on $L^{p(\cdot)}_\omega$. We have the following result from [8, Lemma 5.8.2 and Theorem 5.8.6].

Theorem 3.2. *Let $p \in LH$ with $1 < p_- \leq p_+ < \infty$. Then the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}_\omega$ if and only if $\omega \in \mathcal{A}_{p(\cdot)}$.*

In view of the above theorem, we have $\omega \in \mathcal{A}_{p(\cdot)}$ whenever M is bounded on $L^{p(\cdot)}_\omega$. There are some known results on the boundedness of the maximal operator on $L^{p(\cdot)}_\omega$. For instance, the power weights, $\omega(x) = |x|^a$, $a \in \mathbb{R}$, were considered in [21]. For a more detailed discussion on this issue, the reader is referred to [7, p. 188] and the references given there.

Thus, by Theorems 3.1 and 3.2, when $p \in LH$ with $1 < p_- \leq p_+ < \infty$, we obtain

$$\mathcal{A}_{p(\cdot)} = A_{p(\cdot)}.$$

The set $\mathcal{A}_{p(\cdot)}$ has a crucial property—the left-openness property. We present the left-openness property for the class $\mathcal{A}_{p(\cdot)}$ in the following. It almost immediately follows from [8, Theorem 5.4.15]. The main obstacle is that, using the terminology from [8, Definition 2.7.8], the function $\varphi(x,t) = (t\omega(x))^{p(x)}$ is not necessarily proper. Precisely, the set of simple functions does not necessarily belong to $L^{p(\cdot)}_\omega$.

Theorem 3.3. *Let $p(\cdot) : \mathbb{R}^n \to (1,\infty)$ with $1 < p_- \leq p_+ < \infty$. If $\omega \in \mathcal{A}_{p(\cdot)}$, then there exists a $s_0 \in (0,1)$ such that for any $s \geq s_0$, we have $\omega^{1/s} \in \mathcal{A}_{sp(\cdot)}$.*

Proof. We overcome the above mentioned difficulty by considering the family of functions $\{\varphi_R : R > 0\}$, where

$$\varphi_R(x,t) = \begin{cases} (t\omega(x))^{p(x)}, & x \in B(0,R) \\ 0, & x \in \mathbb{R}^n \backslash B(0,R). \end{cases}$$

The function φ_R is proper with respect to [8, Definition 2.7.8]. Furthermore, according to [8, Definition 2.6.1], the conjugate function of φ_R is given by

$$\varphi^*_R(x,t) = \begin{cases} (t/\omega(x))^{p'(x)}, & x \in B(0,R) \\ 0, & x \in \mathbb{R}^n \backslash B(0,R), \end{cases}$$

where $p'(\cdot)$ is the conjugate function of $p(\cdot)$.

Thus, the $\mathcal{A}$-constant of φ_R and the $\triangle_2$-constant of φ^*_R (see [8, Definition 2.4.1] for the definition of $\triangle_2$-constant) are independent of $R > 0$. Write

$$\omega_R(x) = \begin{cases} \omega(x), & x \in B(0,R) \\ 0, & x \in \mathbb{R}^n \backslash B(0,R). \end{cases}$$

Therefore, in view of [8, Theorem 5.4.15], there exists a s_0 which depends on the $\mathcal{A}$-constant of φ_R and the $\triangle_2$-constant of φ_R^* such that for any $s \geq s_0$, $\omega_R^{1/s} \in \mathcal{A}_{sp(\cdot)}$.

That is, we have a constant $C_0 > 0$ such that for any $R > 0$, for any family of pairwise disjoint open bounded sets $\mathcal{O} = \{O\}$ and $f \in L^{p(\cdot)}_\omega$ with $\operatorname{supp} f \subset B(0, R)$

$$\|T_{\mathcal{O}} f\|_{L^{sp(\cdot)}_{\omega^{1/s}}} \leq C_0 \|f\|_{L^{sp(\cdot)}_{\omega^{1/s}}}. \tag{3.4}$$

Since $T_{\mathcal{O}}$ is a positive operator and $f_R = f\chi_{B(0,R)} \uparrow f$, (3.4) is valid for any $f \in L^{sp(\cdot)}_{\omega^{1/s}}$. Thus, according to Definition 3.2, $\omega^{1/s} \in \mathcal{A}_{sp(\cdot)}$. □

The left-openness property and (3.3) are the essential features to establish the extrapolation theorem for $L^{p(\cdot)}_\omega$.

To precisely state the extrapolation theory for $L^{p(\cdot)}_\omega$, we follow a common practice used in extrapolation theory.

Let $1 \leq p, q < \infty$. Let $\mathcal{F}$ denote a family of ordered pairs of non-negative, Lebesgue measurable functions (f, g). We say that the inequality

$$\int_{\mathbb{R}^n} f(x)^p \omega(x) dx \leq C \int_{\mathbb{R}^n} g(x)^p \omega(x) dx \tag{3.5}$$

holds for any $(f, g) \in \mathcal{F}$ and $\omega \in A_q$ if (3.5) is valid for any pair in $\mathcal{F}$ such that the left-hand side is finite and the constant C depends only on p and $[\omega]_{A_q}$.

The proof of the subsequent extrapolation theory of $L^{p(\cdot)}_\omega$ follows from a simple modification of the classical extrapolation theory [2, 3, 5, 22, 23, 24]. We present the proof for the sake of completeness.

Theorem 3.4. *Given a family $\mathcal{F}$, suppose that for every $\omega_0 \in A_1$ we have*

$$\int_{\mathbb{R}^n} f(x)^{p_0} \omega_0(x) dx \leq C \int_{\mathbb{R}^n} g(x)^{p_0} \omega_0(x) dx, \quad (f, g) \in \mathcal{F}, \tag{3.6}$$

where C depends only on p_0 and $[\omega_0]_{A_1}$.

Let $p \in LH$ with $1 < p_- \leq p_+ < \infty$ and $\omega : \mathbb{R}^n \to (0, \infty)$ be a Lebesgue measurable function. If $\omega \in \mathcal{A}_{p(\cdot)}$, then

$$\|f\|_{L^{p(\cdot)}_\omega} \leq C \|g\|_{L^{p(\cdot)}_\omega}. \tag{3.7}$$

Moreover, for every r, $1 < r < \infty$, and (f_i, g_i), $i \in \mathbb{N}$, satisfying (3.6), we have

$$\Big\| \Big(\sum_{i \in \mathbb{N}} |f_i|^r \Big)^{1/r} \Big\|_{L^{p(\cdot)}_\omega} \leq C \Big\| \Big(\sum_{i \in \mathbb{N}} |g_i|^r \Big)^{1/r} \Big\|_{L^{p(\cdot)}_\omega} \tag{3.8}$$

for some $C > 0$.

Proof. The left-openness property of the class $\mathcal{A}_{p(\cdot)}$, Theorem 3.3, yields a $s_0 \in (0, 1)$ such that for any $s \geq s_0$,

$$\omega^{1/s} \in \mathcal{A}_{sp(\cdot)}. \tag{3.9}$$

For any fixed $\max(s_0, 1/p_-) < s_1 < 1$, define $q(\cdot) = s_1 p(\cdot)$. We have $q \in LH$ and $1 < q_- \leq q_+ < \infty$. Consequently, (3.3) and (3.9) guarantee that

$$u = \omega^{-1/s_1} \in \mathcal{A}_{q'(\cdot)}. \tag{3.10}$$

Then, we apply the Rubio de Francia iteration algorithm presented in [5] to obtain our result.

Without loss of generality, we assume that f is non-negative. It suffices to prove (3.7) as the proof of (3.8) is similar, see [5, Corollary 3.12]. For any non-negative function h, define

$$\mathcal{R}h(x)=\sum_{k=0}^{\infty}\frac{M^k h(x)}{2^k\|M\|^k_{L^{q'(\cdot)}_u}}.$$

We have the following properties for the operator $\mathcal{R}$:

$$h(x)\le \mathcal{R}h(x), \tag{3.11}$$

$$\|\mathcal{R}h\|_{L^{q'(\cdot)}_u}\le 2\|h\|_{L^{q'(\cdot)}_u}, \tag{3.12}$$

$$[\mathcal{R}h]_{A_1}\le 2\|M\|_{L^{q'(\cdot)}_u}. \tag{3.13}$$

The proof of (3.11) is straightforward. The properties (3.12) and (3.13) are consequences of the boundedness of the Hardy–Littlewood maximal operator on $L^{q'(\cdot)}_u$, which is guaranteed by (3.10) and the fact that $q\in LH$ with $1<q_-\le q_+<\infty$.

The standard extrapolation for Lebesgue spaces [5, Theorem 3.9] asserts that, for any $\omega\in A_1$,

$$\int_{\mathbb{R}^n} f(x)^{1/s_1}\omega(x)\,dx\le C\int_{\mathbb{R}^n} g(x)^{1/s_1}\omega(x)\,dx. \tag{3.14}$$

Proposition 2.2 yields

$$\begin{aligned}\|f\|^{1/s_1}_{L^{p(\cdot)}_\omega}&=\|f^{1/s_1}\|_{L^{q(\cdot)}_{u^{-1}}}\\ &\le C\sup\left\{\int_{\mathbb{R}^n} f(x)^{1/s_1}h(x)dx : h\in L^{q'(\cdot)}_u,\ h\ge 0\,\text{a.e. },\|h\|_{L^{q'(\cdot)}_u}\le 1\right\}\end{aligned} \tag{3.15}$$

for some $C>0$ because f is non-negative. For any fixed non-negative $h\in L^{q'(\cdot)}_u$, (3.11) assures that

$$\int_{\mathbb{R}^n} f(x)^{1/s_1}|h(x)|\,dx\le\int_{\mathbb{R}^n} f(x)^{1/s_1}(\mathcal{R}h)(x)\,dx.$$

Lemma 2.1 and (3.12) give

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)^{1/s_1}(\mathcal{R}h)(x)\,dx&\le C\|f^{1/s_1}\|_{L^{q(\cdot)}_{u^{-1}}}\|\mathcal{R}h\|_{L^{q'(\cdot)}_u}\\ &\le C\|f\|^{1/s_1}_{L^{p(\cdot)}_\omega}\|h\|_{L^{q'(\cdot)}_u}<\infty.\end{aligned}$$

In view of (3.13), $\mathcal{R}h\in A_1$. Thus, (3.14) guarantees that

$$\begin{aligned}\int_{\mathbb{R}^n} f(x)^{1/s_1}h(x)\,dx&\le\int_{\mathbb{R}^n} g(x)^{1/s_1}(\mathcal{R}h)(x)\,dx\\ &\le C\|g^{1/s_1}\|_{L^{q(\cdot)}_{u^{-1}}}\|\mathcal{R}h\|_{L^{q'(\cdot)}_u}\le C\|g\|^{1/s_1}_{L^{p(\cdot)}_\omega}\end{aligned} \tag{3.16}$$

for some $C>0$ independent of h. Thus, (3.15) and (3.16) yield (3.7). □

In the following sections, we present several applications of the above extrapolation theorems. More precisely, we study the singular integral and the John–Nirenberg inequalities on $L^{p(\cdot)}_\omega$. We also introduce and study the weighted Triebel–Lizorkin spaces with variable exponents by using Theorem 3.4.

4. Singular integral operators and John–Nirenberg inequalities

We first present the applications of Theorem 3.4 to the boundedness of the singular integral operators and the Fefferman–Stein inequality on $L^{p(\cdot)}_\omega$. Next, we use Theorem 3.4 to obtain the John–Nirenberg inequalities on $L^{p(\cdot)}_\omega$ and study the Triebel–Lizorkin type space associated with $L^{p(\cdot)}_\omega$.

We start with the boundedness of singular integral operators on $L^{p(\cdot)}_\omega$. Let K be a locally integrable function defined on $\mathbb{R}^n\backslash\{0\}$. Assume that K satisfies

$$|K(x)| \leq \frac{C}{|x|^n}, \quad |\nabla K(x)| \leq \frac{C}{|x|^{n+1}}, \quad x \neq 0, \tag{4.1}$$

and the Fourier transform of K is bounded.

For any K satisfying the above conditions, let $Tf(x) = K * f(x)$. For any $1 < p < \infty$ and $\omega \in A_p$, we have

$$\int_{\mathbb{R}^n} |Tf(x)|^p \omega(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^p \omega(x)\, dx. \tag{4.2}$$

For the proof of the above celebrated result the reader may consult [10, 25].

Applying Theorem 3.4 to (4.2), we establish the boundedness of singular integral operators on $L^{p(\cdot)}_\omega$.

Corollary 4.1. *Let $p \in LH$ with $1 < p_- \leq p_+ < \infty$ and $\omega \in \mathcal{A}_{p(\cdot)}$. Let K be a locally integrable function defined on $\mathbb{R}^n\backslash\{0\}$. If K satisfies* (4.1)*, then the singular integral operator $Tf = K * f$ is bounded from $L^{p(\cdot)}_\omega$ to $L^{p(\cdot)}_\omega$.*

We also have the boundedness result for the sharp maximal operator. For any Lebesgue measurable function f, the sharp maximal operator of f is defined by

$$M^\sharp f(x) = \sup_{x \ni B} \frac{1}{|B|} \int_B |f(y) - f_B|\, dy,$$

where the supreme is taken over all $B \in \mathbb{B}$ including x and

$$f_B = \frac{1}{|B|} \int_B f(y)\, dy.$$

The well known Fefferman–Stein inequality states that for any $0 < p < \infty$ and $\omega \in A_\infty$, we have

$$\int_{\mathbb{R}^n} Mf(x)^p \omega(x)\, dx \leq C \int_{\mathbb{R}^n} M^\sharp f(x)^p \omega(x)\, dx.$$

Theorem 3.4 extends the Fefferman–Stein inequality to $L^{p(\cdot)}_\omega$.

Corollary 4.2. *Let $0 < q < \infty$, $p \in LH$ with $1 < p_- \le p_+ < \infty$ and $\omega \in \mathcal{A}_{p(\cdot)}$. Then*

$$\|(Mf)^q\|_{L^{p(\cdot)}_\omega} \le C\|(M^\sharp f)^q\|_{L^{p(\cdot)}_\omega}.$$

We consider the Hardy–Littlewood maximal operator. A remarkable result for the Hardy–Littlewood maximal operator is the weighted inequality. For $1 < p < \infty$, $\omega \in A_p$, we have

$$\int_{\mathbb{R}^n} Mf(x)^p \omega(x)\,dx \le C \int_{\mathbb{R}^n} f(x)^p \omega(x)\,dx.$$

Thus, Theorem 3.4 and the fact $A_1 \subseteq A_p$ yield the weighted Fefferman–Stein vector-valued inequality on Lebesgue spaces with variable exponents.

Corollary 4.3. *Let $1 < r < \infty$, $p \in LH$ with $1 < p_- \le p_+ < \infty$ and $\omega \in \mathcal{A}_{p(\cdot)}$. Then*

$$\Big\|\Big(\sum_{i\in\mathbb{N}} |Mf_i|^r\Big)^{1/r}\Big\|_{L^{p(\cdot)}_\omega} \le C\Big\|\Big(\sum_{i\in\mathbb{N}} |f_i|^r\Big)^{1/r}\Big\|_{L^{p(\cdot)}_\omega}.$$

When $p(\cdot)$ is a constant function, the above inequality becomes the weighted vector-valued maximal inequality obtained in [1].

Next, we present another application of Theorem 3.4. We extend the John–Nirenberg inequalities to $L^{p(\cdot)}_\omega$. The classical John–Nirenberg inequality [11, Theorem 7.1.6] states that for any $\gamma > 0$ and any $B \in \mathbb{B}$,

$$|\{x \in B : |f(x) - f_B| > \gamma\}| \le C_1 e^{-\frac{C_2\gamma}{\|f\|_{\mathrm{BMO}}}} |B|, \quad f \in \mathrm{BMO}\backslash\mathcal{C},$$

where $\mathcal{C}$ denotes the set of constant functions and $C_1, C_2 > 0$ are independent of f and γ.

The John–Nirenberg inequalities had been generalized to some other function spaces such as the weighted Lebesgue spaces [15] and the Lebesgue space with variable exponents [18].

In particular, [15, (3.2)] gives the weighted John–Nirenberg inequalities. For any $\omega \in A_\infty$, there exist constants $C_1, C_2 > 0$ such that for any $B \in \mathbb{B}$

$$\int \chi_{\{y\in B:|f(y)-f_B|>\gamma\}}(x)\omega(x)\,dx \le C_1 e^{-\frac{C_2\gamma}{\|f\|_{\mathrm{BMO}}}} \int \chi_B(x)\omega(x)\,dx, \quad f \in \mathrm{BMO}\backslash\mathcal{C},$$

where C_1 depends on $[\omega]_{A_\infty}$ and n only.

Since $[\omega]_{A_\infty} \le [\omega]_{A_1}$ and $\|\chi_E\|_{L^{sp(\cdot)}_{\omega^{1/s}}} = \|\chi_E\|^{\frac{1}{s}}_{L^{p(\cdot)}_\omega}$ for any bounded Lebesgue measurable set E, Theorem 3.4 establishes the following John–Nirenberg inequalities on weighted Lebesgue spaces with variable exponents.

Theorem 4.4. *Let $p \in LH$ with $1 < p_- \le p_+ < \infty$ and $\omega : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function. If $\omega \in \mathcal{A}_{p(\cdot)}$, then for any $\gamma > 0$ and any $B \in \mathbb{B}$,*

$$\|\chi_{\{x\in B:|f(x)-f_B|>\gamma\}}\|_{L^{p(\cdot)}_\omega} \le C_1 e^{-\frac{C_2\gamma}{\|f\|_{\mathrm{BMO}}}} \|\chi_B\|_{L^{p(\cdot)}_\omega}, \quad f \in \mathrm{BMO}\backslash\mathcal{C},$$

where $C_1, C_2 > 0$ are independent of f and γ.

Theorem 4.4 extends the John–Nirenberg inequalities on Lebesgue spaces with variable exponents established in [18] to $L^{p(\cdot)}_{\omega}$.

With the John–Nirenberg inequalities on $L^{p(\cdot)}_{\omega}$, we can obtain the characterizations of BMO in terms of $L^{p(\cdot)}_{\omega}$. Thus, we recall the definition of BMO. A locally integrable function f belongs to BMO if

$$\|f\|_{\mathrm{BMO}} = \sup_{B\in\mathbb{B}} \frac{\|(f-f_B)\chi_B\|_{L^1}}{\|\chi_B\|_{L^1}} < \infty.$$

For any $p \in LH$ and $\omega \in \mathcal{A}_{p(\cdot)}$, write

$$\|f\|_{\mathrm{BMO}_{L^{p(\cdot)}_{\omega}}} = \sup_{B\in\mathbb{B}} \frac{\|(f-f_B)\chi_B\|_{L^{p(\cdot)}_{\omega}}}{\|\chi_B\|_{L^{p(\cdot)}_{\omega}}}.$$

As a consequence of the John–Nirenberg inequalities, we have the characterizations of BMO in terms of $L^{p(\cdot)}_{\omega}$.

Corollary 4.5. *Let $p \in LH$ with $1 < p_- \leq p_+ < \infty$ and $\omega : \mathbb{R}^n \to (0,\infty)$ be a Lebesgue measurable function. If $\omega \in \mathcal{A}_{p(\cdot)}$, then the norms $\|\cdot\|_{\mathrm{BMO}_{L^{p(\cdot)}_{\omega}}}$ and $\|\cdot\|_{\mathrm{BMO}}$ are mutually equivalent.*

Proof. Lemma 2.1 assures that for any $f \in \mathrm{BMO}$ and $B \in \mathbb{B}$,

$$\frac{1}{|B|}\|(f-f_B)\chi_B\|_{L^1} \leq \frac{1}{|B|}\|(f-f_B)\chi_B\|_{L^{p(\cdot)}_{\omega}}\|\chi_B\|_{L^{p'(\cdot)}_{\omega^{-1}}}.$$

Consequently, (3.2) yields

$$\frac{\|(f-f_B)\chi_B\|_{L^1}}{|B|} \leq C\frac{\|(f-f_B)\chi_B\|_{L^{p(\cdot)}_{\omega}}}{\|\chi_B\|_{L^{p(\cdot)}_{\omega}}} \tag{4.3}$$

for some $C > 0$. By taking supremum over all $B \in \mathbb{B}$, we have

$$\begin{aligned}\|f\|_{\mathrm{BMO}} &= \sup_{B\in\mathbb{B}} \frac{\|(f-f_B)\chi_B\|_{L^1}}{|B|} \\ &\leq C\sup_{B\in\mathbb{B}} \frac{\|(f-f_B)\chi_B\|_{L^{p(\cdot)}_{\omega}}}{\|\chi_B\|_{L^{p(\cdot)}_{\omega}}} = C\|f\|_{\mathrm{BMO}_{L^{p(\cdot)}_{\omega}}}.\end{aligned}$$

For the reverse inequality, we find that for any $f \in \mathrm{BMO}$ and $B \in \mathbb{B}$,

$$\|(f-f_B)\chi_B\|_{L^{p(\cdot)}_{\omega}} \leq \|\chi_B\|_{L^{p(\cdot)}_{\omega}} + \sum_{k=0}^{\infty} 2^{k+1}\|\chi_{\{x\in B:2^k<|f(x)-f_B|\leq 2^{k+1}\}}\|_{L^{p(\cdot)}_{\omega}}.$$

Theorem 4.4 guarantees that

$$\|(f-f_B)\chi_B\|_{L^{p(\cdot)}_{\omega}} \leq \|\chi_B\|_{L^{p(\cdot)}_{\omega}} + C\sum_{k=0}^{\infty} 2^{k+1}e^{-\frac{C_1 2^{k+1}}{\|f\|_{\mathrm{BMO}}}}\|\chi_B\|_{L^{p(\cdot)}_{\omega}}.$$

Since

$$\sum_{k=0}^{\infty} 2^{k+1}e^{-\frac{C_1 2^{k+1}}{\|f\|_{\mathrm{BMO}}}} \leq C\int_0^{\infty} \exp\big(-\frac{C_1 s}{\|f\|_{\mathrm{BMO}}}\big)ds \leq C\|f\|_{\mathrm{BMO}},$$

we obtain

$$\frac{\|(f-f_B)\chi_B\|_{L^{p(\cdot)}_\omega}}{\|\chi_B\|_{L^{p(\cdot)}_\omega}} \leq C\|f\|_{\mathrm{BMO}}$$

for some $C>0$ independent of $B\in\mathbb{B}$ and $f\in\mathrm{BMO}$. By taking supremum over $B\in\mathbb{B}$, we have $\|f\|_{L^{p(\cdot)}_\omega}\leq C\|f\|_{\mathrm{BMO}}$. □

For the characterizations of BMO via Banach function spaces, the reader is referred to [13, 16]. The characterization of BMO can be further extended to vector-valued mean oscillation characterization, see [19].

5. Weighted Triebel–Lizorkin spaces with variable exponents

Finally, we introduce the weighted Triebel–Lizorkin space with variable exponent $\dot{F}^{\alpha,q}_{p(\cdot),\omega}$. We also establish the smooth atomic decomposition for $\dot{F}^{\alpha,q}_{p(\cdot),\omega}$.

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the classes of tempered functions and Schwartz distributions, respectively. Let $\mathcal{P}$ be the class of polynomials on $\mathbb{R}^n$.

We first state the definition of $\dot{F}^{\alpha,q}_{p(\cdot),\omega}$.

Definition 5.1. Let $\alpha\in\mathbb{R}$, $0<q<\infty$, $p(\cdot)\in LH$ and $\omega\in\mathcal{A}_{p(\cdot)}$. The weighted Triebel–Lizorkin space with variable exponent $\dot{F}^{\alpha,q}_{p(\cdot),\omega}$ consists of those $f\in\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ satisfying

$$\|f\|_{\dot{F}^{\alpha,q}_{p(\cdot),\omega}} = \Big\|\big(\sum_{j\in\mathbb{Z}}(2^{j\alpha}|f*\varphi_j)^q|\big)^{1/q}\Big\|_{L^{p(\cdot)}_\omega}<\infty, \tag{5.1}$$

where $\varphi_j(x)=2^{jn}\varphi(2^jx)$ and $\varphi\in\mathcal{S}(\mathbb{R}^n)$ is a real-valued function satisfying

$$\operatorname{supp}\hat{\varphi}\subseteq\{x\in\mathbb{R}^n : 1/2\leq|x|\leq 2\}\quad\text{and}\quad|\hat{\varphi}(\xi)|\geq C,\quad 3/5\leq|x|\leq 5/3, \tag{5.2}$$

for some $C>0$.

The weighted Triebel–Lizorkin space with variable exponents include the weighted Triebel–Lizorkin spaces [9, p. 124] and the variable Triebel–Lizorkin spaces [27].

The definition of $\dot{F}^{\alpha,q}_{p(\cdot),\omega}$ involves the function φ, therefore, to show that $\dot{F}^{\alpha,q}_{p(\cdot),\omega}$ is well defined, we have to show that the space $\dot{F}^{\alpha,q}_{p(\cdot),\omega}$ is independent of the function φ in Definition 5.1. We follow the idea given in [9, Theorem 2.2] and [14, Theorem 3.1].

The study of weighted function spaces has several extensions. For instance, we have the atomic decomposition for weighted Hardy spaces with variable exponents, see [20]. In [17], we have a study of the vector-valued singular integral operators on weighted Morrey spaces with variable exponents; some applications of this study on the Triebel–Lizorkin-Morrey spaces with variable exponents are given there also.

We recall some notations introduced in [9] before we give the statement and proof of the well definition of the weighted Triebel–Lizorkin space with variable exponent.

Let $\mathcal{Q}=\{Q_{i,k} : i\in\mathbb{Z},\ k\in\mathbb{Z}^n\}$ denote the set of dyadic cubes, where $Q_{i,k}=\{(x_1,\dots,x_n)\in\mathbb{R}^n : k_j\leq 2^ix_j<k_j+1,\ j=1,\dots,n\}$ and $k=(k_1,\dots,k_n)$. We denote the Lebesgue measure of $Q\in\mathcal{Q}$ by $|Q|$ and the side length of Q by $l(Q)$.

The φ-ψ transforms consist of two linear operators S_φ and T_ψ generated by a pair of functions $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (5.2) and

$$\sum_{j\in\mathbb{Z}} \hat{\varphi}(2^{-j}\xi)\overline{\hat{\psi}(2^{-j}\xi)} = 1, \quad \xi \neq 0. \tag{5.3}$$

Define $\varphi_\nu(x) = 2^{\nu n}\varphi(2^\nu x)$, $\psi_\nu(x) = 2^{\nu n}\psi(2^\nu x)$ and $\varphi_Q(x) = |Q|^{-1/2}\varphi(2^\nu x - k)$, $\psi_Q(x) = |Q|^{-1/2}\psi(2^\nu x - k)$, $\nu \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ and $Q = Q_{\nu,k}$.

For any $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ and for any complex-valued sequence $s = \{s_Q\}_{Q\in\mathcal{Q}}$, we define

$$\mathrm{S}_\varphi(f) = \{(\mathrm{S}_\varphi f)_Q\}_{Q\in\mathcal{Q}} = \{\langle f, \varphi_Q\rangle\}_{Q\in\mathcal{Q}} \quad \text{and} \quad \mathrm{T}_\psi(s) = \sum_Q s_Q\psi_Q.$$

It is well known that $\mathrm{T}_\psi \circ \mathrm{S}_\varphi = id$ in $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ (see [12], Theorem 6.1).

In order to study the mapping properties of the φ-ψ transforms, we need the following sequence spaces.

Definition 5.2. Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $p(\cdot) \in LH$ and $\omega \in \mathcal{A}_{p(\cdot)}$. The sequence space $\dot{f}^{\alpha,q}_{p(\cdot),\omega}$ is the collection of all complex-valued sequences $s = \{s_Q\}_{Q\in\mathcal{Q}}$ such that

$$\|s\|_{\dot{f}^{\alpha,q}_{p(\cdot),\omega}} = \Big\|\big(\sum_{Q\in\mathcal{Q}}(|Q|^{-\alpha/n}|s_Q|\tilde{\chi}_Q)^q\big)^{1/q}\Big\|_{L^{p(\cdot)}_\omega} < \infty,$$

where $\tilde{\chi}_Q = |Q|^{-1/2}\chi_Q$.

We need some supporting results to establish the well definition of the weighted Triebel–Lizorkin space with variable exponent.

For $0 < a \le 1$ and any locally integrable function g, define the operator M_a as $\mathrm{M}_a(g) = \big[M(|g|^a)\big]^{1/a}$.

Proposition 5.1. *Let $p(\cdot) \in LH$ and $\omega \in \mathcal{A}_{p(\cdot)}$. For any $0 < a < p_-$, we have*

$$\Big\|\big(\sum_{j\in\mathbb{N}}|\mathrm{M}_a f_i|^q\big)^{1/q}\Big\|_{L^{p(\cdot)}_\omega} \le C\Big\|\big(\sum_{j\in\mathbb{N}}|f_i|^q\big)^{1/q}\Big\|_{L^{p(\cdot)}_\omega} \tag{5.4}$$

for some $C > 0$ independent of $\{f_i\}_{i\in\mathbb{N}}$.

Proof. We find that

$$\Big\|\big(\sum_{j\in\mathbb{N}}|\mathrm{M}_a f_i|^q\big)^{1/q}\Big\|_{L^{p(\cdot)}_\omega} = \Big\|\big(\sum_{j\in\mathbb{N}}(M|f_i|^a)^{q/a}\big)^{a/q}\Big\|^{1/a}_{L^{p(\cdot)/a}_{\omega^a}}.$$

Write $\tilde{p}(\cdot) = p(\cdot)/a$. Thus, $\tilde{p} \in LH$ with $1 < \tilde{p}_- \le \tilde{p}_+ < \infty$. We are allowed to apply Corollary 4.3 and conclude that

$$\Big\|\big(\sum_{j\in\mathbb{N}}|\mathrm{M}_a f_i|^q\big)^{1/q}\Big\|_{L^{p(\cdot)}_\omega} \le \Big\|\big(\sum_{j\in\mathbb{N}}(|f_i|^a)^{q/a}\big)^{a/q}\Big\|^{1/a}_{L^{p(\cdot)/a}_{\omega^a}} = \Big\|\big(\sum_{j\in\mathbb{N}}|f_i|^q\big)^{1/q}\Big\|_{L^{p(\cdot)}_\omega}.$$

□

The above proposition is used to overcome the possibility that $p_- = 1$.

The subsequent lemma offers the connection between the Hardy–Littlewood maximal operator and the operator S_φ.

Lemma 5.2. *For any* $0 < a < 1$, *we have*

$$\sum_{|Q|=2^{-jn}} |Q|^{-1/2-\alpha/n} |(\mathrm{S}_\varphi f)_Q| \chi_Q(x) \le C\, \mathrm{M}_a(2^{j\alpha}(\tilde{\varphi} * f))(x),$$

where $\tilde{\varphi}(x) = \overline{\varphi(-x)}$.

Proof. The definition of S_φ yields

$$|Q|^{-1/2} |(\mathrm{S}_\varphi f)_Q| \le \sup_{y\in Q} |\tilde{\varphi} * f(y)|.$$

Let $F(x) = (\tilde{\varphi} * f)(2^{-j}x)$. The above inequality ensures that

$$\begin{aligned} \sum_{|Q|=2^{-jn}} |Q|^{-1/2-\alpha/n} |(\mathrm{S}_\varphi f)_Q| \chi_Q(x) &\le \sum_{|Q|=2^{-jn}} 2^{j\alpha} \sup_{y\in Q} |\tilde{\varphi} * f(y)| \chi_Q(x) \\ &\le \sup_{|z|\le\sqrt{n}2^{-j}} \left\{ (1+2^j|z|)^{-L} |2^{j\alpha}(F(2^j(x-z)))| (1+2^j|z|)^L \right\} \end{aligned}$$

for any $L > 0$. Since $\operatorname{supp} \hat{F} \subseteq \{\xi \in \mathbb{R}^n : 1/2 \le |\xi| \le 2\}$, the results in [26, Sections 1.4.1 and 1.4.2.] lead to

$$\begin{aligned} &\sum_{|Q|=2^{-jn}} |Q|^{-1/2-\alpha/n} |(\mathrm{S}_\varphi f)_Q| \chi_Q(x) \\ &\qquad \le C \sup_{|z|\le\sqrt{n}2^{-j}} (1+2^j|z|)^{-L} \left\{ \sup_{z\in\mathbb{R}^n} \frac{|2^{j\alpha} F(2^j x - z))|}{(1+|z|)^L} \right\} \\ &\qquad \le C\, \mathrm{M}_a(2^{j\alpha} F)(2^j x) = C\, \mathrm{M}_a(2^{j\alpha}(\tilde{\varphi} * f))(x) \end{aligned}$$

for some constant $C > 0$. □

As shown in the proof of [9, Theorem 2.2], to prove the boundedness of T_ψ, for any sequence $\{s_Q\}_{Q\in\mathcal{Q}}$ we need to consider the sequence $s^*_{\lambda,\mu} = \{(s^*_\lambda)^\mu_Q\}_{Q\in\mathcal{Q}}$, $\mu \in \mathbb{Z}$ and $\lambda > n$, defined as

$$(s^*_\lambda)^\mu_Q = \sum_{l(P)=2^{-\mu}} |s_P| \left(1 + \frac{|x_Q - x_P|}{\max(l(Q), l(P))}\right)^{-\lambda}.$$

Similar to the previous Lemma, we have an estimate of $s^*_{\lambda,\mu}$ by M_a. That estimate is given by [9, Lemma A.2 and Remark A.3].

Lemma 5.3. *Let* $\nu, \mu \in \mathbb{Z}$ *and* $0 < a < 1$. *If* $L > n/a$, *we have*

$$\sum_{l(Q)=2^{-\nu}} |Q|^{-\alpha/n} (s^*_\lambda)^\mu_Q \tilde{\chi}_Q \le C(\mu,\nu,\alpha)\, \mathrm{M}_a \Big(\sum_{l(P)=2^{-\mu}} |P|^{-\alpha/n} |s_P| \tilde{\chi}_P \Big),$$

where

$$C(\mu,\nu,\alpha) = \begin{cases} C 2^{(\nu-\mu)(\alpha+\frac{n}{2})}, & \text{if } \mu-\nu \le 0; \\ C 2^{(\mu-\nu)(\frac{n}{a}-\alpha+\frac{n}{2})}, & \text{if } \mu-\nu > 0, \end{cases}$$

for some constant $C > 0$.

We are now ready to state and prove the main result of this section.

Theorem 5.4. *Let* $\alpha \in \mathbb{R}$, $0 < q < \infty$, $p(\cdot) \in LH$ *and* $\omega \in \mathcal{A}_{p(\cdot)}$. *The Triebel–Lizorkin space associated with weighted Lebesgue space with variable exponent* $\dot{F}^{\alpha,q}_{p(\cdot),\omega}$ *is independent of the function* φ *in Definition 5.1. Furthermore, the operators* $\mathrm{S}_\varphi : \dot{F}^{\alpha,q}_{p(\cdot),\omega} \to \dot{f}^{\alpha,q}_{p(\cdot),\omega}$ *and* $\mathrm{T}_\psi : \dot{f}^{\alpha,q}_{p(\cdot),\omega} \to \dot{F}^{\alpha,q}_{p(\cdot),\omega}$ *are bounded. Moreover, we have constants* $C_1 > C_2 > 0$ *such that for any* $f \in \dot{F}^{\alpha,q}_{p(\cdot),\omega}$,

$$C_2\|f\|_{\dot{F}^{\alpha,q}_{p(\cdot),\omega}} \le \|\mathrm{S}_\varphi(f)\|_{\dot{f}^{\alpha,q}_{p(\cdot),\omega}} \le C_1\|f\|_{\dot{F}^{\alpha,q}_{p(\cdot),\omega}}. \tag{5.5}$$

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (5.2). We show that

$$\|\{(\mathrm{S}_\varphi f)_Q\}_Q\|_{\dot{f}^{\alpha,q}_{p(\cdot),\omega}} \le \|f\|_{\dot{F}^{\alpha,q}_{p(\cdot),\omega}(\tilde{\varphi})}, \tag{5.6}$$

where $\|\cdot\|_{\dot{F}^{\alpha,q}_{p(\cdot),\omega}(\tilde{\varphi})}$ denotes the quasi-norm in (5.1) generated by the given function $\tilde{\varphi}$. Notice that the boundedness of S_φ will be proved once we ensure that the space $\dot{F}^{\alpha,q}_{p(\cdot),\omega}$ is independent of the choices of φ.

Lemma 5.2 indicates that

$$\begin{aligned}\|\{(\mathrm{S}_\varphi f)_Q\}_Q\|_{\dot{f}^{\alpha,q}_{p(\cdot),\omega}} &= \left\|\Big(\sum_{Q\in\mathcal{Q}}(|Q|^{-\alpha/n-1/2}|(\mathrm{S}_\varphi f)_Q|\chi_Q(x))^q\Big)^{1/q}\right\|_{L^{p(\cdot)}_\omega} \\ &\le C\left\|\big(\sum_{j\in\mathbb{Z}}(\mathrm{M}_a(2^{j\alpha}|\tilde{\varphi}*f|))^q\big)^{1/q}\right\|_{L^{p(\cdot)}_\omega}.\end{aligned}$$

Proposition 5.1 asserts that $\|\{(\mathrm{S}_\varphi f)_Q\}_Q\|_{\dot{f}^{\alpha,q}_{p(\cdot),\omega}} \le C\|f\|_{\dot{F}^{\alpha,q}_{p(\cdot),\omega}(\tilde{\varphi})}$.

Next, we consider the boundedness of T_ψ.

Suppose that $s = \{s_Q\}$, $f = \mathrm{T}_\psi s = \sum_Q s_Q\psi_Q$ and $x \in Q^* \subseteq Q \subseteq Q^{**}$, where Q^*, Q and Q^{**} are dyadic cubes with $|Q^*| = 2^{-nj-n}$, $|Q| = 2^{-nj}$ and $|Q^{**}| = 2^{-nj+n}$, $j \in \mathbb{Z}$.

Let $0 < a < \min(1,q)$ and $\lambda > n/a$. For any family φ and ψ satisfying (5.2) and (5.3) we have

$$|\tilde{\varphi} * f(x)| \le C|Q|^{-1/2}\left((s^*_\lambda)^{j+1}_{Q^*} + (s^*_\lambda)^{j}_{Q} + (s^*_\lambda)^{j-1}_{Q^{**}}\right).$$

For the proof of the above inequality, see [9, p. 50]. Consequently, we obtain

$$\|\mathrm{T}_\psi s\|_{\dot{F}^{\alpha,q}_{p(\cdot),\omega}(\varphi)} \le C\left(\sum_{j\in\mathbb{Z}}\sum_{l=j-1}^{j+1}\|s^*_{\lambda,l}\|_{\dot{f}^{\alpha,q}_{p(\cdot),\omega}}\right).$$

By applying Proposition 5.1 and Lemma 5.3 with $\mu = j$ and $\nu = j-1, j, j+1$, respectively, to each term of the right-hand side, we have

$$\begin{aligned}\|\mathrm{T}_\psi s\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\varphi)} &\le C\Bigg\|\Bigg(\sum_{j\in\mathbb{Z}}(\mathrm{M}_a(\sum_{|P|=2^{-jn}} 2^{j\alpha}(|s_P|\tilde\chi_P)))^q\Bigg)^{1/q}\Bigg\|_{L^{p(\cdot)}_\omega}\\ &\le C\Bigg\|\Big(\sum_{P\in\mathcal{Q}}(2^{j\alpha}(|s_P|\tilde\chi_P))^q\Big)^{1/q}\Bigg\|_{L^{p(\cdot)}_\omega} = C\|s\|_{\dot f^{\alpha,q}_{p(\cdot),\omega}}.\end{aligned}$$

Next, we show the independence of the choice of φ in the definition of $\dot F^{\alpha,q}_{p(\cdot),\omega}$ and the equivalence of the quasi-norms, $\|\cdot\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}}$ and $\|\cdot\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\varphi)}$. Suppose that φ^l, ψ^l, $l = 1, 2$, are functions that satisfy the conditions (5.2) and (5.3). Therefore,

$$\begin{aligned}\|f\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\varphi^1)} &= \Bigg\|\sum_Q (\mathrm{S}_{\varphi^2} f)_Q \psi_Q^2\Bigg\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\varphi^1)} = \big\|\mathrm{T}_{\psi^2}(\mathrm{S}_{\varphi^2} f)\big\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\varphi^1)}\\ &\le C\|(\mathrm{S}_{\varphi^2} f)_Q\|_{\dot f^{\alpha,q}_{p(\cdot),\omega}} \le C\|f\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\tilde\varphi^2)}\end{aligned}$$

by the boundedness of T_{ψ^2} and (5.6). By taking $\varphi^1 = \varphi^2$, we assert that

$$C^{-1}\|f\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\varphi^1)} \le \|f\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\tilde\varphi^1)}.$$

Reversing the roles of $\tilde\varphi^1$ and φ^1, we find that

$$C^{-1}\|f\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\tilde\varphi^1)} \le \|f\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\varphi^1)}.$$

The above inequalities guarantee that

$$C^{-1}\|f\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\varphi^1)} \le \|f\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\varphi^2)} \le C\|f\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\varphi^1)}.$$

Therefore, the definition of the Littlewood–Paley space is independent of $\varphi \in \mathcal{S}_0(\mathbb{R}^n)$ satisfying (5.2).

Moreover, we have the equivalence of the quasi-norms $\|\cdot\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}}$ and $\|\cdot\|_{\dot F^{\alpha,q}_{p(\cdot),\omega}(\varphi)}$ and the boundedness of S_φ and T_ψ on $\dot F^{\alpha,q}_{p(\cdot),\omega}$ and $\dot f^{\alpha,q}_{p(\cdot),\omega}$, respectively.

The inequalities (5.5) follow from the boundedness of S_φ, T_ψ and the fact that $\mathrm{T}_\psi \circ \mathrm{S}_\varphi = id$. □

When $\omega \equiv 1$, $\dot F^{\alpha,q}_{p(\cdot),\omega}$ reduces to the variable Triebel–Lizorkin spaces studied in [27].

In addition, as stated in [14], the Fefferman–Stein vector-valued maximal inequality on $L^{p(\cdot)}_\omega$ guarantees the validity of the smooth atomic decompositions of $\dot F^{\alpha,q}_{p(\cdot),\omega}$.

We now recall the definition of smooth atom for $\dot F^{\alpha,q}_{p(\cdot),\omega}$. For each dyadic cube Q, a_Q is a smooth N-atom for $\dot F^{\alpha,q}_{p(\cdot),\omega}$, $N \in \mathbb{N}$, if it satisfies

$$\int x^\gamma a_Q(x)\,dx = 0 \quad \text{for } |\gamma| \le N, \gamma \in \mathbb{Z}^n_+,$$
$$\operatorname{supp} a_Q \subseteq 3Q,$$

and, for $\gamma \in \mathbb{Z}_+^n$,

$$|\partial^\gamma a_Q(x)| \le C_\gamma |Q|^{-1/2-|\gamma|/n}.$$

Theorem 5.5 (Smooth atomic decomposition)**.** *Let $\alpha \in \mathbb{R}$, $0 < q < \infty$, $p(\cdot) \in LH$ and $\omega \in \mathcal{A}_{p(\cdot)}$. If $f \in \dot{F}^{\alpha,q}_{p(\cdot),\omega}$, then there exist a sequence, $s = \{s_Q\}_Q \in \dot{f}^{\alpha,q}_{p(\cdot),\omega}$, and smooth N-atoms, $\{a_Q\}$, such that $f = \sum_Q s_Q a_Q$ and $\|s\|_{\dot{f}^{\alpha,q}_{p(\cdot),\omega}} \le C\|f\|_{\dot{F}^{\alpha,q}_{p(\cdot),\omega}}$ for some constant $C > 0$.*

Proof. According to Lemma 5.2, for any $f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$ we have $f = \sum_{Q\in\mathcal{Q}} s_Q a_Q$, where each a_Q is a smooth N-atom and the s_Q satisfy

$$\sum_{|Q|=2^{-jn}} |Q|^{-1/2-\alpha/n}|s_Q|\chi_Q(x) \le C\,\mathrm{M}_a(2^{j\alpha}(\tilde{\varphi}_j * f))(x)$$

for some $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (5.2). Thus, the inequality $\|s\|_{\dot{f}^{\alpha,q}_{p(\cdot),\omega}} \le C\|f\|_{\dot{F}^{\alpha,q}_{p(\cdot),\omega}}$, follows from Proposition 5.1. □

Theorem 5.5 generalizes the smooth atomic decomposition for variable Triebel–Lizorkin spaces obtained in [28].

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