REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 57, No. 1, 2016, Pages 103–109
Published online: March 8, 2016

ANOTHER PROOF OF CHARACTERIZATION OF BMO VIA BANACH FUNCTION SPACES

MITSUO IZUKI

Abstract. Our aim is to give a characterization of the BMO norm via Banach function spaces based on the Rubio de Francia algorithm. Our proof is different from the one by Ho [Atomic decomposition of Hardy spaces and characterization of BMO via Banach function spaces, Anal. Math. 38 (2012), 173–185].

1. Introduction

The BMO space $\mathrm{BMO}(\mathbb{R}^n)$ is known as the dual space of the Hardy space, and plays an important role in real analysis. The space $\mathrm{BMO}(\mathbb{R}^n)$ consists of all locally integrable functions b satisfying that the semi-norm

$$\|b\|_{\mathrm{BMO}} := \sup_{Q:\mathrm{cube}} \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, dx$$

is finite, where, for each cube $Q \subset \mathbb{R}^n$, $|Q|$ is its Lebesgue measure and b_Q is the mean value of the function b on Q, namely

$$b_Q := \frac{1}{|Q|} \int_Q b(y) \, dy.$$

The definition above tells us that every function $b \in \mathrm{BMO}(\mathbb{R}^n)$ has bounded mean oscillation. The semi-norm $\|b\|_{\mathrm{BMO}}$ is called the BMO norm. If $b \in \mathrm{BMO}(\mathbb{R}^n)$, then there exist positive constants C_1 and C_2 such that for all cubes Q and $\lambda > 0$,

$$|\{x \in Q \,:\, |b(x) - b_Q| > \lambda\}| \le C_1 \, |Q| \exp\left(-\frac{C_2 \lambda}{\|b\|_{\mathrm{BMO}}}\right). \tag{1.1}$$

The inequality (1.1) is proved by John and Nirenberg in [11] and implies that the BMO norm $\|b\|_{\mathrm{BMO}}$ is equivalent to

$$\|b\|_{\mathrm{BMO}_{L^p}} := \sup_{Q:\mathrm{cube}} \frac{1}{\|\chi_Q\|_{L^p(\mathbb{R}^n)}} \|(b - b_Q)\chi_Q\|_{L^p(\mathbb{R}^n)} \tag{1.2}$$

for any constant $1 \le p < \infty$, where χ_Q is the characteristic function for Q.

2010 *Mathematics Subject Classification.* 42B35.
Key words and phrases. BMO norm, Banach function space, Rubio de Francia algorithm.
The author was partially supported by Grand-in-Aid for Scientific Research (C), No. 24540185, for the Japan Society for the Promotion of Science.

The author has considered generalization of the equivalent BMO norm and proved the following:

(1) (The author, [7]) By using a bounded measurable function $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ we generalize the semi-norm (1.2) to

$$\|b\|_{\mathrm{BMO}_{L^{p(\cdot)}}} := \sup_{Q:\text{cube}} \frac{1}{\|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \|(b-b_Q)\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}, \tag{1.3}$$

where the variable Lebesgue norm $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ is defined by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} := \inf\left\{\lambda > 0 \,:\, \int_{\mathbb{R}^n} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

The generalized Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ consists of all measurable functions f such that the norm $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ is finite. If $p(\cdot)$ satisfies $1 < \inf p(x)$ and the Hardy–Littlewood maximal operator M, defined by

$$Mf(x) := \sup_{Q:\text{cube},\, Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, then the generalized BMO norm $\|b\|_{\mathrm{BMO}_{L^{p(\cdot)}}}$ is equivalent to the classical one $\|b\|_{\mathrm{BMO}}$.

(2) (The author and Sawano, [9]) If a bounded measurable function $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ satisfies $1 \le \inf p(x)$ and the log-Hölder conditions:

$$|p(x)-p(y)| \le \frac{C}{-\log(|x-y|)} \qquad \text{for } x,\, y \in \mathbb{R}^n,\ |x-y| \le 1/2,$$

$$|p(x)-p_\infty| \le \frac{C}{\log(e+|x|)} \qquad \text{for } x \in \mathbb{R}^n,$$

for some constants C and p_∞ independent of x, y, then $\|b\|_{\mathrm{BMO}_{L^{p(\cdot)}}}$ is equivalent to $\|b\|_{\mathrm{BMO}}$.

(3) (The author, Sawano and Tsutsui, [10]) If a variable exponent $p(\cdot) : \mathbb{R}^n \to [1,\infty)$ is bounded and M is of weak type $(p(\cdot), p(\cdot))$, that is, there exists a constant $C > 0$ such that for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and all $\lambda > 0$,

$$\|\chi_{\{Mf>\lambda\}}\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\,\lambda^{-1}\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)},$$

then $\|b\|_{\mathrm{BMO}_{L^{p(\cdot)}}}$ is equivalent to $\|b\|_{\mathrm{BMO}}$.

On the other hand, Ho [6] has proved an interesting characterization in the context of general function spaces including Lebesgue spaces. Given a Banach function space X equipped with a norm $\|\cdot\|_X$, we define a generalized BMO norm

$$\|b\|_{\mathrm{BMO}_X} := \sup_{Q:\text{cube}} \frac{1}{\|\chi_Q\|_X}\|(b-b_Q)\chi_Q\|_X.$$

If M is bounded on the associate space X', then $\|b\|_{\mathrm{BMO}_X}$ is equivalent to $\|b\|_{\mathrm{BMO}}$.

We remark that Ho's result [6] has included the authors' one [7, 9]. The statements in [7, 9] are deeply depending on Diening's work [3] on variable exponent analysis. On the other hand, Ho's proof is self-contained and obtained as a by-product of the new results about atomic decomposition introduced in [6].

In the present paper we give a new proof of the characterization of the BMO norm initially proved by Ho. We use the Rubio de Francia algorithm [13, 14, 15].

2. Preliminaries

2.1. **Banach function spaces.** We present the definition of Banach function space and some fundamental properties of it based on Bennett and Sharpley [1]. For further informations on the theory of Banach function space including the proof of Lemma 2.1 below we refer to that book.

Definition 2.1. Let $\mathcal{M}$ be the set of all complex-valued measurable functions defined on $\mathbb{R}^n$, and X be a linear subspace of $\mathcal{M}$.

(1) The space X is said to be a Banach function space if there exists a functional $\|\cdot\|_X : \mathcal{M} \to [0, \infty]$ satisfying the following properties: Let f, g, $f_j \in \mathcal{M}$ $(j = 1, 2, \dots)$, then
  (a) $f \in X$ holds if and only if $\|f\|_X < \infty$.
  (b) Norm property:
    (i) Positivity: $\|f\|_X \geq 0$.
    (ii) Strict positivity: $\|f\|_X = 0$ holds if and only if $f(x) = 0$ for almost every $x \in \mathbb{R}^n$.
    (iii) Homogeneity: $\|\lambda f\|_X = |\lambda| \cdot \|f\|_X$ holds for all complex numbers λ.
    (iv) Triangle inequality: $\|f + g\|_X \leq \|f\|_X + \|g\|_X$.
  (c) Symmetry: $\|f\|_X = \||f|\|_X$.
  (d) Lattice property: If $0 \leq g(x) \leq f(x)$ for almost every $x \in \mathbb{R}^n$, then $\|g\|_X \leq \|f\|_X$.
  (e) Fatou property: If $0 \leq f_j(x) \leq f_{j+1}(x)$ for all j and $f_j(x) \to f(x)$ as $j \to \infty$ for almost every $x \in \mathbb{R}^n$, then $\lim_{j\to\infty} \|f_j\|_X = \|f\|_X$.
  (f) For every measurable set $F \subset \mathbb{R}^n$ such that $|F| < \infty$, $\|\chi_F\|_X$ is finite. Additionally there exists a constant $C_F > 0$ depending only on F such that for all $h \in X$,
$$\int_F |h(x)|\, dx \leq C_F \|h\|_X.$$

(2) Suppose that X is a Banach function space equipped with a norm $\|\cdot\|_X$. The associate space X' is defined by
$$X' := \{f \in \mathcal{M} : \|f\|_{X'} < \infty\},$$
where
$$\|f\|_{X'} := \sup_g \left\{ \left| \int_{\mathbb{R}^n} f(x)g(x)\, dx \right| : \|g\|_X \leq 1 \right\}.$$

Lemma 2.1. *Let X be a Banach function space. Then the following hold:*

(1) *(The Lorentz–Luxemburg theorem.) $(X')' = X$ holds; in particular, the norms $\|\cdot\|_{(X')'}$ and $\|\cdot\|_X$ are equivalent.*

(2) *(The generalized Hölder inequality.) If $f \in X$ and $g \in X'$, then we have*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\, dx \le \|f\|_X \|g\|_{X'}.$$

The usual Lebesgue space $L^p(\mathbb{R}^n)$ with constant exponent $1 \le p \le \infty$ is an example of Banach function spaces. Kováčik and Rákosník [12] have proved that the generalized Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent $p(\cdot)$ is a Banach function space and the associate space is $L^{p'(\cdot)}(\mathbb{R}^n)$ with norm equivalence, where $p'(\cdot)$ is the conjugate exponent given by $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

If we assume some conditions for boundedness of the Hardy–Littlewood maximal operator M on X, then the norm $\|\cdot\|_X$ has properties similar to the Muckenhoupt weights.

Lemma 2.2. *Let X be a Banach function space. Suppose that the Hardy–Littlewood maximal operator M is weakly bounded on X, that is, there exists a positive constant C such that*

$$\|\chi_{\{Mf>\lambda\}}\|_X \le C\lambda^{-1}\|f\|_X \tag{2.1}$$

is true for all $f \in X$ and $\lambda > 0$. Then we have

$$\sup_{Q:\text{cube}} \frac{1}{|Q|}\|\chi_Q\|_X \|\chi_Q\|_{X'} < \infty. \tag{2.2}$$

Proof. The proof is found in the author's papers [8, Lemmas 2.4 and 2.5] and [10, Lemmas G' and H]. For the reader's convenience we give the self-contained proof. Take a cube Q and a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Suppose that $|f|_Q > 0$. Because $|f|_Q\chi_Q(x) \le M(f\chi_Q)(x)$ holds for almost every $x \in \mathbb{R}^n$, we obtain $M(f\chi_Q)(x) > \lambda$ for almost every $x \in Q$, where $\lambda := |f|_Q/2$. Hence by assumption (2.1) we get

$$\begin{aligned} |f|_Q\|\chi_Q\|_X &\le |f|_Q\|\chi_{\{M(f\chi_Q)>\lambda\}}\|_X \\ &\le |f|_Q \cdot C\,\lambda^{-1}\|f\chi_Q\|_X \\ &= 2C\,\|f\chi_Q\|_X. \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{1}{|Q|}\|\chi_Q\|_X\|\chi_Q\|_{X'} &= \frac{1}{|Q|}\|\chi_Q\|_X \cdot \sup_g \left\{ \int_{\mathbb{R}^n} |g(x)|\chi_Q(x)\, dx \,:\, g \in X,\ \|g\|_X \le 1 \right\} \\ &= \sup_g \{ |g|_Q \|\chi_Q\|_X \,:\, g \in X,\ \|g\|_X \le 1 \} \\ &\le \sup_g \{ 2C\,\|g\chi_Q\|_X \,:\, g \in X,\ \|g\|_X \le 1 \} \\ &\le 2C. \end{aligned}$$

□

Remark 2.1. If M is bounded on X, that is, there exists a positive constant C such that

$$\|Mf\|_X \le C\,\|f\|_X$$

holds for all $f \in X$, then we can easily check that (2.1) holds. On the other hand, if M is bounded on the associate space X', then Lemma 2.1 shows that (2.2) is true.

2.2. Muckenhoupt weights.

Definition 2.2. Let w be a locally integrable and positive function on $\mathbb{R}^n$. The function w is said to be a Muckenhoupt A_1 weight if there exists a positive constant C_1 such that $Mw(x) \le C_1 w(x)$ holds for almost every $x \in \mathbb{R}^n$. The set A_1 consists of all Muckenhoupt A_1 weights. For every $w \in A_1$, the finite value

$$[w]_{A_1} := \sup_{Q:\text{cube}} \left\{ \frac{1}{|Q|} \int_Q w(x)\, dx \cdot \|w^{-1}\|_{L^\infty(Q)} \right\}$$

is said to be a Muckenhoupt A_1 constant.

We remark that if $w \in A_1$, then

$$\frac{1}{|Q|} \int_Q w(x)\, dx \le [w]_{A_1} \inf_{x \in Q} w(x)$$

for all cubes Q. We will use a classical result on the Muckenhoupt weights.

Lemma 2.3 (Chapter 7 in [4], Chapter 9 in [5])**.** *Let $w \in A_1$. We write $w(Q) := \int_Q w(x)\, dx$ for a cube Q. Then the reverse Hölder inequality holds, that is, there exist positive constants $q > 1$ and C depending on n and $[w]_{A_1}$ such that for all cubes Q,*

$$\left(\frac{1}{|Q|} \int_Q w(x)^q\, dx \right)^{1/q} \le \frac{C}{|Q|} w(Q).$$

3. The main result

Ho [6, Theorem 2.3] has proved the following characterization of the BMO norm via Banach function spaces.

Theorem 3.1. *Let X be a Banach function space. If the Hardy–Littlewood maximal operator M is bounded on the associate space X', then there exist positive constants $C_1 \le C_2$ such that for all $b \in \mathrm{BMO}(\mathbb{R}^n)$,*

$$C_1 \|b\|_{\mathrm{BMO}} \le \sup_{Q:\text{cube}} \frac{1}{\|\chi_Q\|_X} \|(b - b_Q)\chi_Q\|_X \le C_2 \|b\|_{\mathrm{BMO}}. \tag{3.1}$$

Ho's proof is based on the theory of Hardy spaces. We will give another proof of Theorem 3.1 by virtue of the Rubio de Francia algorithm.

Proof. Below we take $b \in \mathrm{BMO}(\mathbb{R}^n)$ arbitrarily and write $f := b - b_Q$ for any cube Q.

We first prove the left hand side inequality in (3.1). Using Lemma 2.2 and Remark 2.1, we get for all cubes Q:

$$\begin{aligned} \frac{1}{|Q|} \int_Q |f(x)|\, dx &\le \frac{1}{|Q|} \|f\chi_Q\|_X \|\chi_Q\|_{X'} \\ &\le C \cdot \frac{1}{\|\chi_Q\|_X} \|f\chi_Q\|_X, \end{aligned}$$

where $C > 0$ is a constant independent of b and Q. This shows the desired estimate.

Next we prove the right hand side inequality in (3.1). Our idea is based on [2, Proof of Lemma 3.3]. Take $g \in X'$ with $\|g\|_{X'} \le 1$. Let $B := \|M\|_{X' \to X'}$ and define a function

$$Rg(x) := \sum_{k=0}^{\infty} \frac{M^k g(x)}{(2B)^k} \quad (g \in X'), \tag{3.2}$$

where

$$M^k g := \begin{cases} |g| & (k=0), \\ Mg & (k=1), \\ M(M^{k-1}g) & (k \ge 2). \end{cases}$$

For every $g \in X$, the function Rg satisfies the following properties:

(1) $|g(x)| \le Rg(x)$ for almost every $x \in \mathbb{R}^n$.
(2) $\|Rg\|_{X'} \le 2\|g\|_{X'}$, namely the operator R is bounded on X'.
(3) $M(Rg)(x) \le 2BRg(x)$, that is, Rg is a Muckenhoupt A_1 weight.

By Lemma 2.3, there exist positive constants $q > 1$ and C independent of g such that for all cubes Q,

$$\left(\frac{1}{|Q|} \int_Q Rg(x)^q \, dx \right)^{1/q} \le \frac{C}{|Q|} Rg(Q).$$

By virtue of the generalized Hölder inequality, we obtain

$$\begin{aligned} \|(Rg)\chi_Q\|_{L^q(\mathbb{R}^n)} &= |Q|^{1/q} \left(\frac{1}{|Q|} \int_Q Rg(x)^q \, dx \right)^{1/q} \\ &\le |Q|^{1/q} \cdot \frac{C}{|Q|} Rg(Q) \\ &\le C\,|Q|^{-1/(q')} \|Rg\|_{X'} \|\chi_Q\|_X \\ &\le C\,|Q|^{-1/(q')} \|\chi_Q\|_X. \end{aligned}$$

Then we have

$$\begin{aligned} \int_Q |f(x)g(x)| \, dx &\le \int_Q |f(x)| Rg(x) \, dx \\ &\le \|f\chi_Q\|_{L^{q'}(\mathbb{R}^n)} \|(Rg)\chi_Q\|_{L^q(\mathbb{R}^n)} \\ &\le C \left(\frac{1}{|Q|} \int_Q |f(x)|^{q'} \, dx \right)^{1/(q')} \|\chi_Q\|_X \\ &\le C\,\|b\|_{\mathrm{BMO}} \|\chi_Q\|_X. \end{aligned}$$

By Lemma 2.1 we get

$$\begin{aligned} \|(b - b_Q)\chi_Q\|_X &= \|f\chi_Q\|_X \\ &\le C \sup_g \left\{ \left| \int_Q f(x)g(x) \, dx \right| \, : \, g \in X', \, \|g\|_{X'} \le 1 \right\} \\ &\le C\,\|b\|_{\mathrm{BMO}} \|\chi_Q\|_X. \end{aligned}$$

Consequently we have proved the theorem. □

Acknowledgment

The author is very grateful to the referee whose comments improved the manuscript.

References

[1] C. Bennett and R. Sharpley, Interpolation of operators, Pure and Applied Mathematics, 129. Academic Press, New York, 1988. MR 0928802.
[2] D. Cruz-Uribe, E. Hernández and J. M. Martell, Greedy bases in variable Lebesgue spaces, Monatsh. Math. 179 (2016), 355–378. MR 3460475.
[3] L. Diening, Maximal function on Musielak-Orlicz spaces and generalized Lebesgue spaces, Bull. Sci. Math. 129 (2005), 657–700. MR 2166733.
[4] J. Duoandikoetxea, Fourier analysis, Graduate Studies in Math. 29, Amer. Math. Soc., Providence, RI, 2001. MR 1800316.
[5] L. Grafakos, Modern Fourier analysis. Second edition, Graduate Texts in Mathematics 250, Springer, New York, 2009. MR 2463316.
[6] K.-P. Ho, Atomic decomposition of Hardy spaces and characterization of BMO via Banach function spaces, Anal. Math. 38 (2012), 173–185. MR 2958353.
[7] M. Izuki, Boundedness of commutators on Herz spaces with variable exponent, Rend. Circ. Mat. Palermo (2) 59 (2010), 199–213. MR 2670690.
[8] M. Izuki, Remarks on Muckenhoupt weights with variable exponent, J. Anal. Appl. 11 (2013), 27–41. MR 3221846.
[9] M. Izuki and Y. Sawano, Variable Lebesgue norm estimates for BMO functions, Czechoslovak Math. J. 62 (2012), 717–727. MR 2984631.
[10] M. Izuki, Y. Sawano and Y. Tsutsui, Variable Lebesgue norm estimates for BMO functions. II, Anal. Math. 40 (2014), 215–230. MR 3240224.
[11] F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math. 14 (1961), 415–426. MR 0131498.
[12] O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, Czechoslovak Math. J. 41 (1991), 592–618. MR 1134951.
[13] J. L. Rubio de Francia, Factorization and extrapolation of weights, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 393–395. MR 0663793.
[14] J. L. Rubio de Francia, A new technique in the theory of A_p weights, Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, 1982), 571–579, Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983. MR 0748875.
[15] J. L. Rubio de Francia, Factorization theory and A_p weights, Amer. J. Math. 106 (1984), 533–547. MR 0745140.

M. Izuki
Department of Mathematics Education, Faculty of Education, Okayama University
Tsushima-naka 3-1-1, Okayama, 700-8530, Japan
izuki@okayama-u.ac.jp

Received: January 13, 2015
Accepted: July 24, 2015