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THE MODULI SPACE OF PRINCIPAL SPIN BUNDLES

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Abstract. We study the geometry of the moduli space of principal Spin bundles through the study of the subvarieties of fixed points of automorphisms of finite order coming from outer automorphisms of the structure group. We describe the nonstable and the singular locus of the moduli space, prove that nonstable bundles are singular points, and identify the fixed points, in the case of $M(\mathrm{Spin}(8,\mathbb{C}))$, in the stable or strictly polystable locus of the moduli space.

Introduction

Let X be a complex algebraic curve of genus $g \geq 2$ and let G be a complex reductive Lie group. A principal G-bundle over X is a fibre bundle $E \to X$ together with a holomorphic right action $E \times G \to E$ such that G preserves the fibres of E and acts freely and transitively on them. If H is a real group, then a principal H-bundle is just a principal bundle whose structure group is the complexification $H^{\mathbb{C}}$ of H. In [17], Ramanathan gave appropriate notions of stability, semistability and polystability for principal G-bundles over X in order to construct the moduli space of polystable principal G-bundles, $M(G)$. The moduli space $M(G)$ is a complex algebraic variety of dimension $\dim G(g-1)+1$ having the subset of stable and simple elements, $M_s(G)$, as an open subset of smooth points.

Principal bundles are of interest in many areas of geometry and physics. For example, for $G = \mathrm{U}(n)$, the unitary group, the moduli space $M(\mathrm{U}(n))$ of semistable vector bundles can be identified by the theorem of Narasimhan and Seshadri [14] with the space $\mathcal{R}(G) = \mathrm{Hom}^{\mathrm{red}}(\pi_1(X), G)/G$, the quotient of the reductive representations of the fundamental group of X in G modulo the action of G by conjugation. These spaces have a very rich topology and geometry and are studied in many areas with several applications.

The geometric study of some families of equations coming from theoretical physics (like Yang-Mills equations or Hitchin equations) resulted in the appearance of pairs of different types having principal bundles as a foundation. Higgs

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bundles, for example, were introduced by Hitchin in [11]. These are pairs of a principal G-bundle together with a holomorphic global section of the adjoint bundle twisted by the canonical bundle. From the point of view of gauge theory, G-Higgs bundles are in correspondence with pairs of a connection together with a differential $(1,0)$-form satisfying Hitchin equations. Moreover, Hitchin gave a notion of stability obtaining that the moduli space of polystable G-Higgs bundles is a complex variety and indeed provides the framework for the generalization of the theorem of Narasimhan and Seshadri to complex reductive Lie groups, as a consequence of theorems by Hitchin [11], Simpson [21, 22, 23], Donaldson [5] and Corlette [4].

A way to explore the moduli space $M(G)$ is to study the subvarieties and automorphisms of $M(G)$, in the spirit of Serman [20] and Kouvidakis and Pantev [13]. Given an automorphism of $M(G)$, it is natural to consider the subvariety of fixed points in $M(G)$ of the automorphism. If G is a simple complex Lie group, then the symmetries of the Dynkin diagram provide us of outer automorphisms of G which, as we will see, induce automorphisms of finite order of the moduli space. In [9], García-Prada studied the moduli space of $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles from this point of view and, in [1], we provided a complete description of the action of $\mathrm{Out}(G)$ in $M(G)$, which we recall here to study the geometry of the subvarieties of fixed points in $M(\mathrm{Spin}(8,\mathbb{C}))$ for involutions and automorphisms of order three coming from the symmetries of the Dynkin diagram D_4. In [1], we studied a family of Spin-groups which admit an outer automorphism of order three (called triality when $G = \mathrm{Spin}(8,\mathbb{C})$) and identified the fixed points in $M(\mathrm{Spin}(n,\mathbb{C}))$ for the action of this automorphism. Here, we extend this study to involutions, simplify some proofs given for triality and study the way in which these fixed points fall in the moduli space for the case $G = \mathrm{Spin}(8,\mathbb{C})$. In particular, we prove that fixed points of the announced type are always singular points and identify the component of the singular locus to which they belong.

To do that, we make a comprehensive study of the geometry of $M(\mathrm{Spin}(n,\mathbb{C}))$ when n is even and of the group of automorphisms of a given Spin-bundle. We give a description of the components of the nonstable locus of $M(\mathrm{Spin}(n,\mathbb{C}))$ in terms of hyperbolic bundles and prove that this nonstable locus is contained in the singular locus of $M(\mathrm{Spin}(n,\mathbb{C}))$ for $n \geq 4$ and $g \geq 3$. We further the study of the singular locus in $M(\mathrm{Spin}(8,\mathbb{C}))$ proving that stable bundles which are singular points of the moduli space are exactly those that admit nontrivial automorphisms. We also prove that the nonstable locus of $M(\mathrm{Spin}(8,\mathbb{C}))$ consists of four irreducible components which we describe in terms of hyperbolic bundles. Then, we identify a family of finite order automorphisms of $M(\mathrm{Spin}(n,\mathbb{C}))$ coming from an action of the group of outer automorphisms of the group, $\mathrm{Out}(G)$, in the moduli space, which we also define. Finally, we prove that fixed points in $M(\mathrm{Spin}(8,\mathbb{C}))$ of triality and outer involutions are exactly reductions of the structure group to the subgroup of fixed points in $\mathrm{Spin}(8,\mathbb{C})$ of the corresponding outer automorphisms and that these subvarieties of fixed points fall, in the case of triality, in the singular locus and, indeed, all in the same component. We will see that three of the four irreducible components of the nonstable locus of $M(\mathrm{Spin}(8,\mathbb{C}))$ are composed of fixed points for an outer involution of $\mathrm{Spin}(8, \mathbb{C})$ and that there are also stable Spin-bundles fixed of that involution. Moreover, we identify the fixed points of the triality automorphism as an irreducible component of the subvariety of strictly polystable fixed points of the involution, so we prove that fixed points of triality are always nonstable and fixed by the outer involutions of the moduli space.

This paper is organized as follows. In Section 1 we recall some preliminary notions about the group Spin which will be relevant for us. In Section 2, we study stability and semistability in the particular case of principal Spin and SO-bundles and in Section 3 we describe polystability for these bundles in terms of hyperbolic bundles. We will see in Section 4 that the context provided by hyperbolic bundles is an appropriate point of view in order to study nonstable locus of $M(\mathrm{Spin}(n, \mathbb{C}))$ and in Sections 5 and 6 we prove that the singular locus of this moduli space consists of the nonstable locus together with the set of stable bundles which admit nontrivial automorphisms. Finally, Section 7 is devoted to the study of the automorphisms of finite order of $M(\mathrm{Spin}(8, \mathbb{C}))$ coming from outer automorphisms of the structure group.

1. Preliminaries on the Spin group

The main group in this work will be $G = \mathrm{Spin}(n, \mathbb{C})$, with n even and $n \geq 4$. This is the simply connected simple complex group with Dynkin diagram of type $D_{\frac{n}{2}}$. Let us establish some steps in the construction of this group which will play a role in our study. It can be constructed by starting with an n-dimensional quadratic space (V, q), by defining first the even Clifford group $\Gamma^+(q)$ to be the subgroup of invertible elements s in the even Clifford algebra $\mathrm{Cl}^+(q)$ which satisfy $sVs^{-1} \subseteq V$ (for details, see [8, Chapter 20] and [1]). One can easily see that there is a canonically defined anti-automorphism $*$ of the Clifford algebra $\mathrm{Cl}(q)$ which restricts to the identity map on V. For every $s \in \Gamma^+(q)$, the element $\mathrm{Nm}(s)$ defined by $s \cdot s^*$ is always a nonzero complex number, called the norm of the element s. It is clear that, so defined, the map Nm defines a homomorphism of groups $\mathrm{Nm} : \Gamma^+(q) \to \mathbb{C}^*$, whose kernel is defined to be the Spin group associated to the quadratic form q, and is denoted by $\mathrm{Spin}(q)$. The norm Nm is also defined in the whole Clifford group $\Gamma(q)$, giving rise to a homomorphism of groups $\mathrm{Nm} : \Gamma(q) \to \mathbb{C}^*$. The kernel of this homomorphism is defined to be the Pin-group. It is denoted by $\mathrm{Pin}(q)$. The group $\mathrm{Pin}(q)$ is a double covering of $O(q)$ and has $\mathrm{Spin}(q)$ as a normal subgroup.

For any complex Lie group G, given an automorphism of G it induces a permutation of the centre, Z, of G. So there is an action of the group of automorphisms of G, $\mathrm{Aut}(G)$, on the centre of G. Hence, we get a homomorphism of $\mathrm{Aut}(G)$ into the group $S(Z^2)$ of permutations of the set Z^2 of elements of order 2 in Z. The subgroup $\mathrm{Int}(G)$ acts trivially on Z, and so we get an induced homomorphism of $\mathrm{Out}(G)$ into the permutation group $S(Z^2)$, which is isomorphic to S_3 in the case of $\mathrm{Spin}(n, \mathbb{C})$ when n is even. This homomorphism is always injective and it is actually an isomorphism.

Let us assume now that $n = 8$, so $G = \mathrm{Spin}(8, \mathbb{C})$. Then there are three mutually non isomorphic irreducible representations of G of dimension 8 each (see [8, Chapter 20]). In each of these, exactly one nontrivial element of Z^2 acts trivially. Observe that in this case $Z = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and, then, Z^2 has three elements. There is a canonical bijection of Z^2 with that set of 8-dimensional irreducible representations which assigns to a representation of G the unique element of Z^2 fixed by that representation.

According to the Dynkin diagram D_4, we have also a canonical bijection of the set of outer involutions of G and Z^2. Thus we have bijections of the set of elements of order 2 in Z, the set of involutions in $\mathrm{Out}(G)$ and the set of isomorphism classes of irreducible 8-dimensional representations of G. Indeed, for each $z \in Z^2$, there exists a unique 8 dimensional irreducible representation ρ_z of G and a unique outer automorphism of order 2, σ_z, such that $\rho_z(z) = 1$ and $\sigma_z(z) = z$.

The representation ρ_z (with associated 8-dimensional complex vector space V_z) descends to a representation

$$\mathrm{Spin}(8, \mathbb{C})/\langle 1, z\rangle \cong \mathrm{SO}(8, \mathbb{C}) \to \mathrm{GL}(V_z),$$

so it induces an invariant nondegenerate quadratic form q_z on V_z and so a short exact sequence of groups

$$1 \to \langle 1, z\rangle \cong \mathbb{Z}_2 \to \mathrm{Spin}(q_z) \to \mathrm{SO}(q_z) \to 1.$$

The involution σ_z fixes z, so it also induces an automorphism of $\mathrm{SO}(q_z)$. This is an outer automorphism of $\mathrm{SO}(q_z)$ because $\sigma_z(-1) = -1$, and it is obviously an involution. It corresponds to a symmetry which fixes a subspace of odd rank (symmetries which fix even rank subspaces are inner). We may suppose that σ_z is a symmetry which fixes a subspace of V_z of rank 1 or 3. Thus outer involutions of $\mathrm{SO}(8, \mathbb{C})$ which lift to outer involutions of $\mathrm{Spin}(8, \mathbb{C})$ correspond, then, to symmetries which leave invariant a subspace of rank 1 or 3. This says that there are two lifts of each σ_z by the morphism $\mathrm{Aut}(\mathrm{SO}(8, \mathbb{C})) \to \mathrm{Out}(\mathrm{SO}(8, \mathbb{C}))$.

Let τ be an element of order three in $\mathrm{Out}(G)$. This is called the triality automorphism. The group $\mathrm{Out}(G)$ is isomorphic to S_3, the group of permutations of three elements, so there are two such automorphisms, τ and $\tau^{-1} = \tau^2$. Let z_1, z_2, z_3 be the three elements of order two in Z. We may assume that they satisfy $\tau(z_1) = z_2$, $\tau(z_2) = z_3$, $\tau(z_3) = z_1$. The election of this order induces an election of an element $z \in Z^2$ of order 2 (say $z = z_1$) and, then, a representation space V and a quadratic form q on V as above. The automorphism τ induces then an outer automorphism of $G/Z \cong \mathrm{PSO}(q)$ of order 3. There must exist subspaces of rank 2, V_1, V_2, V_3 and V_4 such that τ interchanges V_1, V_2 and V_3 and leaves V_4 invariant. This says that triality induces an automorphism of the subgroup of $\mathrm{GL}(2, \mathbb{C})^4$ consisting of matrices $\{(A, B, C, D)\}$ all of whose determinants are the same. There is a natural homomorphism of this group into $\mathrm{SO}(8, \mathbb{C})$, and so into $\mathrm{PSO}(8, \mathbb{C})$, given by

$$(A, B, C, D) \mapsto (A \otimes B^*) \oplus (C \otimes D^*).$$

The kernel of this map is the group of diagonal scalars and the image of $(-1,-1,-1,1)$ is an element of order 2, a, in the conjugacy class. Actually, this group is a subgroup of $\mathrm{Spin}(8,\mathbb{C})$ and is the centralizer of a. Triality acts on this subgroup as we have explained above.

We will now study fixed points in $\mathrm{Spin}(8,\mathbb{C})$ of the triality automorphism. We can establish first some general notions which work for general semisimple Lie groups. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. We denote by $\mathrm{Int}(\mathfrak{g})$ the normal subgroup of $\mathrm{Aut}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$. The group $\mathrm{Out}(\mathfrak{g})$ of outer automorphisms is defined by $\mathrm{Out}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$. We have the following exact sequence of groups:

$$1 \longrightarrow \mathrm{Int}(\mathfrak{g}) \longrightarrow \mathrm{Aut}(\mathfrak{g}) \longrightarrow \mathrm{Out}(\mathfrak{g}) \longrightarrow 1.$$

We have similar definitions for a group G. It is well-known that, if G is the simply connected complex Lie group with Lie algebra $\mathfrak{g}$, then there is a natural isomorphism of short exact sequences of the form

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Int}(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & \mathrm{Out}(G) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Int}(\mathfrak{g}) & \longrightarrow & \mathrm{Aut}(\mathfrak{g}) & \longrightarrow & \mathrm{Out}(\mathfrak{g}) & \longrightarrow & 1. \end{array}$$

This says that we can speak, indistinctly, of automorphisms of G and automorphisms of $\mathfrak{g}$.

The following equivalence relation on $\mathrm{Aut}(\mathfrak{g})$ will also be relevant for us.

Definition 1.1. If $\alpha,\beta \in \mathrm{Aut}(\mathfrak{g})$, we say that $\alpha \sim_i \beta$ if there exists $\theta \in \mathrm{Int}(\mathfrak{g})$ such that $\alpha = \theta \circ \beta \circ \theta^{-1}$.

The relation defined above does not define outer automorphisms, but it is easily seen that, for $\alpha,\beta \in \mathrm{Aut}(\mathfrak{g})$, if $\alpha \sim_i \beta$, then α and β define the same element in $\mathrm{Out}(\mathfrak{g})$ (for details, see [1]). This says that the obvious map

$$\mathrm{Aut}(\mathfrak{g})/\sim_i \to \mathrm{Out}(\mathfrak{g}) \tag{1}$$

is well defined.

We consider now, for $j \geq 0$,

$$\mathrm{Aut}_j(\mathfrak{g}) = \{\alpha \in \mathrm{Aut}(\mathfrak{g}) : \alpha \text{ is of order } j\}.$$

An analogous definition for $\mathrm{Out}_j(\mathfrak{g})$,

$$\mathrm{Out}_j(\mathfrak{g}) = \{\alpha \in \mathrm{Out}(\mathfrak{g}) : \alpha \text{ is of order } j\}$$

and for $(\mathrm{Aut}(\mathfrak{g})/\sim_i)_j$. It is clear that the order of an automorphism of $\mathfrak{g}$ coincides with the order of its class modulo $\sim_i$. Then

$$\mathrm{Aut}_j(\mathfrak{g})/\sim_i = (\mathrm{Aut}(\mathfrak{g})/\sim_i)_j\,.$$

It is clear that, via (1), automorphisms of order $j = 2,3$ are sent to elements of $\mathrm{Out}(\mathfrak{g})$ of order j or to the identity. This says that $\mathrm{Aut}_j(\mathfrak{g})/\sim_i$ is sent onto $\mathrm{Out}_j(\mathfrak{g}) \cup \{1\}$ via the natural map if $j \in \{2,3\}$, that is,

$$\mathrm{Aut}_j(\mathfrak{g})/\sim_i \to \mathrm{Out}_j(\mathfrak{g}) \cup \{1\}, \quad j = 2,3.$$

We will consider this map for $j = 3$, that is,

$$\mathrm{Aut}_3(\mathfrak{g})/\sim_i \to \mathrm{Out}_3(\mathfrak{g}) \cup \{1\}. \tag{2}$$

Take now $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ and $G = \mathrm{Spin}(n, \mathbb{C})$, the simply connected complex Lie group with Lie algebra $\mathfrak{g}$, for n and even integer number with $n \geq 4$. As we have explained above, $\mathrm{Out}(\mathfrak{g}) \cong S_3$. So, there are two outer automorphisms of order three, say $\mathcal{T}$ and its inverse $\mathcal{T}^2 = \mathcal{T}^{-1}$. When $n = 8$, $\mathcal{T}$ is the triality automorphism, τ, explained before. For this, we will also call $\mathcal{T}$ the triality automorphism. There are also three outer involutions obtained from one of them, say $\mathcal{S}$, by composition with $\mathcal{T}$ and $\mathcal{T}^2$.

In general, there are as many involutions of $\mathfrak{g}$ as lifts of $\mathcal{S}$ by (2). In [25, Theorem 5.10] it is proved that, for $n = 8$, there is only one lift of each $\mathcal{S}$, and its subgroup of fixed points is isomorphic to a Cartan real form of $\mathrm{Spin}(8, \mathbb{C})$: $\{1\} \oplus B_7$ if $\mathcal{S}$ is a symmetry over a subspace of complex rank 1 and $B_1 \oplus B_6$ if it is a symmetry over a subspace of rank 3.

There are also as many automorphisms of order three of $\mathfrak{g}$ with the same subgroup of fixed points as lifts of $\mathcal{T}$ by the map (2). In [1] it is proved that the number of elements in the pre-image of $\mathcal{T}$ by the map (2) is equal to $\sqrt{n+1} - 1$ if $n+1$ is a perfect square (which occurs if and only if $n = 4k^2 + 4k$ for some $k \in \mathbb{N}$); 1 otherwise (always with $n \equiv 0 \mod 8$).

In the case in which $n + 1$ is not a perfect square, there is only one possibility for the subalgebra of fixed points of an outer automorphism of order three of $\mathfrak{g}$. In [19] it is proved that the dimension of this subalgebra is 14 and it must be $\mathrm{Fix}(\mathcal{T}) \cong \mathfrak{g}_2$. In terms of the group, this says that $\mathrm{Fix}(\mathcal{T}) \cong G_2$, where $\mathcal{T}$ is seen as an automorphism of G. If $n + 1$ is a perfect square, we obtain that one of the $2k$ lifts of $\mathcal{T}$ by the map (2) has $\mathfrak{g}_2$ as subalgebra of fixed points. So we have the following result.

Proposition 1.1. *There is an outer automorphism of order three of the group $G = \mathrm{Spin}(n, \mathbb{C})$ with subgroup of fixed points isomorphic to G_2. If $n + 1$ is not a perfect square, then G_2 is the subgroup of fixed points of every outer automorphism of order three of G.*

In [25, Theorem 5.5], Wolf and Gray proved that in the case in which $n = 8$ (that is, for $G = \mathrm{Spin}(8, \mathbb{C})$), the two lifts of the map (2) have as subalgebras of fixed points $\mathfrak{g}_2$ and $\mathfrak{a}_2$ (with simply connected subgroups G_2 and $\mathrm{PSL}(3, \mathbb{C})$).

2. Stability of SO and Spin-principal bundles

Hereafter, we will deal with principal bundles over a complex algebraic curve, X. We recall that there is a general notion of semistability and stability for principal bundles with reductive structure group (see [17], [18] or [10]) defined in terms of the reductions of structure group of the bundle to parabolic subgroups of the structure group. In particular, this general notion reduces, in the case of orthogonal bundles, to the following ([15, Definition 4.1]).

Definition 2.1. An orthogonal bundle E with $\mathrm{rk}\, E > 2$ is stable (resp. semistable) if every isotropic subbundle of E has negative (resp. nonpositive) degree.

In [15, Proposition 4.2] it is proved that an orthogonal bundle is semistable if and only if it is semistable as a vector bundle. But the analogue does not hold for stable bundles. In fact we have the following ([15, Proposition 4.5]).

Proposition 2.1. *An orthogonal bundle E is stable if and only if it is an orthogonal sum of orthogonal bundles $E_1, \dots, E_r$, where each E_i is stable as a vector bundle and all of the $E_i, i = 1, \dots, r$ are mutually nonisomorphic.*

An orthogonal bundle may be thought of as a vector bundle V with nondegenerate quadratic forms q on the fibres. Its determinant bundle comes with a canonical orthogonal structure and is in particular a line bundle of order two. We have from Section 1 that the group $\mathrm{Spin}(q)$ is a double cover of the group $\mathrm{SO}(q)$. This says that every principal $\mathrm{Spin}(q)$-bundle E can be seen as a principal $\mathrm{SO}(q)$-bundle, say E_{SO}, and that this E_{SO} admits two different lifts to principal $\mathrm{Spin}(q)$-bundles.

For a group G we denote by $\underline{G}$ the sheaf of functions with values in G (see [24, Chapter II]). Isomorphism classes of principal G-bundles over X are parametrized by the first cohomology group of the sheaf $\underline{G}$, $H^1(\underline{G})$. Suppose that n is even. The natural exact sequence of sheaves

$$1 \to \underline{\mathrm{SO}(n,\mathbb{C})} \to \underline{\mathrm{O}(n,\mathbb{C})} \to \underline{\mathbb{Z}_2} \to 1$$

induces a long exact sequence in cohomology groups from which we see that an orthogonal bundle lifts to a special orthogonal bundle if and only if its determinant bundle is trivial. In fact, since the group $H^0(\mathbb{Z}_2)$ acts on $H^1(\underline{\mathrm{SO}(n,\mathbb{C})})$ transitively, for each orthogonal bundle E with trivial determinant bundle there are exactly two $\mathrm{Spin}(n,\mathbb{C})$-structures. Moreover, these structures are in natural bijection with the orthogonal isomorphisms of $\det(E)$ with the trivial line bundle, $\mathcal{O}$.

Now, the homomorphism of the Clifford group $\Gamma(n)$ onto $\mathrm{SO}(n,\mathbb{C})$ explained in Section 1 has $\mathbb{C}^*$ as kernel. This induces the following exact sequence of sheaves of groups.

$$1 \to \underline{\mathcal{O}^*} \to \underline{\Gamma(n)} \to \underline{\mathrm{SO}(n,\mathbb{C})} \to 1$$

From the induced long exact sequence of cohomology groups it follows that the obstruction for an $\mathrm{SO}(n,\mathbb{C})$-bundle to be liftable to a $\Gamma(n)$-bundle is an element of $H^2(\mathcal{O}^*) = 0$. This says that any special orthogonal bundle can be lifted to a Clifford bundle. Now, the norm $\mathrm{Nm} : \Gamma(n) \to \mathbb{C}^*$, whose kernel is exactly the group $\mathrm{Pin}(n)$, induces an exact sequence of sheaves

$$1 \to \underline{\mathrm{Pin}(n)} \to \underline{\Gamma(n)} \to \mathcal{O}^*,$$

and, then, a morphism $H^1(\Gamma(n)) \to H^1(\mathcal{O}^*)$. Moreover, the Picard group of line bundles, $H^1(\mathcal{O}^*)$, acts on the set of isomorphisms classes of Clifford bundles via the action of $\mathbb{C}^*$ on $\Gamma(n)$. Denote this action by $(L, E) \mapsto L \star E$. Any lift E of an $\mathrm{SO}(n,\mathbb{C})$-bundle is determined up to the action of a line bundle on it. It is clear that $\mathrm{Nm}(L \star E) = \mathrm{Nm}(E) \otimes L^2$, so the degree of the lifted bundle E is defined modulo 2. In particular, the $\mathrm{SO}(n,\mathbb{C})$-bundle can be lifted to a $\mathrm{Spin}(n,\mathbb{C})$-bundle if and only if it can be lifted to a Clifford bundle with norm of even degree. If this is the case, all Spin-structures on it are obtained from one by the action of the Jacobian of order two, J_2.

In [15], appropriate notions of stability and semistability are given for principal $\mathrm{SO}(n,\mathbb{C})$-bundles, which reduce those given for general G-bundles in [17] and [18].

Definition 2.2. A special orthogonal bundle E is called stable (resp. semistable) if for every isotropic subbundle E' of E we have $\deg E' < 0$ (resp. $\deg E' \leq 0$).

The following characterization is proved in [15].

Proposition 2.2. *An orthogonal bundle is stable if and only if it is an orthogonal sum of orthogonal bundles which are stable as vector bundles and are mutually nonisomorphic.*

Denote by $\pi : \mathrm{Spin}(n,\mathbb{C}) \to \mathrm{SO}(n,\mathbb{C})$ the projection which allows us to see $\mathrm{Spin}(n,\mathbb{C})$ as a double cover of $\mathrm{SO}(n,\mathbb{C})$. As both groups have the same Lie algebra and there is a bijection between Borel subgroups and Borel subalgebras of a group (because Borel subgroups are connected), Borel subgroups of $\mathrm{Spin}(n,\mathbb{C})$ correspond exactly to Borel subgroups of $\mathrm{SO}(n,\mathbb{C})$ via π. Moreover, $\ker\pi$ is contained in every Borel subgroup of $\mathrm{Spin}(n,\mathbb{C})$, so the same is true for parabolic subgroups. From this, it is not difficult to verify from the notion of stability given in [17] for general reductive groups (see [1]) that a principal $\mathrm{Spin}(n,\mathbb{C})$-bundle E is stable (resp. semistable, polystable) if and only if the corresponding $\mathrm{SO}(n,\mathbb{C})$-bundle is so. Then we can define stability for principal Spin-bundles in this way.

Definition 2.3. A $\mathrm{Spin}(n,\mathbb{C})$-bundle is stable (resp. semistable) if the associated $\mathrm{SO}(n,\mathbb{C})$-bundle is so.

3. Hyperbolic bundles and polystability

Recall the classical notion of hyperbolic bundle.

Definition 3.1. If V is a vector bundle, then the orthogonal bundle $V \oplus V^*$ with the natural quadratic form is called the hyperbolic bundle associated to V and is denoted by $H(V)$.

A hyperbolic bundle $H(V)$ with $\mathrm{rk}\, V = n$ comes with a canonical trivialization of its determinant bundle and with a natural quadratic form, so it is canonically an $\mathrm{SO}(2n,\mathbb{C})$-bundle.

Observe that V and V^* are maximal isotropic subbundles of $H(V)$, so it is clear that $H(V)$ is semistable (resp. stable) as an special orthogonal bundle if and only if V is so as a vector bundle and $\deg V = 0$ (resp. V and V^* are not isomorphic).

If V is a stable vector bundle of rank n, then the $\mathrm{SO}(2n,\mathbb{C})$-bundle $H(V)$ admits a canonical Clifford structure as well. In fact, the natural homomorphism $\mathrm{GL}(n,\mathbb{C}) \to \mathrm{SO}(2n,\mathbb{C})$ given by

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$$

associates to the vector bundle V the special orthogonal bundle $H(V)$, so it has a canonical lift as a homomorphism $\mathrm{GL}(n,\mathbb{C}) \to \Gamma(n)$. Therefore, $H(V)$ admits a $\mathrm{Spin}(2n,\mathbb{C})$-structure if and only if $\det V$ has even degree. In this case, any $\mathrm{Spin}(2n,\mathbb{C})$-structure on $H(V)$ is given by the choice of a square root of $\det V$.

It is well-known that a vector bundle of degree 0 is polystable if and only if it is a direct sum of stable bundles of degree 0. An orthogonal bundle is polystable if it is the orthogonal direct sum of stable orthogonal bundles. With these facts, we can give the following notion of polystability.

Definition 3.2. An orthogonal bundle E is said to be polystable if it is the orthogonal sum of s stable orthogonal bundle F and a hyperbolic bundle $H(V)$ with V polystable of degree 0.

Since $H(V)$ has a Spin-structure, a bundle E as in the definition admits a Spin-structure if and only if the bundle F does.

The orthogonal sum of hyperbolic bundles is hyperbolic. On the other hand, the direct sum of stable orthogonal bundles is a stable orthogonal bundle and the direct sum of polystable vector bundles of degree 0 is polystable of degree 0. Hence, the orthogonal sum of polystable bundles as in the definition is polystable. The following result follows the spirit of Ramanan in [15, Proposition 4.2].

Proposition 3.1. *An orthogonal bundle is polystable if and only if it is polystable as a vector bundle.*

Proof. If the announced bundle is orthogonally polystable, then by definition it is isomorphic to some $H(V) \oplus F$ as above, with V and F both polystable as vector bundles. We will prove the converse by induction on the rank. Assume that an orthogonal bundle E is polystable as a vector bundle. If it does not admit an isotropic subbundle of degree 0 then it is stable as an orthogonal bundle and we have the result. Take I an isotropic subbundle of degree 0 and minimal rank (we suppose $\operatorname{rk} I > 0$). Then I is a stable vector bundle. Hence the vector bundle E is the direct sum

$$E = I^{\perp} \oplus I^{*} = I \oplus I^{\perp}/I \oplus I^{*} = H(I) \oplus \left(I^{\perp}/I\right),$$

and the subbundles $H(I)$ and $I^{\perp}/I$ are mutually orthogonal. Now, the orthogonal bundle $I^{\perp}/I$ is polystable as a vector bundle and orthogonal, so by induction assumption, it is polystable as an orthogonal bundle as well. This proves our statement. □

Recall the construction given in the preceding proof. If an orthogonal bundle is semistable, then there is a filtration of E by isotropic subbundles of degree 0

$$0 = E_0 \subset E_1 \subset \cdots \subset E_k$$

such that each of the bundles E_{i+1}/E_i is stable as a vector bundle for $i \leq k-1$ and $E_k^{\perp}/E_k$ is stable as an orthogonal bundle. This filtration is not necessarily unique, but the orthogonal bundles $G(E_i) = H(E_{i+1}/E_i)$ and $G_k = E_k^{\perp}/E_k$ are uniquely determined up to isomorphism and order. Thus the isomorphism class of the polystable orthogonal bundle

$$G(E) = H\left(\bigoplus_{i=1}^{k-1} G(E_i)\right) \oplus G_k$$

is well defined. Two semistable orthogonal bundles E and E' are said to be S-equivalent if the corresponding polystable bundles $G(E)$ and $G(E')$ are isomorphic.

4. Nonstable locus in the moduli space of Spin-bundles

The set of S-equivalence classes of semistable orthogonal (resp. special orthogonal, Spin) bundles form a reduced projective algebraic scheme called the moduli space, which we will denote $M(\mathrm{O}(n,\mathbb{C}))$ (resp. $M(\mathrm{SO}(n,\mathbb{C}))$, $M(\mathrm{Spin}(n,\mathbb{C}))$).

Since $H^2(\mathrm{SO}(n,\mathbb{C}))=0$, from the natural exact sequence of sheaves

$$1 \to \underline{\mathrm{SO}(n,\mathbb{C})} \to \underline{\mathrm{O}(n,\mathbb{C})} \to \underline{\mathbb{Z}_2} \to 1$$

we see that the induced map $H^1(\underline{\mathrm{O}(n,\mathbb{C})}) \to H^1(\underline{\mathbb{Z}_2})$ is surjective. This means that $M(\mathrm{O}(n,\mathbb{C}))$ is not irreducible.

Even $M(\mathrm{SO}(n,\mathbb{C}))$ is not irreducible. We have seen that for every orthogonal bundle, one may associate an obstruction class in $H^2(\underline{\mathbb{Z}_2})=\mathbb{Z}_2$ to be liftable to a $\mathrm{Spin}(n,\mathbb{C})$-bundle. It can be shown that each space of bundles with one of these two obstructions is irreducible. Accordingly, $M(\mathrm{O}(n,\mathbb{C}))$ consists of 2^{2g+1} components, while $M(\mathrm{SO}(n,\mathbb{C}))$ has two components. Finally, $M(\mathrm{Spin}(n,\mathbb{C}))$ is irreducible.

We have explained that if E is a principal SO-bundle and then an orthogonal bundle with trivial determinant, there are two possible identifications of $\det E$ with the trivial line bundle $\mathcal{O}$, which give rise to the two possible reductions of structure group of E to the special orthogonal group. These special orthogonal structures are equivalent if the rank n of E is odd, since $-\mathrm{id} : E \to E$ takes one of the identifications to the other. If n is even and the bundle E does not have any automorphism of determinant -1 (this is the case, for example, if E is stable as a vector bundle), then the two special orthogonal structures on E are inequivalent. Then the morphism $M(\mathrm{SO}(n,\mathbb{C})) \to M(\mathrm{O}(n,\mathbb{C}))$ is a finite surjective map onto one of the components, generically 2-sheeted if n is even. Again, the morphism $M(\mathrm{Spin}(n,\mathbb{C})) \to M(\mathrm{SO}(n,\mathbb{C}))$ is a finite surjective morphism, generically 2^{2g}-sheeted when n is even.

We will now determine the irreducible components of the nonstable locus of $M(\mathrm{Spin}(n,\mathbb{C}))$. The main ideas of the following construction can be found in [18].

For each r with $1 \leq r \leq \frac{n-3}{2}$, let $M(\mathrm{U}(r))$ be the moduli space of polystable unitary bundles of rank r and J be the Jacobian. Consider the morphisms $\det : M(\mathrm{U}(r)) \to J$ and $J \to J$ given by $L \mapsto L^2$. Let $M'(\mathrm{U}(r))$ be their fibre product over J. So we have a commutative diagram of the form

$$\begin{array}{ccc} M'(\mathrm{U}(r)) = M(\mathrm{U}(r)) \times_J J & \longrightarrow & J \\ \big\downarrow & & \big\downarrow{\scriptstyle L^2} \\ J & \xrightarrow{\det} & J \end{array}$$

We consider the morphism

$$\mathcal{F}_r : M'(\mathrm{U}(r)) \times M(\mathrm{Spin}(n-2r,\mathbb{C})) \to M(\mathrm{Spin}(n,\mathbb{C}))$$

given by $\mathcal{F}_r(V,F) = H(V) \oplus F$, the induced Spin structure on $H(V)$ being given by the square root of $\det V$ provided by the projection $M(\mathrm{U}(r)) \to J$. Denote by $N(r)$ the image of the morphism $\mathcal{F}$.

Suppose that n is even. Take $r = \frac{n}{2}$. We take two morphisms, $\mathcal{G}_1, \mathcal{G}_2 : M'(\mathrm{U}(r)) \to M(\mathrm{Spin}(n,\mathbb{C}))$ given by the maps $V \mapsto H(V)$, the latter being provided with the two Spin structures. Denote by $N(\frac{n}{2}) = \operatorname{Im}\mathcal{G}_1 \cup \operatorname{Im}\mathcal{G}_2$.

The assertion that there are morphisms such as described above follows from the fact that families of semistable bundles V_t and F_t parametrized by a space T induce the existence of a family of semistable special orthogonal bundles of the form $H(V_t) \oplus F_t$. These are all principal $\mathrm{Spin}(n,\mathbb{C})$-bundles if the bundles F_t are so and hence this induces a classifying morphism into $M(\mathrm{Spin}(n,\mathbb{C}))$. These morphisms are functorial on T and by the defining property of the coarse moduli schemes $M(\mathrm{U}(r))$ and $M(\mathrm{Spin}(n,\mathbb{C}))$, this induces a morphism as announced.

From this, we can prove the following result describing the nonstable locus in $M(\mathrm{Spin}(n,\mathbb{C}))$.

Proposition 4.1. *If n is even, the nonstable locus in $M(\mathrm{Spin}(n,\mathbb{C}))$ is the union of all the subvarieties $N(r)$, $1 \le r \le \frac{n}{2}$. In particular, the nonstable locus of $M(\mathrm{Spin}(8,\mathbb{C}))$ consists of four components.*

Proof. Any nonstable $\mathrm{Spin}(n,\mathbb{C})$-bundle is up to S-equivalence, a $\mathrm{Spin}(n,\mathbb{C}))$-bundle E with an orthogonal decomposition $E = H(V) \oplus F$, with V polystable of rank r, $1 \le r \le \frac{n}{2}$ and degree 0, and F stable orthogonal.

Assume first that $r \neq \frac{n}{2}$. The $\mathrm{SO}(n,\mathbb{C})$-structure on E is given by an orthogonal isomorphism of $\det E$ with the trivial bundle. Since $\det H(V)$ is canonically trivial, this is equivalent to an $\mathrm{SO}(n-2r,\mathbb{C})$-structure on F. Given any square root L of $\det V$, we get a lift of the $\mathrm{SO}(2r,\mathbb{C})$-structure on $H(V)$ to $\mathrm{Spin}(2r,\mathbb{C})$. From this we conclude that there is also a canonical $\mathrm{Spin}(n-2r,\mathbb{C})$-structure on F. Clearly, E is the image of the point $((V,L),F) \in M'(\mathrm{U}(r)) \times M(\mathrm{Spin}(n-2r,\mathbb{C}))$ under the morphism $\mathcal{F}_r$.

If $r = \frac{n}{2}$, it may happen that the $\mathrm{SO}(n,\mathbb{C})$-structure on $H(V)$ is not the one on E. In any case, E is in the image of one of the two copies of $M'(\mathrm{U}(\frac{n}{2}))$ in $M(\mathrm{Spin}(n,\mathbb{C}))$, so $E \in \operatorname{Im}\mathcal{G}_1 \cup \operatorname{Im}\mathcal{G}_2$.

Finally, it is easy to see that these components are distinct. □

5. Singular locus in $M(\mathrm{Spin}(n,\mathbb{C}))$

We will now parametrize semistable $\mathrm{Spin}(n,\mathbb{C})$-bundles E which are S-equivalent to a given polystable bundle of the form $H(V) \oplus F$ as above.

We will make the following genericity assumptions. Firstly let V be a stable vector bundle and F a stable orthogonal bundle. We will also assume that no two of F, V, V^* are isomorphic.

Any semistable bundle which is S-equivalent to $H(V) \oplus F$ contains V or V^*, say V^*, as an isotropic subbundle. We will denote the quadratic form on F by q and the associated bilinear form by B. Locally we may choose a suitable trivialization so that the quadratic form is given by $(w,w',f) \mapsto (w,w') + q(f)$.

Two such trivializations differ by an automorphism which leaves the Jordan–Hölder filtration $V^* \subset V^* \oplus F \subset E$ invariant. Also, this automorphism preserves the quadratic form. Hence it is given by

$$(w, w', f) \mapsto (w, b(w) + c(f) + w', a(w) + f)$$

for some $a : V \to F$, $b : V \to V^*$ and $c : F \to V^*$. The orthogonality of the homomorphism implies that $(w, w') + q(f) = (w, b(w) + c(f) + w') + q(a(w) + f)$, or, what is the same, $q(a(w)) + B(a(w), f) + (w, b(w)) + (w, c(f)) = 0$.

Taking $f = 0$, we obtain

$$q(a(w)) + (w, b(w)) = 0 \quad \forall w \in V$$

and, then,

$$B(a(w), f) + (w, c(f)) = 0 \quad \forall f \in F.$$

Let us look at the first equality. We have

$$\begin{aligned} q(a(w)) + (w, b(w)) &= B(a(w), a(w)) + (w, b(w)) \\ &= (w, a^t \cdot a(w)) + (w, b(w)) \\ &= (v, (b + a^t \cdot a)(w)) = 0. \end{aligned}$$

In other words, the map $b + a^t \cdot a : V \to V^*$ is alternating. The second condition says that $a + c^t = 0$. In particular, $c = -a^t$, so c is determined by a.

Consider the set of all transformations of the above type. This is the set of pairs (a, b) with $a \in \mathrm{Hom}(V, F)$ and $b \in \mathrm{Hom}(V, V^*)$ such that $b + a^t \cdot a$ is alternating. Define a composition law on this set by the prescription

$$(a, b) * (a', b') = (a + a', b + b' - a^t \cdot a').$$

It is easy to see that this is a group scheme $\mathcal{A}^2$ over X. The family of bundles we have in mind is parametrized by $H^1(\mathcal{A}^2)$.

Note that there is a natural homomorphism $\mathcal{A}^2 \to \mathrm{Hom}(V, F)$ mapping (a, b) on a, the kernel being the space $\bigwedge^2 V$ of alternating bilinear forms on V. This says that there is a central extension given by the exact sequence

$$0 \to \bigwedge^2 V \to \mathcal{A}^2 \to \mathrm{Hom}(V, F) \to 0.$$

We consider now the induced long exact sequence of cohomology groups. From the general theory of non abelian cohomology ([6, 7]) it is clear that the induced map $H^1(\mathcal{A}^2) \to H^1(\mathrm{Hom}(V, F))$ is surjective. Moreover, the additive group $H^1(\bigwedge^2 V)$ acts on $H^1(\mathcal{A}^2)$ freely and the above map induces a bijection of the quotient with $H^1(\mathrm{Hom}(V, F))$, so $H^1(\mathcal{A}^2)/H^1(A^2(V)) \cong H^1(\mathrm{Hom}(V, F))$.

We have parametrized the elements in the S-equivalence class of a given polystable bundle $H(V) \oplus F$. We also wish to treat $H^1(\mathcal{A}^2)$ as a variety and construct a universal orthogonal extension over it.

Any semistable vector bundle E which contains V^* as an isotropic vector bundle and is S-equivalent to $H(V) \oplus F$ is obtained from the following construction, which involves two types of extensions. First, we fix a bundle Q which is an extension of F by V^*. Second, the bundle E can be seen as an extension of of $E/Q = V$ by Q.

The first step can be constructed as a family of extensions parametrized by the vector space $T = H^1(\mathrm{Hom}(F, V^*))$. In other words, there is a universal extension on $T \times X$. Every bundle Q so obtained comes with a quadratic form pulled from F and it has V^* as its radical. The problem is to parametrize all extensions of V by Q to which the quadratic form extends as a nondegenerate one on the whole bundle E so that E becomes an special orthogonal bundle (and so it induces the natural duality $V \cong V^*$). This bundle E has always an induced Spin-structure and is clearly S-equivalent to the previous one.

We first take the extension x, which we can be seen as an element of the group $H^1(\mathrm{Hom}(F, V^*)) = H^1(\mathrm{Hom}(V, F))$, corresponding to the bundle Q. We have an exact sequence of bundles

$$0 \to V^* \to Q \to F \to 0.$$

So we have that V^* is a subbundle of Q and that there is a canonical quadratic form on Q, which comes from pulling back the one of F. Now, the bundle E is an extension of V by Q

$$0 \to Q \to E \to V \to 0,$$

so it is given by an element $y \in H^1(\mathrm{Hom}(V, Q))$.

Now, since the extension of Q by V, E, is to be an orthogonal bundle, so E must be canonically isomorphic to E^*, there should also be a dual homomorphism $E \to Q^*$. The restriction of this homomorphism to Q gives rise to the quadratic form on Q pulled back from F just mentioned. Hence this homomorphism maps Q on F and induces the identity on the quotient, V. In other words, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q & \longrightarrow & E & \longrightarrow & V & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & F & \longrightarrow & Q^* & \longrightarrow & V & & \end{array}$$

This shows that the extension $y \in H^1(\mathrm{Hom}(V, Q))$ corresponding to E is a lift of x for the following exact sequence:

$$0 \to H^1(\mathrm{Hom}(V, V^*)) \to H^1(\mathrm{Hom}(V, Q)) \to H^1(\mathrm{Hom}(V, F)) \to 0.$$

This exact sequence follows by applying the functor $\mathrm{Hom}(V, -)$ on the exact sequence

$$0 \to V^* \to Q \to F \to 0.$$

The surjectivity on $H^1(\mathrm{Hom}(V, Q)) \to H^1(\mathrm{Hom}(V, F))$ is due to the fact that H^2 is zero for coherent sheaves, because all these sheaves are over the same variety X which is a curve. The injectivity on $H^1(\mathrm{Hom}(V, V^*)) \to H^1(\mathrm{Hom}(V, Q))$ follows from the stability of the special orthogonal bundle F, being inequivalent to V, which implies $H^0(\mathrm{Hom}(V, F)) = 0$.

The vector bundle so obtained E may still not be orthogonal. We know that the bundle Q is equipped with an orthogonal structure (we can see it as a symmetric bilinear form $Q \to Q^*$ or as a quadratic form on Q) and that the induced symmetric bilinear form on Q, $Q \to Q^*$, can be extended to a homomorphism $E \to Q^*$. But, by construction, Q is a subbundle of E, so this homomorphism gives rise to a bilinear form on $E \times Q$. Our goal now is to lift this to a symmetric map $E \to E^*$.

Consider the bundle $\operatorname{Sym}^2 V^*$ of symmetric bilinear forms $V \to V^*$. Let B be the bundle of symmetric bilinear forms, $\beta : E \to E^*$, on E such that there exists a complex number λ satisfying the following: firstly, the restriction of β to Q is λ times the given form on Q; secondly, the form β maps V^* into V^* and this restriction is λ times the identity.

Consider the bundle B_Q of bilinear forms on $E \times Q$ or, what is the same, morphisms $E \to Q^*$, whose restrictions to Q coincide with λ times the given bilinear form on Q.

Consider the short exact sequence

$$0 \to \operatorname{Sym}^2 V^* \to B \to B_Q \to 0.$$

We have already given a global section of the bundle B_Q which we want to lift to a global section of B. The obstruction for this is given by an element $\gamma(y) \in H^1(\operatorname{Sym}^2 V^*)$. Moreover, as we have assumed that V is stable, $H^0(\operatorname{Sym}^2 V^*) = 0$, so this lift to a symmetric form is unique. If there exists such a lift, it is obviously nondegenerate and provide the orthogonal extension sought.

We want now to determine the lifts y for which $\gamma(y) = 0$. Since we want lifts of the same $x \in H^1(\operatorname{Hom}(V, F))$, the only choice we have is to add a term $z \in H^1(\operatorname{Hom}(V, V^*))$. We must compute $\gamma(y + z)$.

Proposition 5.1. *The obstruction $\gamma(y+z)$ corresponding to the lift $y+z$ over $x \in H^1(\operatorname{Hom}(V,F))$ with $y \in H^1(\operatorname{Hom}(V,Q))$ and $z \in H^1(\operatorname{Hom}(V,V^*))$ is given by $\gamma(y+z) = \gamma(y) + z + z^t$.*

Proof. The element z corresponds to an extension

$$0 \to V^* \to I \to V \to 0.$$

Let E be the extension of V by Q corresponding to y. Denote by $\alpha : I \to V$ and $\beta : E \to V$ the surjective morphisms considered. The extension corresponding to $y + z$ can be explicitly written down as follows. Consider the bundle

$$P = I \oplus_V E = \{(v_I, v_E) \in I \oplus E : \alpha(v_I) = \beta(v_E)\}.$$

This contains V^* as a subbundle imbedded by the map $w \mapsto (w, -w)$, by identifying V^* as a subbundle of Q (observe that $\alpha(w) = 0 = \beta(w)$ for $w \in V^*$). Take $R = P/V^*$ to get an extension

$$0 \to Q \to R \to V \to 0.$$

This is the extension corresponding to $y + z$. The map $f : E \to Q^*$ which extends the bilinear form on Q gives rise canonically to a map $g : R \to Q^*$ by defining $g(v_I, v_E) = g(v_E)$. Hence g extends the given form on Q to a homomorphism of R to Q^*.

We can now compute the obstruction for R in terms of that for E and I. Let $\{U_l\}_{l \in L}$ be an open covering with splittings $t_l : I \to V^*$ and bilinear forms β_l on

$E|_{U_l}$ which lift f. We will construct similar bilinear forms on $R|_{U_l}$. For $v_I, v'_I \in I$ and $v_E, v'_E \in E$, we set

$$\beta'_l((v_I, v_E), (v'_I, v'_E)) = (t_l(v_I), \alpha(v'_I)) + (t_l(v'_I), \alpha(v_I)) + \beta_l(v_E, v'_E).$$

If we take $(v_I, v_E) = (w, -w)$ with $w \in V^*$, we have $\beta'_l((v_I, v_E), (v'_I, v'_E)) = (w, \alpha(v'_I)) - \beta_l(w, v'_E)$. But by the property of β_l we have $\beta_l(w, v'_E) = (w, \beta(v'_E))$ for all $w \in V^*$. Since we have $\alpha(v'_I) = \beta(v'_E)$, it follows that β'_l is zero on the subbundle V^* and it induces a bilinear form on $R|_{U_l}$. Now the required obstruction is given by the Cěch cocycle $\gamma'_{lm} = \beta'_l - \beta'_m$ on $U_l \cap U_m$. By definition, it is

$$\gamma_{lm}((v_I, v_E), (v'_I, v'_E)) = ((t_l - t_m)(v_I), \alpha(v'_I)) + ((t_l - t_m)(v'_I), \alpha(v_I)) + \gamma_{lm}(v_E, v'_E),$$

where γ_{lm} is the Cěch cocycle representing $\gamma(y)$. But $t_l - t_m$ represents the extension cocycle of I, so z. This proves our assertion. □

It follows from the preceding result that if we set $y = -\frac{1}{2}\gamma(x)$, then $\gamma(x+y) = 0$ and so we have extensions of the type we were looking for. Moreover, the set of all such extensions is canonically bijective with

$$\{y' \in H^1(\mathrm{Hom}(V, V^*)) : y' = -y - \gamma(x)\} \cong H^1(\mathrm{Hom}(V^*, V)).$$

We are finally ready to construct the desired family. We first construct the universal extension, namely a bundle Ω over $T \times X$, where $T = H^1(\mathrm{Hom}(F, V^*))$ is an affine space. This gives us all possible bundles of the form Q. On this universal family we may consider the first direct image of $\mathrm{Hom}(\pi_X^*(V), \Omega)$, where $\pi_X : V \to X$ is the natural projection. This is locally free over T, thanks to the constancy of the ranks of the cohomology spaces, which parametrizes all possible extensions of type E. The subvariety of this vector bundle over T consisting of those bundles which have an orthogonal structure is given by the affine space of extensions which lie over the universal extension and satisfy the condition $\gamma = 0$. This gives rise to an affine variety which projects over $T = H^1(\mathrm{Hom}(F, V^*))$ with fibres isomorphic to $H^1(\mathrm{Hom}(V^*, V))$. All this proves the following statement.

Theorem 5.1. *Let V be a vector bundle of rank r and F a $\mathrm{Spin}(n-2r, \mathbb{C})$-bundle. Then there exists an affine variety $\mathcal{U}(V, F)$ of dimension*

$$\dim H^1(\mathrm{Hom}(F, V^*)) + \dim H^1(\mathrm{Sym}^2 V^*)$$

such that every semistable bundle containing V^* as an isotropic subbundle and S-equivalent to $H(V) \oplus F$ occurs as a unique point of $\mathcal{U}(V, F)$.

The following result will be also relevant for us later.

Proposition 5.2. *Let V a stable vector bundle of degree* 0 *and F a stable* Spin-*bundle such that V and V^* are not isomorphic and the group of automorphisms of F is trivial. Let $t \in \mathcal{U}(V, F)$ correspond to a nontrivial extension of V by F. Then the automorphism group of the corresponding semistable bundle E is trivial.*

Proof. Take an automorphism of the extension E. The bundle Q corresponding to the extension t is preserved by the automorphism. Then, as E is an extension of V by Q, V is also preserved and so is V^*. In other words, the automorphism preserves the filtration $0 \subset V^* \subset Q \subset E$. It also preserves the quotient $Q/V^* = F$. The induced map on V^* is a nonzero scalar λ, and it is $\mu = \pm 1$ on F by assumption. In view of the duality, it must be λ^{-1} on V. As V^* is an isotropic subbundle of E, it must be $|\lambda|^2 = 1$, so $\lambda = \pm 1$. Then $\lambda = \lambda^{-1}$. Finally, if $\lambda\mu = -1$, we have a direct sum decomposition $E = E' \oplus F$ where E' is an extension of V by V^*. This means that the extension t is trivial. This contradicts our assumption. So $\lambda\mu = 1$ and we have the result. □

The following is the main result of the section.

Theorem 5.2. *If n is even, $n \geq 4$ and $g \geq 3$, then the nonstable locus in $M(\mathrm{Spin}(n,\mathbb{C}))$ is contained in its singular locus.*

Proof. We have noted (Proposition 4.1) that the nonstable locus is covered by components $N(r)$, $1 \leq r \leq \frac{n}{2}$. These varieties are images of $M'(\mathrm{U}(r)) \times M(\mathrm{Spin}(n-2r,\mathbb{C}))$. What we want to prove is that the generic point of this variety goes into the singular locus.

Take $E \in M(\mathrm{Spin}(n,\mathbb{C}))$ represented by the polystable bundle $H(V) \oplus F$, with V, V^*, F satisfying the genericity assumptions. It is enough to show that it is a singular point.

Consider a point of $\mathcal{U}(V,F)$ which gives a nontrivial extension, first of F by V^*. By Proposition 5.2, its automorphism group is trivial. Hence we can construct a universal family parametrized by a smooth variety T consisting of semistable bundles whose automorphism group is trivial such that any two bundles E_t, $E_{t'}$, for $t \neq t'$ in the family are nonisomorphic. By the universal property of coarse moduli scheme, there is a morphism of T into $M(\mathrm{Spin}(n,\mathbb{C}))$, which is injective when restricted to the open subset of stable points. If the point $H(V)\oplus F$ is smooth, then the set of points at which the differential map is noninjective is a divisor. We have seen that this degeneracy locus is contained in the set of nonstable points in the family. Hence it is image of the universal family, O, of extensions given by Theorem 5.1 above. From Proposition 4.1 and Theorem 5.1, we conclude that the dimension of this degeneracy locus is not greater than

$$\dim M'(\mathrm{U}(r))(r) + \dim M(\mathrm{Spin}(n-2r,\mathbb{C})) \\ + \dim H^1(\mathrm{Hom}(F,V^*)) + \dim H^1(\mathrm{Sym}^2 V^*).$$

By the Riemann-Roch theorem we obtain that this equals

$$r^2(g-1) + 1 + \frac{(n-2r)(n-2r-1)}{2}(g-1) + (n-2r)r(g-1) + \frac{r(r-1)}{2}(g-1) \\ = \frac{(n-2r)(n-1) + r(3r-1)}{2}(g-1) + 1.$$

So, the dimension of the local moduli space being $\frac{n(n-1)}{2}(g-1)$, the codimension of the degeneracy locus is at least

$$\frac{(2n-3r-1)r}{2}(g-1) - 1.$$

As $1 \leq r \leq \frac{n}{2}$, we have that this codimension is at least

$$\left(\frac{n}{2} - 1\right)(g-1) - 1.$$

By hypothesis, $n \geq 4$ and $g \geq 3$, so this codimension is at least 1. This contradicts the fact that E is smooth, so we conclude that E is necessarily singular. □

6. Automorphisms of stable Spin-bundles

We now turn to a study of the automorphisms of a given stable $\mathrm{Spin}(8,\mathbb{C})$-bundle. We are interested in this question because, as we will see, there are stable Spin-bundles in the singular locus of $M(\mathrm{Spin}(8,\mathbb{C}))$, precisely those which admit nontrivial automorphisms. It is not true in general that a stable orthogonal or Spin-bundle has only trivial automorphisms, in contrast to the case of vector bundles ([16]), as Ramanan observed in [15]. We are also interested because the group of automorphisms of Spin-bundles plays a role in the study of fixed points for the action of triality in the moduli space.

First, we establish this general property, which is true for a complex reductive Lie group G.

Lemma 6.1. *Let G be a complex reductive Lie group and let E be a stable principal G-bundle. Then the group of automorphisms of E is naturally embedded in G.*

Proof. As the group G is semisimple, it admits faithful matrix representation, so we can see G as a group of matrices. Fix $x \in X$ and ξ in the fibre of x in E. We define the morphism of groups $f : \operatorname{Aut} E \to G$ as follows: if $a \in \operatorname{Aut} E$, then $f(a)$ will be the unique element of G such that

$$a(\xi) = \xi f(a).$$

If we take another element in the same fibre of ξ for the definition of f, then we obtain conjugate elements of G. To see this, take $\eta_1, \eta_2 \in E$ elements of the same fibre of E and, for $a \in \operatorname{Aut} E$, denote by $f_1(a), f_2(a) \in G$,

$$a(\eta_i) = \eta_i f_i(a).$$

There exists $g \in G$ such that $\eta_2 = \eta_1 g^{-1}$. Then

$$\begin{aligned} a(\eta_2) = a(\eta_1 g^{-1}) &= a(\eta_1) g^{-1} = \eta_1 f_1(a) g^{-1} \\ &= \eta_1 g^{-1} g f_1(a) g^{-1} = \eta_2 i_g(f_1(a)), \end{aligned}$$

where i_g denotes the inner automorphism of G given by g. From this, $f_2(a) = i_g(f_1(a))$ and these elements are conjugated.

Moreover, this conjugation class does not depend on the fibre. To see that, observe that, for each x, if p is an invariant polynomial of G for the adjoint representation, then the map $p(f(a)) : X \to \mathbb{C}$ is holomorphic and bounded, so constant. This proves that the conjugacy class of $f(a)$ is constant.

Take $a \in \operatorname{Aut} E$ such that $f(a) = 1$. Then as the conjugacy class of $f(a)$ is constant and the conjugacy class of the element 1 is $\{1\}$, we see that f must be the identity map. This proves that f is an injective map. Obviously, it is a morphism of groups, so, finally, it is an embedding of groups, as we wanted to see. □

Observe that the centre of the structure group G does act on the principal bundle and we regard these as trivial automorphisms.

We make now some observations concerning orthogonal bundles.

Proposition 6.1. *A stable orthogonal bundle admits no nontrivial automorphisms if and only if it is stable as a vector bundle.*

Proof. Any automorphism of an orthogonal bundle which is stable as a vector bundle is a scalar multiple of the identity and hence equal to $\pm\,\mathrm{id}$, so it is trivial. Conversely, if an orthogonal bundle is a nontrivial orthogonal sum of two subbundles, the automorphism which is 1 on one of them and -1 on the other is a nontrivial automorphism. □

Proposition 6.2. *If E is a stable orthogonal bundle, then there exists $r \geq 1$ such that $\operatorname{Aut} E \cong (\mathbb{Z}_2)^r$. In particular, every automorphism of E is of order 2 and the group of automorphisms of E is abelian.*

Proof. If E is a stable orthogonal bundle, then it is an orthogonal sum of stable vector bundles. Every automorphism of E induces an automorphism on each summand, which must be $\pm\,\mathrm{id}$. Thus the group of automorphisms of E is isomorphic to $(\mathbb{Z}_2)^r$, where r is the number of summands in the orthogonal decomposition of E into stable vector bundles. □

Observe that a stable $\mathrm{SO}(n,\mathbb{C})$ bundle may not be stable as a vector bundle but may still have a trivial group of automorphisms. For example, if n is even (which will be our case) and the bundle is an orthogonal sum of two nonisomorphic stable orthogonal bundles of odd rank, then all automorphisms of the orthogonal bundle have determinant -1, so the special orthogonal bundle has no nontrivial automorphisms.

Of course, the group of automorphisms of a stable $\mathrm{SO}(n,\mathbb{C})$-bundle is a subgroup of that of the corresponding orthogonal bundle, which is also stable. Hence the group of automorphisms of a stable $\mathrm{SO}(n,\mathbb{C})$-bundle is always isomorphic to $(\mathbb{Z}_2)^r$ for some $r \geq 1$, perhaps less than the one given in the proposition above.

Now, let E be a stable $\mathrm{Spin}(n,\mathbb{C})$-bundle. Any automorphism of E gives rise to an automorphism of the corresponding $\mathrm{SO}(n,\mathbb{C})$-bundle, say E_{SO}, inducing a homomorphism $\operatorname{Aut} E \to \operatorname{Aut} E_{\mathrm{SO}}$. The kernel of this homomorphism is clearly the kernel of the projection $\mathrm{Spin}(n,\mathbb{C}) \to \mathrm{SO}(n,\mathbb{C})$, which is isomorphic to $\mathbb{Z}_2$. So we have seen an analogous result for stable $\mathrm{Spin}(n,\mathbb{C})$-bundles.

Proposition 6.3. *If E is a stable $\mathrm{Spin}(n,\mathbb{C})$-bundle, then there exists an integer r, with $1 \leq r \leq n$, such that $\operatorname{Aut} E \cong (\mathbb{Z}_2)^r$. In particular, every automorphism of E is of order 2 and the group of automorphisms of E is abelian.*

We are now in a position to study the particular case of $\mathrm{Spin}(8,\mathbb{C})$-bundles.

Proposition 6.4. *Let E be a stable $\mathrm{Spin}(8,\mathbb{C})$-bundle and let E_{SO} be the corresponding $\mathrm{SO}(8,\mathbb{C})$-bundle. If E admits nontrivial automorphisms, then E_{SO} can be written as an orthogonal sum*

$$E_{\mathrm{SO}} = (V_1 \otimes V_2^*) \oplus (V_3 \otimes V_4^*),$$

where V_1, V_2, V_3, V_4 are stable vector bundles of rank 2 and with the same determinant bundle.

Proof. Let a be a nontrivial automorphism of E, which has order two. Lemma 6.1 says that a acts, in some fibre, as the product by an element of $\mathrm{Spin}(8,\mathbb{C})$ and a can be seen naturally as an element of $\mathrm{Spin}(8,\mathbb{C})$ (by choosing a fibre and fixing an element of the fibre). Denote by $Z(a)$ the centralizer of a in $\mathrm{Spin}(8,\mathbb{C})$, that is,

$$Z(a) = \{g \in \mathrm{Spin}(8,\mathbb{C}) : ga = ag\}.$$

Then the structure group of E can be reduced to $Z(a)$. To see this, consider

$$F = \{\chi \in E : a(\xi) = \xi f(a)^{-1}\},$$

where $f : \operatorname{Aut} E \to \mathrm{Spin}(8,\mathbb{C})$ is the natural inclusion defined in Lemma 6.1. The subvariety F of E is invariant under the action of $Z(a)$: if $b \in Z(a)$ and $\xi \in F$,

$$a(b(\xi)) = b(a(\xi)) = b(\xi)f(a)^{-1}.$$

This shows that $Z(a)$ acts on F. We can see that this action is simply transitive. Take $e_1, e_2 \in F$ in the same fibre. Then as the action of $\mathrm{Spin}(8,\mathbb{C})$ is simply transitive, there exists a unique $g \in \mathrm{Spin}(8,\mathbb{C})$ such that $e_2 = e_1 g$. Then

$$a(e_2) = a(e_1 g) = a(e_1)g = e_1 a^{-1} g,$$

But, as $e_2 \in F$,

$$a(e_2) = e_2 a^{-1} = e_1 g a^{-1},$$

so $ga^{-1} = a^{-1}g$ and $g \in Z(a)$. This proves that E admits a reduction of the structure group to $Z(a)$.

The automorphism a gives a decomposition of the bundle E_{SO} into the orthogonal direct sum of two vector bundles (in fact, orthogonal bundles), V^+ and V^-, which are the eigenspaces for the eigenvalues 1 and -1 of a, respectively (observe that a has order 2). The automorphism a acts in V^- as a change of sign, and this automorphism lifts to an automorphism of Spin-bundles in two ways. If $\dim V^- = m$ and $s = e_1, \ldots, e_m$ is an orthogonal basis of V^-, then the action of the element of the Spin group $e_1 \cdots e_m$ on an element $x \in V$ is

$$sxs^{-1} = e_1 \cdots e_m x e_m \cdots e_1.$$

Using that $e_i e_j = -e_j e_i$ we have that $sxs^{-1} = -x$ if and only if m is even and, in this case, the elements of the Spin group $\pm e_1 \cdots e_m$ lift the automorphism -1 in V^-. Then we have proved that V^+ and V^- are of even dimension, say $m = 2r$.

All this proves that, if π denotes the canonical projection of $\mathrm{Spin}(8,\mathbb{C})$ over $\mathrm{SO}(8,\mathbb{C})$,

$$\pi(Z(a)) = Z(\overline{a}) = \mathrm{S}(\mathrm{O}(2,\mathbb{C}) \times \mathrm{O}(6,\mathbb{C}))$$

or

$$\pi(Z(a)) = Z(\overline{a}) = \mathrm{S}(\mathrm{O}(4,\mathbb{C}) \times \mathrm{O}(4,\mathbb{C})).$$

An $\mathrm{SO}(2,\mathbb{C})$-bundle cannot appear in this decomposition, because it would be hyperbolic and hence not stable. Then we must have

$$\pi(Z(a)) = Z(\overline{a}) = \mathrm{S}(\mathrm{O}(4,\mathbb{C}) \times \mathrm{O}(4,\mathbb{C})).$$

We now compute explicitly the inverse image of $Z(\overline{a})$ in order to give an explicit description of $Z(a)$ (observe that, in this case, $\pi^{-1}\pi(Z(a)) = Z(a)$ because, if $g \in Z(a)$, then $-g \in Z(a)$ and $\{g, -g\} = \pi^{-1}(g)$).

The Clifford group $\Gamma(4)$ is the quotient of the subgroup $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})$ consisting of matrices (A, B) such that $\det A = \det B$ by the subgroup of scalars (λ, λ^{-1}), $\lambda \in \mathbb{C}^*$. From this, one can deduce that $Z(a)$ is the quotient of the subgroup of $\mathrm{GL}(2,\mathbb{C})^4$ of quadruples of matrices all having the same determinant, by the group of diagonal scalars.

As E admits a reduction of its structure group to $Z(\overline{a})$ and $\mathrm{Spin}(8,\mathbb{C})$ admits a restriction to $\mathrm{GL}(2,\mathbb{C})^4$ given by the representation

$$(A, B, C, D) \mapsto A \otimes B^* \oplus C \otimes D^*,$$

E decomposes as an orthogonal direct sum

$$E = (V_1 \otimes V_2^*) \oplus (V_3 \otimes V_4^*),$$

where all the V_i are stable vector bundles of rank two with the same determinant. □

From Proposition 6.4, we see that the group of orthogonal automorphisms of a stable $\mathrm{Spin}(8,\mathbb{C})$-bundle admitting nontrivial automorphisms is $\mathbb{Z}_2^2$. Then $\mathrm{Aut}\, E$ is an extension of groups of the form

$$1 \to \mathbb{Z}_2 \to \mathrm{Aut}\, E \to \mathbb{Z}_2^2 \to 1.$$

So, we have seen the following proposition.

Proposition 6.5. *If E is a stable $\mathrm{Spin}(8,\mathbb{C})$-bundle that admits nontrivial automorphisms, then $\mathrm{Aut}\, E \cong \mathbb{Z}_2^3$.*

We now conclude that any stable $\mathrm{Spin}(8,\mathbb{C})$-bundle which admits nontrivial automorphisms is a singular point of $M(\mathrm{Spin}(8,\mathbb{C}))$. Indeed, if $Z = Z(\mathrm{Spin}(8,\mathbb{C}))$, the group $\mathrm{Aut}\, E/Z$ acts in a natural way on $H^1(\mathrm{ad}\, E)$ and a formal neighborhood of E in $M(\mathrm{Spin}(8,\mathbb{C}))$ is isomorphic to the quotient of the affine space $H^1(\mathrm{ad}\, E)$ by this action. Chevalley proved in [3, Theorem B] that the quotient is smooth if and only if the action is generated by reflections in hyperplanes. All elements are of order 2 and none of them has a fixed subspace of codimension 1. Thus from this observation and Theorem 5.2, we have the following theorem.

Theorem 6.1. *The singular locus of $M(\mathrm{Spin}(8,\mathbb{C}))$ is the union of the nonstable locus and the locus of stable bundles which admit nontrivial automorphisms.*

7. Automorphisms of finite order of $M(\mathrm{Spin}(8,\mathbb{C}))$

We recall the action of the group $\mathrm{Aut}(G)$ of automorphisms of the Lie group G on the set of principal G-bundles for a complex reductive Lie group G established in [1].

Definition 7.1. Let E be a principal G-bundle. If $A \in \mathrm{Aut}(G)$ and $\{(U_i, \varphi_i)\}_i$ is a trivializing covering of E, then $\{(U_i, \mathrm{id}_{U_i} \times A \circ \varphi_i)\}_i$ is a trivializing cover of a certain principal G-bundle, where

$$\pi^{-1}(U_i) \xrightarrow{\varphi_i} U_i \times G \xrightarrow{\mathrm{id}_{U_i} \times A} U_i \times G.$$

We define $A(E)$ to be this principal G-bundle.

In fact, if $\{\psi_{ij}\}_{ij}$ is a family of transition functions of E associated to the covering given in the definition, the transition functions of the new bundle $A(E)$ associated to this covering are $\{A \circ \psi_{ij}\}_{ij}$. Observe that these functions verify the cocycle conditions.

When A is an inner automorphism, E and $A(E)$ are isomorphic. Indeed, the map $f : E \to A(E)$ defined by $f(e) = eg_0^{-1}$ is an isomorphism of principal G-bundles (see [12]). Then $\mathrm{Out}(G)$ acts on the set of isomorphism classes of principal G-bundles in the following way: if $\sigma \in \mathrm{Out}(G)$ and $A \in \mathrm{Aut}(G)$ is an automorphism of G representing σ, then $\sigma(E) = A(E)$.

Take now $G = \mathrm{Spin}(n, \mathbb{C})$ for any n. Let E be a principal $\mathrm{Spin}(n, \mathbb{C})$-bundle. Thanks to the equivalence between stability for Spin-bundles and SO-bundles established above, $A(E)$ will be stable (resp. semistable, polystable) if and only if it is so seen as an $\mathrm{SO}(n, \mathbb{C})$-bundle. Since the action of an automorphism, A, of $\mathrm{Spin}(n, \mathbb{C})$ gives rise to a bijective correspondence between isotropic subbundles of E and isotropic subbundles of $A(E)$ preserving the degrees, we have that E is stable (resp. semistable, polystable) if and only if $A(E)$ is so. We have then seen that A induces an automorphism of $M(\mathrm{Spin}(n, \mathbb{C}))$ and then $\mathrm{Out}(G)$ acts on the moduli space.

Take now $G = \mathrm{Spin}(8, \mathbb{C})$. All outer automorphisms of G have finite order (2 or 3). In order to study fixed points for the induced automorphisms of the moduli space, there are only three of such automorphisms which are relevant for us: symmetries over a 1-dimensional subspace, symmetries over a 3-dimensional subspace, and the triality automorphism.

Proposition 7.1. *Let σ be an outer involution of $\mathrm{Spin}(8, \mathbb{C})$. Let $E \in M(\mathrm{Spin}(8, \mathbb{C}))$. The bundle E is fixed for the action of σ if and only if the corresponding orthogonal bundle reduces its structure group to $S(\mathrm{O}(1, \mathbb{C}) \times \mathrm{O}(3, \mathbb{C}))$ or to $S(\mathrm{O}(3, \mathbb{C}) \times \mathrm{O}(5, \mathbb{C}))$. That is, it is of type $B_r \oplus B_{3-r}$ for $r = 0, 1$.*

Proof. As we have stated in Section 1, the involution σ can be seen as an outer involution of $\mathrm{SO}(8, \mathbb{C})$ acting as a symmetry through a rank 1 or rank 3 subspace. Take the induced decomposition of the representation vector space, $V = V_1 \oplus V_2$. The bundle E is fixed for the action of σ if and only if the cocycle $\{(\phi^1_{ij}, -\phi^2_{ij})\}_{ij}$ represents the same bundle E, $\{(\phi^1_{ij}, \phi^2_{ij})\}_{ij}$ being a family of transition functions of E. This is only possible if E admits a reduction of structure group to the subgroup of fixed points in $\mathrm{Spin}(8, \mathbb{C})$ of the symmetry σ. As we have stated, this subgroup of fixed points is one of the two groups announced. □

This gives a result in the spirit of the work by García-Prada in [9] for real involutions of $\mathrm{SL}(n, \mathbb{C})$-bundles, which gives rise to reductions of structure group to the Cartan real forms of the group. We can rephrase this result in the following way.

For each $k \leq 8$, we have the exact sequence of groups

$$1 \to \mathbb{Z}_2 \to \mathrm{Pin}(k, \mathbb{C}) \to \mathrm{O}(k, \mathbb{C}) \to 1$$

and, then, the long exact sequence of cohomology groups

$$H^1(\mathrm{Pin}(k, \mathbb{C})) \to H^1(\mathrm{O}(k, \mathbb{C})) \stackrel{d_k}{\to} \mathbb{Z}_2.$$

Analogously, for $r = 0, 1$, we have

$$H^1(\mathrm{Pin}(2r+1, \mathbb{C}) \times \mathrm{Pin}(8-2r-1, \mathbb{C})) \\ \to H^1(\mathrm{O}(2r+1, \mathbb{C}) \times \mathrm{O}(8-2r-1, \mathbb{C})) \stackrel{(d_{2r+1}, d_{2n-2r-1})}{\to} \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

It is easy to see that an element of $H^1(\mathrm{O}(2r+1, \mathbb{C}) \times \mathrm{O}(8-2r-1, \mathbb{C}))$ lifts to an element of $H^1(\mathrm{Spin}(8, \mathbb{C}))$ if and only if this element annihilates $d_{2r+1} + d_{8-2r-1}$. We denote by M_r the set

$$\{(E_1, E_2) \in M(S(\mathrm{O}(2r+1, \mathbb{C}) \times \mathrm{O}(8-2r-1, \mathbb{C}))) : d_{2r+1}(E_1) + d_{8-2r-1}(E_2) = 0\}.$$

We have a well-defined morphism $M_r \to M(\mathrm{SO}(8, \mathbb{C}))$.

Consider the projection map

$$\pi : M(\mathrm{Spin}(8, \mathbb{C})) \to M(\mathrm{SO}(8, \mathbb{C})).$$

We define

$$N'(r) = \pi^{-1}(\mathrm{Im}(M_r \to M(\mathrm{SO}(8, \mathbb{C})))).$$

We can now reformulate our result in the following way.

Proposition 7.2. *Let σ be a nontrivial outer involution of D_4. Then the subvariety of fixed points in $M(\mathrm{Spin}(8, \mathbb{C}))$ for the action of σ is*

$$N'(0) \cup N'(1).$$

The expression stated in Proposition 7.2 consists exactly of two components. One of them is given by symmetries through a rank 1 subspace and the other by symmetries through a rank 3 subspace.

It is clear that an isotropic subbundle of a fixed point in $M(\mathrm{Spin}(8, \mathbb{C})$ is necessarily a direct sum of isotropic subbundles of its decomposition of type $B_0 \oplus B_3$ or $B_1 \oplus B_2$. It is then easily seen that a fixed point so obtained is stable as a Spin-bundle if and only if it can be written as a direct sum of two stable orthogonal bundles of the convenient ranks.

For each $r = 0, 1$, denote by M_r^s the open subvariety of M_r consisting of stable points of M_r. Let $M^s(\mathrm{SO}(8, \mathbb{C})$ the open subset of stable $\mathrm{SO}(8, \mathbb{C})$-bundles. Then it is clear that the image of the map $M_r^s \to M(\mathrm{SO}(8, \mathbb{C}))$ is contained in $M^s(\mathrm{SO}(8, \mathbb{C})$. We define

$$N_s'(r) = \pi^{-1}(\mathrm{Im}(M_r^s \to M^s(\mathrm{SO}(8, \mathbb{C})))).$$

From what we have explained, $N_s'(r)$ consists of stable fixed points of σ. Every other fixed point so obtained is strictly polystable seen as a $\mathrm{Spin}(8, \mathbb{C})$-bundle because it is polystable as an orthogonal bundle. Thus these fixed points fall in the nonstable locus, and then in the singular locus, of $M(\mathrm{Spin}(8,\mathbb{C}))$. Denote by

$$N'_{ps}(r) = \pi^{-1}(\mathrm{Im}(M_r \setminus M_r^s \to M(\mathrm{SO}(8,\mathbb{C})))).$$

Finally, Proposition 7.2 can be reformulated as follows.

Proposition 7.3. *Let σ be a nontrivial outer involution of D_4. Then the subvariety of stable fixed points in $M(\mathrm{Spin}(8,\mathbb{C}))$ for the action of σ is*

$$N'_s(0) \cup N'_s(1)$$

and the subvariety of strictly polystable fixed points of σ is

$$N'_{ps}(0) \cup N'_{ps}(1).$$

Moreover, these strictly polystable fixed points are singular points of $M(\mathrm{Spin}(8,\mathbb{C}))$.

We will now determine how the polystable points fall in the nonstable locus of the moduli space.

Let E be an element of $M(\mathrm{Spin}(8,\mathbb{C}))$ fixed for σ. Then E comes from an element of $M_0 \setminus M_0^s$ or $M_1 \setminus M_1^s$. The bundle E is then of the form $E = E_1 \oplus E_2$, with E_1 and E_2 orthogonal bundles, $\mathrm{rk}\, E_1 = 1$ and $\mathrm{rk}\, E_2 = 7$, or $\mathrm{rk}\, E_1 = 3$ and $\mathrm{rk}\, E_2 = 5$. There are three possibilities:

(1) $\mathrm{rk}\, E_1 = 1$, $\mathrm{rk}\, E_2 = 7$ and E_2 strictly polystable. Then E_2 is of the form $E_2 = H(V) \oplus F$ for some V with $\mathrm{rk}\, V = k$, $1 \le k \le 3$. Then $E \in N(1) \cup N(2) \cup N(3)$.
(2) $\mathrm{rk}\, E_1 = 3$, $\mathrm{rk}\, E_2 = 5$, E_1 strictly polystable and E_2 stable. As before, $E \in N(1)$.
(3) $\mathrm{rk}\, E_1 = 3$, $\mathrm{rk}\, E_2 = 5$, E_2 strictly polystable. Then, as before, $E \in N(1) \cup N(2)$.

This says that the subvariety of strictly polystable fixed points of σ satisfies

$$N'_{ps}(0) \cup N'_{ps}(1) \subseteq N(1) \cup N(2) \cup N(3).$$

In fact, as every point in $N(1) \cup N(2) \cup N(3)$ comes with a reduction of structure group to the subgroup of fixed points of σ, it is obviously fixed by σ, so we have the equality

$$N'_{ps}(0) \cup N'_{ps}(1) = N(1) \cup N(2) \cup N(3).$$

Finally, we have seen that he whole nonstable locus of $M(\mathrm{Spin}(8,\mathbb{C}))$ is composed of fixed points of σ except for the irreducible component $N(4)$, which has no fixed points. We have finally proved the following proposition.

Proposition 7.4. *The irreducible components $N(1)$, $N(2)$, $N(3)$ of the nonstable locus of the moduli $M(\mathrm{Spin}(8,\mathbb{C}))$ are composed of fixed points of σ. Moreover, every nonstable fixed point of σ is in $N(1)\cup N(2)\cup N(3)$, so the irreducible component $N(4)$ has no fixed points of σ.*

Let us study the case of triality. We will formulate the result in an analogous way as the involutions case above and bring a shorter proof than the one presented in [1], thanks to the observations made in Section 1.

Theorem 7.1. *Let $\mathcal{T} \in \mathrm{Out}_3(\mathrm{Spin}(8,\mathbb{C}))$ be the triality automorphism. The fixed point set $M^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C}))$ is given by*

$$M^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C})) = \mathrm{Im}\,(M(G_2) \to M(\mathrm{Spin}(8,\mathbb{C}))) \\ \cup \mathrm{Im}\,(M(\mathrm{PSL}(3,\mathbb{C}))) \to M(\mathrm{Spin}(8,\mathbb{C}))) \,.$$

Proof. Let E be a principal $\mathrm{Spin}(8,\mathbb{C})$-bundle fixed for τ. Suppose first that E is stable. As E is fixed by $\mathcal{T}$, there exists an automorphism of E, $f_0 : E \to \mathcal{T}(E)$. Let $f = f_0 \circ \mathcal{T}(f_0) \circ \mathcal{T}^2(f_0)$. Then f is an automorphism of E not coming from the centre of $\mathrm{Spin}(8,\mathbb{C})$ (this is easily seen from the action of triality on the centre of $\mathrm{Spin}(8,\mathbb{C})$ explained in Section 1). Then E is a stable $\mathrm{Spin}(8,\mathbb{C})$-bundle which admits nontrivial automorphisms, so by Proposition 6.4 E is of the form

$$E = (V_1 \otimes V_2^*) \oplus (V_3 \otimes V_4^*)$$

for certain stable vector bundles of rank 2 with the same determinant bundle. The vector bundles $V_1 \otimes V_2^*$ and $V_3 \otimes V_4^*$ are stable principal $\mathrm{SO}(4,\mathbb{C})$-bundles and the direct sum is orthogonal. Moreover, the triality automorphism acts on the subgroup $\mathrm{GL}(2,\mathbb{C})^4$ of $\mathrm{Spin}(8,\mathbb{C})$ by fixing one of the components and interchanging the other three. This means that a stable fixed point for the action of $\mathcal{T}$, E is of the form $(W \otimes V) \oplus (V \otimes V)$, that is, induces a reduction of the structure group of E to the subgroup of fixed points of the triality automorphism, $\mathrm{Fix}(\mathcal{T})$. As observed in Section 1, there are two possible lifts of $\mathcal{T}$ by (2) and so two possibilities for $\mathrm{Fix}(\mathcal{T})$, which are identified in [25, Theorem 5.5]: G_2 and $\mathrm{PSL}(3,\mathbb{C})$.

This completes the case in which E is stable.

If E is polystable, then its Jordan–Hölder reduction allows us to reduce its structure group to the centralizer of a torus of $\mathrm{PSO}(8,\mathbb{C})$ (see [10] for details), this reduction being stable. It is easy to see that the centralizer of a maximal torus of $\mathrm{SO}(8,\mathbb{C})$ is of the form $\mathrm{S}(\mathrm{O}(2,\mathbb{C})^4)$, so we are in the preceding situation. □

Observe that, if E is a polystable fixed point for triality, then $E \in N(r)$ for some r with $1 \le r \le 4$ (see Proposition 4.1). Triality induces an automorphism of order three of E, so it is clear that it must be $r = 3$.

In any case, we see that the fixed points in $M(\mathrm{Spin}(8,\mathbb{C}))$ of triality are the reductions of structure group to G_2 or $\mathrm{PSL}(3,\mathbb{C})$.

The group G_2 has two irreducible representations, called the fundamental representations. These are the adjoint representation, of dimension 14, and its action on the imaginary octonions, of dimension 7. The last representation is an orthogonal representation

$$\rho : G_2 \to \mathrm{SO}(7,\mathbb{C}).$$

Via this representation, G_2 can be seen as the group of automorphisms of $\mathbb{C}^7$ which preserve a non-degenerate 3-form (see [2]). Then a principal G_2 bundle is a rank 7 complex vector bundle E over X together with a holomorphic global non-degenerate 3-form $\omega \in H^0(X, \bigwedge^3 E^*)$.

Now, the inclusion $G_2 \to \mathrm{SO}(8,\mathbb{C})$ admits a factorization through the fundamental 7-dimensional representation of G_2, which is also an orthogonal representation. All this shows that the corresponding principal $\mathrm{SO}(8,\mathbb{C})$-bundle is $E \oplus L$ for some line bundle L, the orthogonal bundle associated to E via the homomorphism of groups $G_2 \to \mathrm{SO}(7, \mathbb{C})$ stated before. Then it is clear that E is not stable. Moreover, the fundamental 7-dimensional representation of G_2 is of the form $W \oplus W^* \oplus \mathbb{C}$ (see [8, Chapter 22]) with W of rank 3. This says that E is always polystable and $E \in N(3)$, as we have seen before.

In the case of $\mathrm{PSL}(3, \mathbb{C})$, it is clear that the morphism

$$\mathrm{SL}(3, \mathbb{C}) \to \mathrm{PSL}(3, \mathbb{C}) \to \mathrm{Spin}(8, \mathbb{C}) \to \mathrm{SO}(8, \mathbb{C})$$

maps a vector bundle of rank 3, W, to an orthogonal bundle $(W \oplus L) \oplus (W^* \oplus L^*)$, L being a degree 0 line bundle. This says that the image in $M(\mathrm{Spin}(8, \mathbb{C}))$ of a $\mathrm{PSL}(3, \mathbb{C})$-bundle is always polystable and indeed contained in $N(3)$.

We have seen that every fixed point of triality is an element of $N(3)$. We will now see the converse, that is, that every $\mathrm{Spin}(8, \mathbb{C})$-bundle in $N(3)$ is fixed by $\mathcal{T}$.

Proposition 7.5. *Let $E \in N(3)$. Then $\mathcal{T}(E) = E$.*

Proof. As $E \in N(3)$, E can be seen as a bundle of the form

$$E = H(V) \oplus F,$$

where F is an orthogonal line bundle and V is a rank 3 complex vector bundle. $H(V)$ comes with the natural quadratic form q given by

$$q(x + \alpha) = \alpha(x)$$

for $x \in V$ and $\alpha \in V^*$. A homomorphism of its fibres must be of the form

$$\begin{pmatrix} f & 0 \\ 0 & f^t \end{pmatrix}$$

with $f : V \to V$ a homomorphism, or of the form

$$\begin{pmatrix} 0 & g \\ h & 0 \end{pmatrix},$$

where $g : V^* \to V$ and $h : V \to V^*$. These transformations shall preserve q, which imposes some conditions to f, g and h. In particular,

$$q(f(x) + f^t(\alpha)) = f^t(\alpha)(f(x)) = \alpha(f^2(x)),$$

so it must be $f^2 = 1$, and

$$q(h(\alpha) + g(x)) = g(x)(h(\alpha)) = g(x)(\alpha \circ h^t) = \alpha(h^t g(x)),$$

so $h^t g = 1$. It is easily seen that a transformation of the form

$$\begin{pmatrix} f & g \\ h & f^t \end{pmatrix}$$

should not be possible, because q has the form

$$q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and, then,

$$\begin{pmatrix} f & g \\ h & f^t \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} f^t & h^t \\ g^t & f \end{pmatrix} = \begin{pmatrix} gf^t + fg^t & 2 \\ 2 & f^t h^t + hf \end{pmatrix} \neq q.$$

In light of the first condition on f, $f^2 = 1$, we see that f shall be an element of $\mathrm{O}(3)$, which is naturally isomorphic to $\mathrm{SU}(2)$. From the condition on g, we see that $g : V \to$ defines a symmetric bilinear form on V, that is, an orthogonal structure, too. All this means that E reduces its structure group to $\mathrm{SU}(2)\times\mathrm{SU}(2)/\{(-1,-1)\}$, which is precisely the maximal compact subgroup of G_2. This proves that E is fixed by $\mathcal{T}$. $\square$

We may paraphrase our results in the following statement.

Theorem 7.2. *Let $\mathcal{T} \in \mathrm{Out}_3(\mathrm{Spin}(8,\mathbb{C}))$ be the triality automorphism. Let $M^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C}))$ be the set of fixed points in $M(\mathrm{Spin}(8,\mathbb{C}))$ for the action of $\mathcal{T}$. Then $M^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C})) = N(3)$. In particular, $M^{\mathcal{T}}(\mathrm{Spin}(8,\mathbb{C}))$ falls in the singular locus of $M(\mathrm{Spin}(8,\mathbb{C}))$.*

From Proposition 7.4, $N(3)$ is a component of the subset of nonstable fixed points for the action of the outer involution σ, so in particular we have seen the following proposition.

Proposition 7.6. *Every fixed point of $\mathcal{T}$ is also a nonstable fixed point of σ.*

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