REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 57, No. 2, 2016, Pages 53–62
Published online: June 8, 2016

INEQUALITIES FOR GENERALIZED δ-CASORATI CURVATURES OF SUBMANIFOLDS IN REAL SPACE FORMS ENDOWED WITH A SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. We study two sharp inequalities involving the instrinsic scalar curvature and extrinsic generalized normalized δ-Casorati curvature of submanifolds of real space forms endowed with a semi-symmetric metric connection, which are the generalization of some recent results related to the Casorati curvature for submanifolds in a real space form with a semi-symmetric metric connection, obtained by Lee et al. [*J. Inequal. Appl.* **2014**, 2014:327].

Introduction

The theory of Chen invariants (or δ-invariants), initiated by Chen ([4]) in a seminal paper published in 1993, is presently one of the most interesting research topics in differential geometry of submanifolds. Chen established a sharp inequality for a submanifold in a real space form using the scalar curvature and the sectional curvature, and the squared mean curvature. That is, he established in [5] simple relationships between the main intrinsic invariants and the main extrinsic invariants of a submanifold in real space forms with any condimensions. Many famous results concern Chen invariants and inequalities for the different classes of submanifolds in various ambient spaces, like complex space forms ([6, 7, 21]). Recently, in [18, 19], Özgür and Mihai proved Chen inequalities for submanifolds of real, complex, and Sasakian space forms endowed with semi-symmetric metric connections, and in [22, 23], Özgür and Murathan gave Chen inequalities for submanifolds of a locally conformal almost cosymplectic manifold and a cosymplectic space form endowed with semi-symmetric metric connections. Moreover, Zhang et al.[28] obtained Chen-like inequalities for submanifolds of a Riemannian manifold of quasi-constant curvature endowed with a semi-symmetric metric connection by using an algebraic approach.

2010 *Mathematics Subject Classification.* 53C40, 53B05.
Key words and phrases. Casorati curvature, real space form, semi-symmetric metric connection.

Instead of concentrating on the sectional curvature with the extrinsic squared mean curvature, the Casorati curvature of a submanifold in a Riemannian manifold was considered as an extrinsic invariant defined as the normalized square of the length of the second fundamental form. The notion of Casorati curvature extends the concept of the principal direction of a hypersurface of a Riemannian manifold. Several geometers in [1, 3, 12, 25, 26] found geometrical meaning and the importance of the Casorati curvature. Therefore, it is of great interest to obtain optimal inequalities for the Casorati curvatures of submanifolds in different ambient spaces. Some inequalities involving Casorati curvatures were proved in [8, 9, 16, 11, 24] in real, complex and quaternionic space forms. Recently, Lee et al. in [17] obtained optimal inequalities for submanifolds in real space forms, endowed with a semi-symmetric metric connection.

In this paper, we establish two optimal inequalities involving the generalized normalized δ-Casorati curvatures for submanifolds in real space forms with semi-symmetric metric connections and also characterize those submanifolds for which the equalities hold, generalizing some recent results from [17].

1. Preliminaries

This section gives several basic definitions and notations for our framework based on [16, 17]. We will consider a Riemannian manifold N^m endowed with a semi-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection, denoted by $\mathring{\widetilde{\nabla}}$. Let M be an n-dimensional submanifold of an m-dimensional Riemannian manifold N. On the submanifold M, we consider the induced semi-symmetric metric connection, denoted by ∇, and the induced Levi-Civita connection, denoted by $\mathring{\nabla}$. Let $\widetilde{R}$ be the curvature tensor of N with respect to $\widetilde{\nabla}$ and $\mathring{\widetilde{R}}$ the curvature tensor of N with respect to $\mathring{\widetilde{\nabla}}$. We also denote by R and $\mathring{R}$ the curvature tensors of ∇ and $\mathring{\nabla}$, repectively, on M.

The Gauss formulas with respect to ∇ and $\mathring{\nabla}$, respectively, can be written as

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad X, Y \in \chi(M),$$
$$\mathring{\widetilde{\nabla}}_X Y = \mathring{\nabla}_X Y + \mathring{h}(X,Y), \quad X, Y \in \chi(M),$$

where $\mathring{h}$ is the second fundamental form of M in N and h is a $(0,2)$-tensor on M. According to the formula (7) from [20], h is also symmetric. One denotes by $\mathring{H}$ the mean curvature vector of M and N. Let $N(c)$ be a real space form of constant sectional curvature c endowed with a semi-symmetric metric connection $\widetilde{\nabla}$. Then, the curvature tensor $\widetilde{R}$ with respect to the semi-symmetric metric connection $\widetilde{\nabla}$ on $N(c)$ can be written as (see [15]):

$$\begin{aligned}\widetilde{R}(X,Y,Z,W) = {} & \mathring{\widetilde{R}}(X,Y,Z,W) - \alpha(Y,Z)g(X,W) \\ & + \alpha(X,Z)g(Y,W) - \alpha(X,W)g(Y,Z) \\ & + \alpha(Y,W)g(X,Z),\end{aligned}$$

for any vector fields X, Y, Z, $W \in \chi(M)$, where α is a $(0,2)$-tensor field defined by

$$\alpha(X,Y) = \left(\overset{\circ}{\widetilde{\nabla}}_X \phi\right) Y - \phi(X)\phi(Y) + \frac{1}{2}\phi(P)g(X,Y), \forall X, Y \in \chi(M).$$

It follows that the curvature tensor $\widetilde{R}$ can be expressed as

$$\begin{aligned} \widetilde{R}(X,Y,Z,W) = {} & c\{g(X,W)g(Y,Z) - g(X,Z)g(Y,W)\} \\ & - \alpha(Y,Z)g(X,W) + \alpha(X,Z)g(Y,W) \\ & - \alpha(X,W)g(Y,Z) + \alpha(Y,W)g(X,Z). \end{aligned} \tag{1.1}$$

The Gauss equation for the submanifold M into the real space form $N(c)$ is

$$\begin{aligned} \overset{\circ}{\widetilde{R}}(X,Y,Z,W) = {} & \overset{\circ}{R}(X,Y,Z,W) + g\left(\overset{\circ}{h}(X,Z), \overset{\circ}{h}(Y,W)\right) \\ & - g\left(\overset{\circ}{h}(X,W), \overset{\circ}{h}(Y,Z)\right). \end{aligned}$$

Let $\pi \subset T_xM$, $x \in M$, be a 2-plane section. Denote by $K(\pi)$ the sectional curvature of M with respect to the induced semi-symmetric metric connection ∇. For any orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_xM and $\{e_{n+1}, \ldots, e_m\}$ an orthonormal basis of the normal space $T_x^\perp M$, the scalar curvature τ at x is defined by

$$\tau(x) = \sum_{1 \le i < j \le n} K\left(e_i \wedge e_j\right),$$

and the normalized scalar curvature ρ of M is defined by

$$\rho = \frac{2\tau}{n(n-1)}.$$

We denote by H the mean curvature vector, that is,

$$H(x) = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

and we also set

$$h_{ij}^{\alpha} = g\left(h(e_i, e_j), e_\alpha\right), \qquad i, j \in \{1, \ldots, n\},\ \alpha \in \{n+1, \ldots, m\}.$$

Then, the squared mean curvature of the submanifold M in N is defined by

$$\|H\|^2 = \frac{1}{n^2}\sum_{\alpha=n+1}^{m}\left(\sum_{i=1}^{n} h_{ii}^{\alpha}\right)^2,$$

and the squared norm of h over dimension n is denoted by $\mathcal{C}$ and is called the *Casorati curvature* of the submanifold M. Therefore, we have

$$\mathcal{C} = \frac{1}{n}\sum_{\alpha=n+1}^{m}\sum_{i,j=1}^{n}\left(h_{ij}^{\alpha}\right)^2.$$

The submanifold M is called *invariantly quasi-umbilical* if there exist $m-n$ mutually orthogonal unit normal vectors $\xi_{n+1}, \ldots, \xi_m$ such that the shape operators with respect to all directions ξ_α have an eigenvalue of multiplicity $n-1$ and that for each ξ_α the distinguished eigendirection is the same ([2]).

Suppose now that L is an r-dimensional subspace of T_xM, $r \geq 2$, and $\{e_1, \dots, e_r\}$ is an orthonormal basis of L. Then the scalar curvature $\tau(L)$ of the r-plane section L is given by

$$\tau(L) = \sum_{1\leq\alpha<\beta\leq r} K\left(e_\alpha \wedge e_\beta\right),$$

and the Casorati curvature $\mathcal{C}(L)$ of the subspace L is defined as

$$\mathcal{C}(L) = \frac{1}{r} \sum_{\alpha=n+1}^{m} \sum_{i,j=1}^{r} \left(h_{ij}^\alpha\right)^2.$$

The normalized δ-Casorati curvature $\delta_c(n-1)$ and $\hat{\delta}_c(n-1)$ are given by

$$[\delta_c(n-1)]_x = \frac{1}{2}\mathcal{C}_x + \frac{n+1}{2n} \inf\{\mathcal{C}(L)|L : \text{a hyperplane of } T_xM\},$$

and

$$[\hat{\delta}_c(n-1)]_x = 2\mathcal{C}_x - \frac{2n-1}{2n} \sup\{\mathcal{C}(L)|L : \text{a hyperplane of } T_xM\}.$$

For any positive real number $r\ (\neq n(n-1))$, set

$$a(r) := \frac{(n-1)(n+r)(n^2-n-r)}{rn},$$

in order to define the generalized normalized δ-Casorati curvatures $\delta_C(r; n-1)$ and $\hat{\delta}_C(r; n-1)$ of the submanifold M^n as follows:

$$\left[\delta_C(r; n-1)\right]_p = r\mathcal{C}_p + a(r) \cdot \inf\{\mathcal{C}(L) \mid L : \text{a hyperplane of } T_pM\},$$

if $0 < r < n^2 - n$, and

$$\left[\hat{\delta}_C(r; n-1)\right]_p = r\mathcal{C}_p + a(r) \cdot \sup\{\mathcal{C}(L) \mid L : \text{a hyperplane of } T_pM\},$$

if $r > n^2 - n$.

2. Main theorem

Theorem 2.1. *Let M be a submanifold of a real space form $N(c)$. Then:*

(i) *The generalized normalized δ-Casorati curvature $\delta_C(r; n-1)$ satisfies*

$$\rho \leq \frac{\delta_C(r; n-1)}{n(n-1)} + c - \frac{2}{n}\lambda, \tag{2.1}$$

for any real number r such that $0 < r < n(n-1)$.

(ii) *The generalized normalized δ-Casorati curvature $\hat{\delta}_C(r; n-1)$ satisfies*

$$\rho \leq \frac{\hat{\delta}_C(r; n-1)}{n(n-1)} + c - \frac{2}{n}\lambda, \tag{2.2}$$

for any real number $r > n(n-1)$, where λ is the trace of α.

Moreover, the equality sign holds in the inequalities (2.1) and (2.2) if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $N(c)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1, \dots, \xi_n\}$ and orthonormal normal frame $\{\xi_{n+1}, \dots, \xi_m\}$, the shape operators $A_r \equiv A_{\xi_r}$, $r \in \{n+1, \dots, m\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 & \dots & 0 & 0 \\ 0 & a & 0 & \dots & 0 & 0 \\ 0 & 0 & a & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & a & 0 \\ 0 & 0 & 0 & \dots & 0 & \frac{n(n-1)}{r}a \end{pmatrix}, \quad A_{n+2} = \cdots = A_m = 0.$$

Proof. (i) Let $x \in M$ and $\{e_1, e_2, \dots, e_n\}$ and $\{e_{n+1}, \dots, e_m\}$ be orthonormal basis of T_xM and $T_x^{\perp}M$, respectively. For $X = W = e_i$, $Y = Z = e_j$, $i \neq j$, from equation (1.1) it folllows that

$$\widetilde{R}(e_i, e_j, e_j, e_i) = c - \alpha(e_i, e_i) - \alpha(e_j, e_j). \tag{2.3}$$

From (2.3) and the Gauss equation with the semi-symmetric metric connection, we get

$$\begin{aligned} c - \alpha(e_i, e_i) - \alpha(e_j, e_j) \\ = R(e_i, e_j, e_j, e_i) + g(h(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h(e_j, e_j)). \end{aligned}$$

By summation over $1 \leq i, j \leq n$, it follows from the previous relation that

$$2\tau = n^2\|H\|^2 - n\mathcal{C} + n(n-1)c - 2(n-1)\lambda,$$

where λ is the trace of α.

We define now the following function $\mathcal{P}$, which is a quadratic polynomial in the components of the second fundamental form:

$$\mathcal{P} = r\mathcal{C} + a(r) \cdot \mathcal{C}(L) - 2\tau + n(n-1)c - 2(n-1)\lambda.$$

Without loss of generality, by assuming that L is spanned by $e_1, \dots, e_{n-1}$, one derives that

$$\mathcal{P} = \frac{n+r}{n} \sum_{\alpha=n+1}^{m} \left(\sum_{ij=1}^{n} (h_{ij}^{\alpha})^2 \right) + \frac{a(r)}{n-1} \sum_{\alpha=n+1}^{m} \left(\sum_{i,j=1}^{n-1} (h_{ij}^{\alpha})^2 \right) - \sum_{\alpha=n+1}^{m} \left(\sum_{i=1}^{n} h_{ii}^{\alpha} \right)^2,$$

and now we obtain easily that

$$\begin{aligned} \mathcal{P} = & \sum_{\alpha=n+1}^{m} \sum_{i=1}^{n-1} \left[\left(\frac{n+r}{n} + \frac{a(r)}{n-1} \right) (h_{ii}^{\alpha})^2 + \frac{2(n+r)}{n} (h_{in}^{\alpha})^2 \right] \\ & + \sum_{\alpha=n+1}^{m} \left[2 \left(\frac{n+r}{n} + \frac{a(r)}{n-1} \right) \sum_{i<j=1}^{n-1} (h_{ij}^{\alpha})^2 - 2 \sum_{i<j=1}^{n} h_{ii}^{\alpha} h_{jj}^{\alpha} + \frac{r}{n} (h_{nn}^{\alpha})^2 \right]. \end{aligned} \tag{2.4}$$

From (2.4), it follows that the critical points

$$h^c = \left(h_{11}^{n+1}, h_{12}^{n+1}, \dots, h_{nn}^{n+1}, \dots, h_{11}^{m}, \dots, h_{nn}^{m}\right)$$

of $\mathcal{P}$ are the solutions of the following system of linear homogeneous equations:

$$\begin{cases} \dfrac{\partial \mathcal{P}}{\partial h_{ii}^{\alpha}} = 2\left(\dfrac{n+r}{n} + \dfrac{a(r)}{n-1}\right) h_{ii}^{\alpha} - 2\displaystyle\sum_{k=1}^{n} h_{kk}^{\alpha} = 0, \\ \dfrac{\partial \mathcal{P}}{\partial h_{nn}^{\alpha}} = \dfrac{2r}{n} h_{nn}^{\alpha} - 2\displaystyle\sum_{k=1}^{n-1} h_{kk}^{\alpha} = 0, \\ \dfrac{\partial \mathcal{P}}{\partial h_{ij}^{\alpha}} = 4\left(\dfrac{n+r}{n} + \dfrac{a(r)}{n-1}\right) h_{ij}^{\alpha} = 0, \\ \dfrac{\partial \mathcal{P}}{\partial h_{in}^{\alpha}} = \dfrac{4(n+r)}{n} h_{in}^{\alpha} = 0, \end{cases} \tag{2.5}$$

with $i, j \in \{1, \dots, n-1\}$, $i \neq j$, and $\alpha \in \{n+1, \dots, m\}$. Thus, every solution h^c has $h_{ij}^{\alpha} = 0$ for $i \neq j$, and the determinant which corresponds to the first two sets of equations of the above system is zero (there exist solutions for non-totally geodesic submanifolds). Moreover, it is easy to see that the Hessian matrix of $\mathcal{P}$ has the form

$$\mathcal{H}(\mathcal{P}) = \begin{pmatrix} H_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & H_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & H_3 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} 2\left(\frac{n+r}{n} + \frac{a(r)}{n-1}\right) - 2 & -2 & \dots & -2 & -2 \\ -2 & 2\left(\frac{n+r}{n} + \frac{a(r)}{n-1}\right) - 2 & \dots & -2 & -2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2 & -2 & \dots & 2\left(\frac{n+r}{n} + \frac{a(r)}{n-1}\right) - 2 & -2 \\ -2 & -2 & \dots & -2 & \frac{2r}{n} \end{pmatrix},$$

$\mathbf{0}$ denotes the null matrix of corresponding dimensions, and H_2, H_3 are the diagonal matrices

$$H_2 = \operatorname{diag}\left(4\left(\frac{n+r}{n} + \frac{a(r)}{n-1}\right), 4\left(\frac{n+r}{n} + \frac{a(r)}{n-1}\right), \dots, 4\left(\frac{n+r}{n} + \frac{a(r)}{n-1}\right)\right)$$

and

$$H_3 = \operatorname{diag}\left(\frac{4(n+r)}{n}, \frac{4(n+r)}{n}, \dots, \frac{4(n+r)}{n}\right).$$

Therefore, we find that $\mathcal{H}(\mathcal{P})$ has the following eigenvalues:

$$\lambda_{11}=0,\quad \lambda_{22}=2\left(\frac{2r}{n}+\frac{a(r)}{n-1}\right),\quad \lambda_{33}=\cdots=\lambda_{nn}=2\left(\frac{n+r}{n}+\frac{a(r)}{n-1}\right),$$

$$\lambda_{ij}=4\left(\frac{n+r}{n}+\frac{a(r)}{n-1}\right),\quad \lambda_{in}=\frac{4(n+r)}{n},\qquad \forall i,j\in\{1,\ldots,n-1\},\ i\neq j.$$

Therefore, $\mathcal{P}$ is parabolic and reaches a minimum $\mathcal{P}(h^c)=0$ for the solution h^c of the system (2.5). It follows that $\mathcal{P}\geq 0$, and hence

$$2\tau\leq r\mathcal{C}+a(r)\cdot\mathcal{C}(L)+n(n-1)c-2(n-1)\lambda.$$

Hence, we deduce that

$$\rho\leq\frac{r}{n(n-1)}\mathcal{C}+\frac{a(r)}{n(n-1)}\mathcal{C}(L)+c-\frac{2}{n}\lambda,$$

for every tangent hyperplane L of M. Taking the infinum over all tangent hyperplanes L, the theorem trivially follows. Moreover, we can easily check that the equality sign holds in the theorem if and only if

$$h^{\alpha}_{ij}=0,\quad \forall i,j\in\{1,\ldots,n\},\ i\neq j,\ \text{and}\ \alpha\in\{n+1,\ldots,m\}\tag{2.6}$$

and

$$h^{\alpha}_{nn}=2h^{\alpha}_{11}=\cdots=2h^{\alpha}_{n-1\,n-1},\quad \forall\alpha\in\{n+1,\ldots,m\}.\tag{2.7}$$

From (2.6) and (2.7) we conclude that the equality holds if and only if the submanifold M is invariantly quasi-umbilical with trivial normal connection in N such that with respect to suitable orthonormal tangent and orthonormal normal frames.

In a similar way one can prove (ii). ☐

Corollary 2.2. *Let M be a submanifold of a real space form $N(c)$ with a semi-symmetric metric connection. Then:*

(i) *The normalized δ-Casorati curvature $\delta_c(n-1)$ satisfies*

$$\rho\leq\delta_c(n-1)+c-\frac{2}{n}\lambda.$$

Moreover, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $N(c)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1,\ldots,\xi_n\}$ and orthonormal normal frame $\{\xi_{n+1},\ldots,\xi_m\}$ the shape operators $A_r\equiv A_{\xi_r}$, $r\in\{n+1,\ldots,m\}$, take the following forms:

$$A_{n+1}=\begin{pmatrix} a & 0 & 0 & \ldots & 0 & 0\\ 0 & a & 0 & \ldots & 0 & 0\\ 0 & 0 & a & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & a & 0\\ 0 & 0 & 0 & \ldots & 0 & 2a\end{pmatrix},\quad A_{n+2}=\cdots=A_m=0.$$

(ii) *The normalized δ-Casorati curvature $\hat{\delta}_c(n-1)$ satisfies*

$$\rho \leq \hat{\delta}_c(n-1) + c - \frac{2}{n}\lambda.$$

Moreover, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $N(c)$, such that with respect to suitable orthonormal tangent frame $\{\xi_1, \ldots, \xi_n\}$ and orthonormal normal frame $\{\xi_{n+1}, \ldots, \xi_m\}$ the shape operators $A_r \equiv A_{\xi_r}$, $r \in \{n+1, \ldots, m\}$, take the following forms:

$$A_{n+1} = \begin{pmatrix} 2a & 0 & 0 & \ldots & 0 & 0 \\ 0 & 2a & 0 & \ldots & 0 & 0 \\ 0 & 0 & 2a & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & 2a & 0 \\ 0 & 0 & 0 & \ldots & 0 & a \end{pmatrix}, \quad A_{n+2} = \cdots = A_m = 0.$$

Proof. (i) It is easy to see that the following relation holds:

$$\left[\delta_C\left(\frac{n(n-1)}{2}; n-1\right)\right]_p = n(n-1)\left[\delta_C(n-1)\right]_p \tag{2.8}$$

in any point $p \in M$. Therefore, taking $r = \frac{n(n-1)}{2}$ in (2.1) and making use of (2.8) we obtain the conclusion.

(ii) The following relation can be easily verified:

$$\left[\hat{\delta}_C\left(2n(n-1); n-1\right)\right]_p = n(n-1)\left[\hat{\delta}_c(n-1)\right]_p, \quad \forall p \in M.$$

Replacing $r = 2n(n-1)$ in (2.1), we derive the conclusion. □

Remark. We have a slightly modified coefficient in the definition of $\delta_C(n-1)$; in fact, we used the coefficient $\frac{n+1}{2n(n-1)}$, as in [8, 9, 24], instead of $\frac{n+1}{2n}$, like in the present paper because we are working on the generalized normalized δ-Casorati curvature $\delta_C(r; n-1)$ for a positive real number $r \neq n(n-1)$, as in [16].

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Received: August 19, 2014
Accepted: April 6, 2016