REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 57, No. 2, 2016, Pages 85–94
Published online: June 14, 2016

AN EXTENSION OF SOME PROPERTIES FOR THE FOURIER TRANSFORM OPERATOR ON $L^p(\mathbb{R})$ SPACES

M. GUADALUPE MORALES, JUAN H. ARREDONDO, AND FRANCISCO J. MENDOZA

Abstract. In this paper the Fourier transform is studied using the Henstock–Kurzweil integral on $\mathbb{R}$. We obtain that the classical Fourier transform $\mathcal{F}_p : L^p(\mathbb{R}) \to L^q(\mathbb{R})$, $1/p + 1/q = 1$ and $1 < p \leq 2$, is represented by the integral on a subspace of $L^p(\mathbb{R})$, which strictly contains $L^1(\mathbb{R}) \cap L^p(\mathbb{R})$. Moreover, for any function f in that subspace, $\mathcal{F}_p(f)$ obeys a generalized Riemann–Lebesgue lemma.

1. Introduction

If f belongs to the space of real valued Lebesgue integrable functions, $L^1(\mathbb{R})$, its classical Fourier transform is defined for every real number s as

$$\mathcal{F}_1(f)(s) := \int_{\mathbb{R}} e^{-ist} f(t)\, dt. \tag{1.1}$$

Here the integral is taken in the Lebesgue sense. When f is in $L^2(\mathbb{R})$, the Fourier transform of f is defined by

$$\mathcal{F}_2(f)(s) := \lim_{n\to\infty} \int_{\mathbb{R}} e^{-ist} f_n(t)\, dt, \tag{1.2}$$

where the limit is taken in the topology of the norm on $L^2(\mathbb{R})$ and (f_n) is a sequence in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ obeying $\|f_n - f\|_2 \to 0$, as $n \to \infty$.

For any unbounded set $X \subseteq \mathbb{R}$, $C_\infty(X)$ denotes the complex valued continuous functions on X vanishing at infinity (see [14]).

In [18] the Henstock–Kurzweil integral was employed to study the Fourier transform. In [11, 12] it was proved that (1.1) makes sense as a Henstock–Kurzweil integral over $BV_0(\mathbb{R})$, and defines a function in $C_\infty(\mathbb{R} \setminus \{0\})$.

In this paper we prove that the Fourier transform operator on $L^p(\mathbb{R})$, for $1 < p \leq 2$, is represented by a Henstock–Kurzweil integral on a subspace of $L^p(\mathbb{R})$, implying an extension of some properties for the operator.

2010 *Mathematics Subject Classification.* Primary 26A39, 43A32; Secondary 26A42.

Key words and phrases. Fourier transform, $L^p(\mathbb{R})$ space, Henstock–Kurzweil integral.

This work is partially supported by CONACyT-SNI and VIEP-BUAP (México). M.G.M. acknowledges support from a CONACyT postdoctoral fellowship.

2. Preliminaries

The characteristic function of a set E will be denoted by $\chi_E(x) = 1$ if $x \in E$ and zero otherwise. The Lebesgue measure of the set E will be denoted by $m(E)$.

Definition 2.1. Let $0 < p < \infty$ and $X \subset \mathbb{R}$. For any Lebesgue measurable function $f : X \to \mathbb{R}$ we define

$$\|f\|_p := \left[\int_X |f|^p \, dm\right]^{1/p}.$$

The set of functions f such that $\|f\|_p < \infty$ is a vector space denoted by $\mathcal{L}^p(X)$. $\mathcal{W}_p$ denotes the subspace of functions on which $\|\cdot\|_p$ vanishes.

For $p \geq 1$, $\|\cdot\|_p$ is a seminorm on $\mathcal{L}^p(X)$ and induces a norm on the quotient space $\mathcal{L}^p(X)/\mathcal{W}_p$. This space with induced norm is denoted by $L^p(X)$. Similarly, for $p \geq 1$ we define $\mathcal{L}^p(X, \mathbb{C})$ and $L^p(X, \mathbb{C})$ by considering complex valued instead of real valued functions.

For $p = \infty$ and $f : X \to \mathbb{R}$, we define $\|f\|_\infty$ to be the essential supremum of $|f|$, and $\mathcal{L}^\infty(\mathbb{R})$ is the vector space of all Lebesgue measurable functions f for which $\|f\|_\infty < \infty$.

Now we will introduce the definition of the Henstock–Kurzweil integral. Let $\overline{\mathbb{R}}$ be defined as $\mathbb{R} \cup \{\pm\infty\}$. For an interval $[a, b] \subset \overline{\mathbb{R}}$, a gauge function is a map $\gamma : [a, b] \to (0, \infty)$.

Definition 2.2. Given a gauge function $\gamma : [a, b] \to (0, \infty)$, one says that a tagged partition $P = \{([x_{i-1}, x_i],\ t_i);\ t_i \in [x_{i-1}, x_i]\}_{i=1}^n$ of $[a, b]$ is γ-fine according to the following cases.

For $a \in \mathbb{R}$ and $b = \infty$:

(1) $a = x_0$, $b = x_n = t_n = \infty$.
(2) $[x_{i-1}, x_i] \subset [t_i - \gamma(t_i), t_i + \gamma(t_i)]$, for all $i = 1, 2, \ldots, n-1$.
(3) $[x_{n-1}, \infty] \subset [1/\gamma(t_n), \infty]$.

For $a = -\infty$ and $b \in \mathbb{R}$:

(1) $a = x_0 = t_1 = -\infty$, $b = x_n$.
(2) $[x_{i-1}, x_i] \subset [t_i - \gamma(t_i), t_i + \gamma(t_i)]$, for all $i = 2, \ldots, n$.
(3) $[-\infty, x_1] \subset [-\infty, -1/\gamma(t_1)]$.

For $a = -\infty$ and $b = \infty$:

(1) $b = x_n = t_n = \infty$, $a = x_0 = t_1 = -\infty$.
(2) $[x_{i-1}, x_i] \subset [t_i - \gamma(t_i), t_i + \gamma(t_i)]$, for all $i = 2, \ldots, n-1$.
(3) $[x_{n-1}, \infty] \subset [1/\gamma(t_n), \infty]$ and $[-\infty, x_1] \subset [-\infty, -1/\gamma(t_1)]$.

For $a, b \in \mathbb{R}$:

(1) $[x_{i-1}, x_i] \subset [t_i - \gamma(t_i), t_i + \gamma(t_i)]$, for all $i = 1, 2, \ldots, n$.

Definition 2.3. A function $f : [a, b] \to \mathbb{R}$ is said to be Henstock–Kurzweil integrable iff there exists $A \in \mathbb{R}$ such that, for each $\varepsilon > 0$, there is a gauge function $\gamma_\varepsilon : [a, b] \to (0, \infty)$ such that if $P = \{([x_{i-1}, x_i], t_i);\ t_i \in [x_{i-1}, x_i]\}_{i=1}^n$ is a γ_ε-fine partition of $[a,b]$, then

$$\left|\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) - A\right| < \varepsilon. \tag{2.1}$$

The number A is the integral of f over $[a,b]$ and it is denoted by $\int_a^b f = A$. Using the convention $0 \cdot (\pm\infty) = 0$, an extra condition for f is that $f(\pm\infty) = 0$ (see [1]).

The space of Henstock–Kurzweil integrable functions defined on an interval I will be denoted by $\mathcal{HK}(I)$. This space is a semi normed space with the Alexiewicz seminorm, which is defined as

$$\|f\|_A = \sup\left\{\left|\int_c^d f(x)\,dx\right| : [c,d] \subset I\right\}.$$

The quotient space $\mathcal{HK}(I)/\mathcal{W}(I)$ will be denoted by $HK(I)$. Here $\mathcal{W}(I)$ is the subspace of $\mathcal{HK}(I)$ for which the Alexiewicz seminorm vanishes (see [2]).

Let I be any interval, then $L^1(I) \subset HK(I)$ and the Lebesgue integral coincides with the Henstock–Kurzweil integral ([1, 4, 10]). When necessary we will emphasize which integral is being used.

A characterization of $\mathcal{HK}(\mathbb{R})$ is given by the following theorem.

Theorem 2.4 (Hake's Theorem, [1])**.** *Let $f : \mathbb{R} \to \mathbb{R}$. The following statements are equivalent:*

(1) *$f \in \mathcal{HK}(\mathbb{R})$ has integral A in the sense of Henstock–Kurzweil.*
(2) *$f \in \mathcal{HK}([a,b])$ for every compact interval $[a,b]$ and*

$$\lim_{a\to-\infty, b\to\infty} \int_a^b f = A = \int_{-\infty}^{\infty} f,$$

where the integrals are in the sense of Henstock–Kurzweil.

Definition 2.5. Let g be a real valued function over $\mathbb{R}$. It is said that g is a bounded variation function over $[a,b]$ iff

$$\mathrm{Var}(g,[a,b]) := \sup_P \sum_{i=1}^{n} |g(t_i) - g(t_{t-i})| < \infty,$$

where the supreme is taken over all possible partitions on $[a,b]$. It is said that g is a bounded variation function over $\mathbb{R}$ iff

$$\mathrm{Var}(g,\mathbb{R}) := \lim_{t\to\infty, s\to-\infty} \mathrm{Var}(g,[s,t])$$

exists in $\mathbb{R}$. We will denote the set of bounded variation functions over an interval I as $BV(I)$. $BV_0(\mathbb{R})$ will denote the bounded variation functions over $\mathbb{R}$ having limits equal to zero at $\pm\infty$.

We remark that the Multiplier Theorem ([1, Theorem 10.12]) states that $f \in \mathcal{HK}(I)$ and $g \in BV(I)$ implies $fg \in \mathcal{HK}(I)$. See also [3].

Definition 2.6. For given $p, q > 0$, we define $\mathcal{L}^p(\mathbb{R}) + \mathcal{L}^q(\mathbb{R})$ as the vector space of functions $f = f_p + f_q$, where $f_p \in \mathcal{L}^q(\mathbb{R})$ and $f_q \in \mathcal{L}^q(\mathbb{R})$. Inductively, we can define the finite sum $\sum_{i=1}^{n} \mathcal{L}^{p_i}(\mathbb{R})$. Also, $\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R})$ is similarly defined.

Given a Lebesgue measurable function f defined on a measurable set X such that $m(\mathbb{R} \setminus X) = 0$, we will denote by the same symbol f the trivial extension of f to a (measurable) function on $\mathbb{R}$. That is, we extend the function as zero on $\mathbb{R} \setminus X$. Furthermore, for a function $f \in \mathcal{L}^p(\mathbb{R})$, or $f \in \mathcal{L}^p(\mathbb{R}, \mathbb{C})$ we will denote by the same symbol f the (unique) element that defines the function in $L^p(\mathbb{R})$ or in $L^p(\mathbb{R}, \mathbb{C})$, respectively.

In [7, 13] a decomposition of $L^p(\mathbb{R})$ is introduced in order to define the Fourier transform operator on $L^p(\mathbb{R})$. Given f in $L^p(\mathbb{R})$ with $1 < p < 2$, the set $E = \{x : |f(x)| > 1\}$ has finite Lebesgue measure and $f = f_1 + f_2$, where $f_1(x) = f\chi_E(x)$ and $f_2 = f - f_1$. It is easy to see that $f_1 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$ and $f_2 \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$. In fact, by Hölder's inequality we have

$$\|f_1\|_1 = \int_{\mathbb{R}} |f\chi_E| \leq m(E)^{1/q} \|f\|_p < \infty,$$

where $\frac{1}{p} + \frac{1}{q} = 1$. On the other hand, due to $|f(x)| \leq 1$ for $x \in E^c$,

$$\|f_2\|_2^2 = \int_{\mathbb{R}} |f\chi_{E^c}|^2 \leq \int_{E^c} |f|^p < \infty.$$

Therefore, $L^p(\mathbb{R}) = L^1(\mathbb{R}) \cap L^p(\mathbb{R}) + L^2(\mathbb{R}) \cap L^p(\mathbb{R})$.

Now we can define the Fourier transform for functions in $L^p(\mathbb{R})$, $1 < p < 2$, as a sum of classical Fourier transforms.

Definition 2.7. The Fourier transform for functions in $L^p(\mathbb{R})$, $1 < p < 2$, is given as

$$\begin{aligned} \mathcal{F}_p &: L^p(\mathbb{R}) \to L^q(\mathbb{R}, \mathbb{C}) \\ \mathcal{F}_p(f) &:= \mathcal{F}_1(f_1) + \mathcal{F}_2(f_2), \end{aligned} \tag{2.2}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, $f_1 \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, $f_2 \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$, and $f = f_1 + f_2$. Furthermore, $\mathcal{F}_1(f_1)$ and $\mathcal{F}_2(f_2)$ are defined by (1.1) and (1.2). $\mathcal{F}_p$ is a linear continuous operator from L^p into L^q.

Remark 2.8. If $f \in L^1(\mathbb{R}) \cap L^p(\mathbb{R})$, we can take $f_1 = f$ and $f_2 = 0$. Besides, for $f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$, we can take $f_1 = 0$ and $f_2 = f$. In those cases the Fourier transform reduces to $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. The operator $\mathcal{F}_p$ defines the classical Fourier transform on $L^p(\mathbb{R})$, by the Stein interpolation theorem ([14, 17]).

Remark 2.9. With respect to the above definition, if $f = f_1 + f_2$ and $f = f_1' + f_2'$ then $f_1 - f_1' = f_2' - f_2$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are injective linear operators on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we obtain

$$\mathcal{F}_1(f_1) + \mathcal{F}_2(f_2) = \mathcal{F}_1(f_1') + \mathcal{F}_2(f_2').$$

Thus expression (2.2) is well defined.

If $f \in \mathcal{L}^2(\mathbb{R})$, then $f(x)e^{-isx}$ is not necessarily Lebesgue integrable. However, for $f \in BV_0(\mathbb{R})$ the integral in equation (1.1) is well defined as a Henstock–Kurzweil integral for $s \neq 0$. See [11, 12].

Definition 2.10. The HK Fourier transform exists for every $s \neq 0$, and is defined by

$$\mathcal{F}_{\mathrm{HK}} : \mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R}) \to C_\infty(\mathbb{R} \setminus \{0\}),$$
$$\mathcal{F}_{\mathrm{HK}}(f)(s) := \int_{\mathbb{R}} e^{-isx} f(x)\, dx,$$

where the integral is a Henstock–Kurzweil integral.

We say "HK Fourier transform" in order to emphasize the use of the Henstock–Kurzweil integral (see [18]). Moreover, $\mathcal{F}_{\mathrm{HK}}(f)$ is pointwise defined and continuous except possibly at zero, vanishing at infinity (see [12]).

There are functions $f \in \mathcal{L}^2(\mathbb{R}) \cap BV_0(\mathbb{R})$ with $\mathcal{F}_{\mathrm{HK}}(f)(s)$ not being continuous at zero; see Example 3(d) in [18]. On the other hand, for $f \in \mathcal{HK}(\mathbb{R}) \cap BV(\mathbb{R}) \subset \mathcal{L}^2(\mathbb{R}) \cap BV_0(\mathbb{R})$, $\mathcal{F}_{\mathrm{HK}}(f)(s)$ exists on $\mathbb{R}$ and is continuous on $\mathbb{R} \setminus \{0\}$. As far as we know, continuity at zero is an open question.

3. Henstock–Kurzweil Fourier transform and $\mathcal{L}^p(\mathbb{R})$ spaces

In accordance with Definitions 2.7 and 2.10, there are several spaces where the Fourier transform can be defined. Theorem 3.3 and Corollary 1 establish relations between the operators $\mathcal{F}_p$ and $\mathcal{F}_{\mathrm{HK}}$.

We require the following basic result (see [16]), in order to prove Theorem 3.3.

Lemma 3.1. *If $1 \leq p \leq \infty$ and (φ_n) is a Cauchy sequence in $\mathcal{L}^p(\mathbb{R})$, with limit f, then (φ_n) has a subsequence which converges pointwise almost everywhere to f.*

The following results are new.

Lemma 3.2. *For $p > 1$, $\mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^p(\mathbb{R})$ is a proper subset of $\mathcal{L}^p(\mathbb{R}) \cap (\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R}))$.*

Proof. It is easy to see that $\mathcal{L}^p(\mathbb{R}) \cap BV_0(\mathbb{R}) \not\subset \mathcal{L}^1(\mathbb{R})$: take for example the function $f(x) = 1/x$, $|x| > 1$ and zero otherwise. □

Theorem 3.3. *If $f \in \mathcal{L}^2(\mathbb{R}) \cap (\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R}))$, then*

$$\mathcal{F}_{\mathrm{HK}}(f) \in \mathcal{L}^2(\mathbb{R}, \mathbb{C}) \cap C_\infty(\mathbb{R} \setminus \{0\}).$$

Moreover,

$$\mathcal{F}_2(f)(s) = \mathcal{F}_{\mathrm{HK}}(f)(s) \quad a.e. \tag{3.1}$$

Proof. For $f \in \mathcal{L}^2(\mathbb{R}) \cap (\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R}))$, and for each $n \in \mathbb{N}$, let us define the sequence (φ_n), where $\varphi_n(x) = f\chi_{[-n,n]}(x)$. Then $\varphi_n \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^2(\mathbb{R})$, and

$$\mathcal{F}_1(\varphi_n)(s) = \mathcal{F}_{\mathrm{HK}}(\varphi_n)(s) = \mathcal{F}_2(\varphi_n)(s),$$

pointwise for every $s \in \mathbb{R}$. By the Plancherel theorem, we get that

$$\lim_{n\to\infty} \|\mathcal{F}_{\mathrm{HK}}(\varphi_n) - \mathcal{F}_2(f)\|_2 = \lim_{n\to\infty} \|\mathcal{F}_2(\varphi_n) - \mathcal{F}_2(f)\|_2 = \lim_{n\to\infty} \|\varphi_n - f\|_2 = 0.$$

Let $f = f_1 + f_0$, where $f_1 \in \mathcal{L}^1(\mathbb{R})$ and $f_0 \in BV_0(\mathbb{R})$. Theorem 3.2 in [11] and Theorem 2.4 above yield

$$\begin{aligned}\lim_{n\to\infty} \mathcal{F}_{\mathrm{HK}}(\varphi_n)(s) &= \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-isx} f_1\chi_{[-n,n]}(x)\,dx + \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{-isx} f_0\chi_{[-n,n]}(x)\,dx \\ &= \mathcal{F}_{\mathrm{HK}}(f)(s),\end{aligned}$$

pointwise for every $s \neq 0$. An application of Lemma 3.1 gives equality (3.1). Also by Theorem 3.2 in [11], it follows that $\mathcal{F}_{\mathrm{HK}}(f) \in \mathcal{L}^2(\mathbb{R},\mathbb{C}) \cap C_\infty(\mathbb{R} \setminus \{0\})$. □

According to our notation, the set

$$\mathcal{L}^p(\mathbb{R}) \cap (\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R}))$$

defines a dense subspace of $L^p(\mathbb{R})$ which strictly contains $L^p(\mathbb{R}) \cap L^1(\mathbb{R})$.

Corollary 1. For $f \in \mathcal{L}^p(\mathbb{R}) \cap (\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R}))$, $1 < p < 2$, the formula

$$\mathcal{F}_p(f)(s) = \int_{\mathbb{R}} e^{-isx} f(x)\,dx$$

holds true pointwise a.e., where the integral is a Henstock–Kurzweil integral. Also, $\mathcal{F}_p(f)(s)$ is equal a.e. to a function belonging to $C_\infty(\mathbb{R} \setminus \{0\})$.

Proof. Let $f \in \mathcal{L}^p(\mathbb{R}) \cap (\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R}))$. Then

$$f = f_1 + f_2 = f_1' + f_2',$$

where $f_1, f_1' \in \mathcal{L}^1(\mathbb{R})$, $f_2 \in \mathcal{L}^2(\mathbb{R})$ and $f_2' \in BV_0(\mathbb{R})$. Thus, $f_2 = f_1' - f_1 + f_2'$ belongs to $\mathcal{L}^2(\mathbb{R}) \cap (\mathcal{L}^1(\mathbb{R}) + BV_0(\mathbb{R}))$. So, Theorem 3.3 proves the corollary. □

This result is a generalization of Corollary 31.15 and Theorem 31.20 of [9] in the case that $X = \mathbb{R}$.

For more properties of the Fourier transform on L^p the reader may see [5, 6, 8, 15], among others.

4. The spaces $\mathcal{HK}(\mathbb{R})$ and $\mathcal{L}^p(\mathbb{R}) + \mathcal{L}^q(\mathbb{R})$

It is easy to see that $\mathcal{HK}(\mathbb{R}) \not\subset \mathcal{L}^1(\mathbb{R}) + \mathcal{L}^2(\mathbb{R})$. Thus, the Fourier transform cannot be fixed on $\mathcal{HK}(\mathbb{R})$ by means of equation (2.2). On the other hand, $\mathcal{L}^p(\mathbb{R}) \not\subset \mathcal{HK}(\mathbb{R})$, for $1 \neq p > 0$, so $\mathcal{L}^p(\mathbb{R}) + \mathcal{L}^\infty(\mathbb{R}) \not\subset \mathcal{HK}(\mathbb{R})$. In this section we prove that $\mathcal{HK}(\mathbb{R})$ is not contained in a finite sum of $\mathcal{L}^{p_i}(\mathbb{R})$, $0 < p_i \leq \infty$.

Let $A = \sum_{i=1}^{\infty} (-1)^{i-1} a_i$ be a conditionally convergent series, with (a_i) a decreasing sequence with $0 < a_i < 1$, for every $i \in \mathbb{N}$. We define

$$H(x) = \begin{cases} (-1)^{i-1} & x \in [2^i, a_i + 2^i],\ i \in \mathbb{N}, \\ 0 & \text{otherwise}. \end{cases}$$

Theorem 4.1. *The function H belongs to $(\mathcal{L}^\infty(\mathbb{R}) \cap \mathcal{HK}(\mathbb{R})) \setminus BV(\mathbb{R})$, but not to $\mathcal{L}^p(\mathbb{R}) + \mathcal{L}^q(\mathbb{R})$, $0 < p, q < \infty$.*

Proof. First we prove that $H \in \mathcal{HK}(\mathbb{R})$. For $c \in (4, \infty)$, there exists a $k \in \mathbb{N}$ such that $c \in (2^k, a_k + 2^k)$ or $c \in [a_{k-1} + 2^{k-1}, 2^k]$. If $c \in (2^k, a_k + 2^k)$,

$$\int_0^c H(x)\,dx = \int_0^{a_{k-1}+2^{k-1}} H(x)\,dx + \int_{2^k}^c H(x)\,dx.$$

For $c \in [a_{k-1} + 2^{k-1}, 2^k]$,

$$\int_0^c H(x)\,dx = \int_0^{a_{k-1}+2^{k-1}} H(x)\,dx.$$

Without loss of generality let us assume that $c \in [a_{k-1} + 2^{k-1}, 2^k]$. Therefore,

$$\int_0^c H(x)\,dx = \int_0^{a_{k-1}+2^{k-1}} H(x)\,dx = \sum_{i=1}^{k-1} \int_{2^i}^{a_i+2^i} (-1)^{i-1}\,dx = \sum_{i=1}^{k-1} (-1)^{i-1} a_i < \infty.$$

Hence, $H \in \mathcal{HK}([0, c])$ for each $c > 0$ and

$$\lim_{c\to\infty} \int_0^c H(x)\,dx = \lim_{k\to\infty} \sum_{i=1}^{k-1} (-1)^{i-1} a_i = A \in \mathbb{R}.$$

Therefore, Theorem 2.4 implies that H belongs to $\mathcal{HK}(\mathbb{R})$ with integral A.

Let us see that $H \notin BV(\mathbb{R})$. Take $x_0 < x_1 < \cdots < x_k$ as a finite collection with $x_i \in [2^i, a^i + 2^i]$, $i = 1, \ldots, k$ and $x_0 = 0$.

$$\begin{aligned}\sum_{i=0}^{k} |H(x_i) - H(x_{i+1})| &= 1 + \sum_{i=1}^{k} |H(x_i) - H(x_{i+1})| \\ &= 1 + \sum_{i=1}^{k} 2 = 1 + 2k.\end{aligned}$$

Therefore, $\operatorname{Var}(H, [0, a_k + 2^k]) \geq 1 + 2k$, and $H \notin BV(\mathbb{R})$.

Suppose $H = f_p + f_q$, where $f_p \in \mathcal{L}^p(\mathbb{R})$ and $f_q \in \mathcal{L}^q(\mathbb{R})$. We denote $E_{2^i} = \{x \in [2^i, 2^i + 1] : |f_q| \geq 1/2\}$ and $A_i = [2^i, a_i + 2^i]$.

Observe that

$$\begin{aligned}\int_0^\infty |f_p|^p\,dm &= \int_0^\infty |H - f_q|^p\,dm \\ &\geq \sum_{i=0}^{\infty} \int_{(E_{2^i}\cap A_i)\cup(E^c_{2^i}\cap A_i)} |H - f_q|^p\,dm.\end{aligned}$$

Since $f_p \in \mathcal{L}^p(\mathbb{R})$,

$$\begin{aligned}\infty > \int_0^\infty |f_p|^p\,dm &\geq \sum_{i=0}^{\infty} \int_{E^c_{2^i}\cap A_i} |H - f_q|^p\,dm \\ &\geq \sum_{i=0}^{\infty} \frac{1}{2^p} m(E^c_{2^i} \cap A_i).\end{aligned} \tag{4.1}$$

On the other hand,

$$\begin{aligned}\infty > \int_0^\infty |f_q|^q\,dm &\geq \sum_{i=0}^\infty \int_{E_{2^i}\cap A_i} |f_q|^q\,dm \\ &\geq \sum_{i=0}^\infty \frac{1}{2^q} m(E_{2^i}\cap A_i).\end{aligned} \tag{4.2}$$

By (4.1) and (4.2),

$$\infty > \sum_{i=0}^\infty m[(E_{2^i}^c \cap A_i)\cup (E_{2^i}\cap A_i)] = \sum_{i=0}^\infty m(A_i) = \sum_{i=0}^\infty a_i.$$

This is a contradiction, since this series does not converge absolutely. Therefore,

$$\mathcal{HK}(\mathbb{R}) \not\subset \mathcal{L}^p(\mathbb{R}) + \mathcal{L}^q(\mathbb{R}). \qquad \square$$

Corollary 2. $\mathcal{HK}(\mathbb{R}) \not\subset \bigcup_{0<p,q<\infty} (\mathcal{L}^p(\mathbb{R}) + \mathcal{L}^q(\mathbb{R}))$.

It is well known that $\mathcal{L}^1(\mathbb{R})$ is not contained in $\mathcal{L}^\infty(\mathbb{R})$, so neither is $\mathcal{HK}(\mathbb{R})$ in $\mathcal{L}^\infty(\mathbb{R})$. In fact, for $q=\infty$ we have the following result.

Proposition 1. $\mathcal{HK}(\mathbb{R}) \not\subset \mathcal{L}^p(\mathbb{R}) + \mathcal{L}^\infty(\mathbb{R})$, for $0<p<1$.

Proof. Let $\sum_{k=1}^\infty \frac{1}{k^s}$ $(s>1)$ be a convergent series with limit A. We choose a sequence $c_1=0$, $c_2=\frac{1}{A2^s}$ and $c_k = c_{k-1} + \frac{1}{Ak^s}$, $k>2$. Now let us define the function

$$h(x) := \begin{cases} (-1)^k k^r & \text{if } x\in[c_{k-1},c_k),\ k\geq 2,\\ 0 & \text{otherwise.}\end{cases}$$

Note that

$$\int_{-\infty}^\infty h = \lim_{n\to\infty} \frac{1}{A}\sum_{k=2}^n (-1)^k k^{r-s},$$

implying that $h\in\mathcal{HK}(\mathbb{R})$ for $r<s$, because of Leibniz's criterion. Moreover, h is not in $\mathcal{L}^p(\mathbb{R})$ iff $\frac{-1+s}{p}\leq r$. In fact,

$$\int_{-\infty}^\infty |h|^p = \lim_{n\to\infty}\sum_{k=2}^n \int_{c_{k-1}}^{c_k} |h|^p = \lim_{n\to\infty}\sum_{k=2}^n \frac{1}{A}k^{rp-s} = \infty,$$

for $-1\leq rp-s$. Note that for $0<p<1$,

$$1 < \frac{-1+s}{p} < s, \quad \text{when } 1+p<s<\frac{1}{1-p}.$$

Given $p>0$, we choose $s\in(1+p, 1/(1-p))$, and then we fix $r=(-1+s)/p$. Therefore, $h\notin\mathcal{L}^p(\mathbb{R})$. For these values of r and s, we show that the function h is not equal to f_p+f_∞, for any $f_p\in\mathcal{L}^p(\mathbb{R})$ and $f_\infty\in\mathcal{L}^\infty(\mathbb{R})$. If it were, note that

$$h|_{[0,1]} \notin \mathcal{L}^\infty([0,1]).$$

For $g := (h-f_p)|_{[0,1]} = f_\infty|_{[0,1]}$, then $g\in\mathcal{L}^p([0,1])$, yielding $h|_{[0,1]} = g + f_p|_{[0,1]} \in \mathcal{L}^p([0,1])$, which is a contradiction. $\square$

Note that the function $h|_{[0,1]}$ above is not in $\mathcal{L}^1([0,1]) \subset \mathcal{L}^p([0,1])$.

Corollary 3. $\mathcal{HK}(\mathbb{R}) \not\subset \sum_{i=1}^{n} \mathcal{L}^{p_i}(\mathbb{R})$, for $0 < p_i \leq \infty$, $i = 1, \ldots, n$ and $n \in \mathbb{N}$.

Proof. Observe that $\mathcal{L}^{p_i}(\mathbb{R}) \subset \mathcal{L}^{\infty}(\mathbb{R}) + \mathcal{L}^p(\mathbb{R})$, where $p = \min(p_i)$. For any bounded interval I and $p \geq 1$ we have

$$\mathcal{HK}(I) \not\subset \mathcal{L}^p(I) + \mathcal{L}^{\infty}(I) = \mathcal{L}^p(I) \subset \mathcal{L}^1(I),$$

so that $\mathcal{HK}(\mathbb{R}) \not\subset \mathcal{L}^p(\mathbb{R}) + \mathcal{L}^{\infty}(\mathbb{R})$. For $0 < p < 1$ the result follows by Proposition 1. □

References

[1] R. G. Bartle. A modern theory of integration. Graduate Studies in Mathematics, 32. American Mathematical Society, Providence, RI, 2001. MR 1817647.

[2] B. Bongiorno, T. V. Panchapagesan. *On the Alexiewicz topology of the Denjoy space*. Real Anal. Exchange. **21** (1995/96), 604–614. MR 1407272.

[3] V. G. Čelidze, A. G. Džvaršeĭšvili. The theory of the Denjoy integral and some applications. Series in Real Analysis, 3. World Scientific, Teaneck, NJ, 1989. MR 1036270.

[4] M. Federson. *Some peculiarities of the Henstock and Kurzweil integrals of Banach space-valued functions*. Real Anal. Exchange. **29** (2003/04), 439–460. MR 2063085.

[5] G. B. Folland. Real analysis. Modern techniques and their applications. Second edition. Pure and Applied Mathematics. A Wiley-Interscience Publication. John Wiley & Sons, New York, 1999. MR 1681462.

[6] T. L. Gill, W. W. Zachary. *A new class of Banach spaces*. J. Phys. A **41** (2008) 495206, 15 pp. MR 2515900.

[7] L. Grafakos. Classical Fourier analysis. Third edition. Graduate Texts in Mathematics, 249. Springer, New York, 2014. MR 3243734.

[8] E. Hewitt. *Fourier transforms of the class* $\mathfrak{L}_p$. Ark. Mat. **2** (1954), 571–574. MR 0061275.

[9] E. Hewitt, K. A. Ross. Abstract harmonic analysis II. Grundlehren der matematischen Wissenschaften, 152. Springer, New York, 1970. MR 0262773.

[10] Z. M. Ma, P. Y. Lee. *An overview of classical integration theory*. Math. Medley **18** (1990), 49–61. MR 1106162.

[11] F. J. Mendoza Torres. *On pointwise inversion of the Fourier transform of* BV_0 *functions*. Ann. Funct. Anal. **1** (2010), 112–120. MR 2772044.

[12] F. J. Mendoza Torres, M. G. Morales Macías, J. A. Escamilla Reyna, J. H. Arredondo Ruiz. *Several aspects around the Riemann–Lebesgue lemma*. J. Adv. Res. Pure Math. **5** (2013), 33–46. MR 3041342.

[13] M. A. Pinsky. Introduction to Fourier analysis and wavelets. Brooks/Cole Series in Advanced Mathematics. Brooks/Cole, Pacific Grove, CA, 2002. MR 2100936.

[14] M. Reed, B. Simon. Methods of modern mathematical physics. II: Fourier analysis, self-adjointness. Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London, 1975. MR 0493420.

[15] W. Rudin. Fourier analysis on groups. Interscience Tracts in Pure and Applied Mathematics, No. 12. Interscience Publishers, New York–London, 1962. MR 0152834.

[16] W. Rudin. Real and complex analysis. Third edition. McGraw-Hill, New York, 1987. MR 0924157.

[17] E. M. Stein, G. Weiss. Introduction to Fourier analysis on Euclidean spaces. Princeton Mathematical Series, No. 32. Princeton University Press, Princeton, N.J., 1971. MR 0304972.

[18] E. Talvila. *Henstock–Kurzweil Fourier transforms.* Illinois J. Math. **46** (2002), 1207–1226. MR 1988259.

M. G. Morales
Departamento de Matemáticas, Universidad Autónoma Metropolitana - Iztapalapa
Av. San Rafael Atlixco 186, México Distrito Federal, 09340, México.
mgumm@xanum.uam.mx

J. H. Arredondo ✉
Departamento de Matemáticas, Universidad Autónoma Metropolitana - Iztapalapa
Av. San Rafael Atlixco 186, México Distrito Federal, 09340, México.
iva@xanum.uam.mx

F. J. Mendoza
Facultad de Ciencias Físico Matemáticas, Benemérita Universidad de Puebla
Av. San Claudio y 18 Sur S/N, Puebla, Puebla, 72570, México.
jmendoza@fcfm.buap.mx

Received: July 15, 2015
Accepted: February 16, 2016