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KHOVANOV HOMOLOGY OF BRAID LINKS

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ABSTRACT. Although computing the Khovanov homology of links is common in literature, no general formulae have been given for all families of knots and links. We give the general formulae of the Khovanov homology of the family of 2-strand braid links $\widehat{x_1^n}$ and the family of 3-strand braid links $\widehat{\Delta^{2k}}$, where $\Delta = x_1x_2x_1$.

1. Introduction

Khovanov homology is an invariant for oriented links which was introduced by Mikhail Khovanov in 2000 as a categorification of the Jones polynomial [8].

Khovanov assigned a bigraded chain complex $C^{i,j}(L)$ to the oriented link diagram L whose differential is graded of bidegree $(1,0)$ and whose homotopy type depends only on the isotopy class of L. The bigraded homology group $H^{i,j}(D)$ of the chain complex $C^{i,j}(D)$ provides an invariant of oriented links, now known as Khovanov homology.

In 2002 Bar-Natan increased the interest of people in Khovanov homology by showing its close relationship with the Kauffman bracket; he revealed that Khovanov actually replaced the Kauffman bracket with a new bracket, which he called the Khovanov bracket [4]. Later in 2007 Bar-Natan gave a new algorithm which drastically reduces the computations of the Khovanov homology [5]. In 2005 Marko proved that the first homology group of the positive braid knot is trivial [12], which was conjectured by Khovanov [9]; all our results about positive braid knots agree with this result.

In the present paper we compute the Khovanov homology of the family of 2-strand braid links $\widehat{x_1^n}$ (Theorem 6.1), and the family of 3-strand braid links Δ^{2k} (Theorem 6.2).

This paper is organized as follows: Section 2 is devoted to links and Section 3 is devoted to braids. The Kauffman bracket and the Jones polynomial are presented in Section 4. Basic material relevant to Khovanov homology is given in Section 5. Moreover, a full example of computing the Khovanov homology of the trefoil knot $\widehat{x_1^3}$ is also given here. The last section contains our main results.

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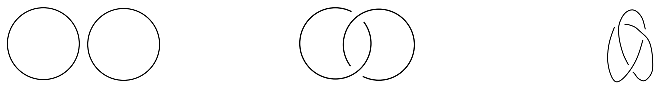
2. Links and link invariants

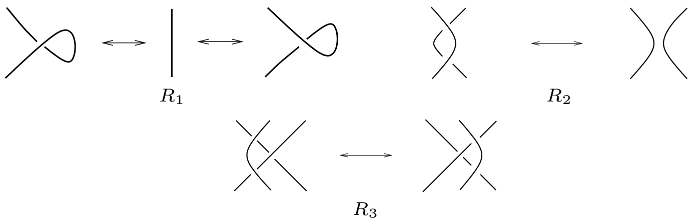
A *link* in $\mathbb{R}^3$ is a finite collection of disjoint circles smoothly embedded into $\mathbb{R}^3$. These circles are called the *components* of the link. If an orientation of the components is specified, we say that the link is oriented. A link consisting of only one component is called a *knot*.

Links are usually studied via projecting them on the plane. A projection with information of over- and under-crossing is called a link *diagram*. Throughout this article by a link we shall mean its diagram.



Trivial 2-component link Hopf link Trefoil knot

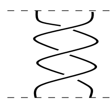
Reidemeister gave a fundamental result about the equivalence of two links: Links are equivalent if and only if one can be transformed into the other by a finite sequence of ambient isotopies of the plane and the local Reidemeister moves [11]:



To classify links one needs a link invariant, which is a function I : Links $\rightarrow$ $\{$numbers, polynomials, colours, etc$\}$ that gives the same value for all links in an isotopy class of links. To check whether a function is a link invariant one has to show that it is invariant under all the Reidemeister moves.

3. Braids

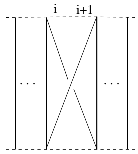
An n-strand *braid* is a set of n non intersecting smooth paths connecting n points on a horizontal plane to n points exactly below them on another horizontal plane in an arbitrary order. The smooth paths are called *strands* of the braid.



A 2-strand braid

The *product* ab of two n-strand braids, a and b, is defined by putting b below a and then gluing their common end points.

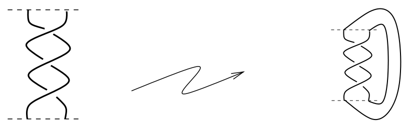
A braid with only one crossing is called the *elementary* braid; the i-th elementary braid x_i with n strands is:



x_i

A useful property of elementary braids is that every braid can be written as a product of elementary braids. For instance, the above 2-strand braid is $x_1^3 = (x_1)(x_1)(x_1)$.

The *closure* of a braid b is the link $\widehat{b}$ obtained by connecting the lower ends of b with the corresponding upper ends.



b $\widehat{b}$

Remark 3.1. All the braids are oriented from top to bottom.

An important result by Alexander connecting knots and braids is:

Theorem 3.2 ([1]). *Each link can be represented as the closure of a braid.*

4. The Kauffman bracket and the Jones polynomial

In 1985, V. F. R. Jones revolutionized knot theory by defining the Jones polynomial as a knot invariant via Von Neumann algebras [6]. However, in 1987 L. H. Kauffman introduced a state-sum model construction of the Jones polynomial that was purely combinatorial and remarkably simple [7].

A Kauffman state s of a link L is obtained by replacing each crossing (╳) of L with the 0-smoothing)(or the 1-smoothing ≍ (so that the result is a disjoint union of circles embedded in the plane). We denote by $\mathcal{S}(L)$ the set of all Kauffman states of L.

Let s be a state in $\mathcal{S}(L)$, $\gamma(s)$ the number of circles in the state, and $\alpha(s)$ and $\beta(s)$ the numbers of crossings in states 0 and 1. Then the Kauffman bracket for L is defined by the relation

$$\langle L \rangle = \sum_s q^{\alpha(s)-\beta(s)}(-q^2 - q^{-2})^{\gamma(s)-1}.$$

It is well known that the Kauffman bracket satisfies the relations:

$$\langle L \rangle = q\langle L_0 \rangle + q^{-1}\langle L_1 \rangle$$
$$\langle L \sqcup \bigcirc \rangle = (-q^2 - q^{-2})\langle L \rangle$$
$$\langle \bigcirc \rangle = 1$$

This bracket is not invariant under the first Reidemeister move; see, for instance, [10]. To overcome this difficulty, one needs something more: Let us consider that the link diagram L is now oriented. Then each crossing appears either as ╳, which is called a *positive crossing*, or as $\times$, which is called a *negative crossing*. If we denote the number of positive crossings by n_+ and the number of negative crossings by n_-, then the *unnormalized Jones polynomial* is defined by the relation $\widehat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle L \rangle$ and its normalized version by the relation $J(L) = \frac{1}{q+q^{-1}} \widehat{J}(L)$. Since this polynomial is invariant under all three Reidemeister moves, it is an invariant for oriented links.

5. Khovanov homology

Khovanov gives the idea of a graded chain complex whose differentials provide a new invariant for oriented knots and links, known as Khovanov homology, which is combinatorial and computable.

Definition 5.1. Let $V = \bigoplus_n V_n$ be a *graded vector space* with homogeneous components $\{V_n\}$.

Definition 5.2. The *degree* of the tensor product of graded vector spaces $V_1 \otimes V_2$ is the sum of the degrees of the homogeneous components of the graded vector spaces, V_1 and V_2.

Here the graded vector space V has basis $\langle v_+, v_- \rangle$, degree $p(v_\pm) = \pm 1$, and has q-dimension $q + q^{-1}$.

Definition 5.3. The *degree shift* $.\{l\}$ operation on a graded vector space $V = \bigoplus V_n$ is defined by

$$\Big(V.\{l\}\Big)_n = V_{n-l}.$$

Before we proceed, we need some preliminary definitions: Let χ be the set of crossings of L, let $n = |\chi|$, let us number the elements of χ from 1 to n in some arbitrary way, and let us write $n = n_+ + n_-$, where $n_+(n_-)$ is the number of right-handed (left-handed) crossings in χ. Label the crossing (fix an ordering) of χ. We call the $)($ the 0-smoothing and $\asymp$ the 1-smoothing of $\times$. Consider the n-cube of complete smoothing $\{0,1\}^\chi$ of a link. With every vertex α of the cube $\{0,1\}^{\mathcal{X}}$ we relate the graded vector space $V_\alpha(L) := V^{\otimes m}\{r\}$, where m is the number of cycles in the smoothing of L corresponding to α and r; $|\alpha| = \Sigma_i \alpha_i$ of α. The direct sum of the vector spaces come out in the cube, along the column, is the chain groups $[[L]]^r$, that is, $r : [[L]]^r := \bigoplus_{\alpha : r = |\alpha|} V_\alpha(L)$. Here r $(0 \le r \le n)$ is the height of a smoothing, the number of 1-smoothings in it.

Definition 5.4. The *chain complex* $\overline{C}$ of graded vector spaces $\overline{C^r}$ is defined as:

$$\ldots \to \overline{C}^{r+1} \xrightarrow{d^{r+1}} \overline{C}^r \xrightarrow{d^r} \overline{C}^{r-1} \xrightarrow{d^{r-1}} \ldots,$$

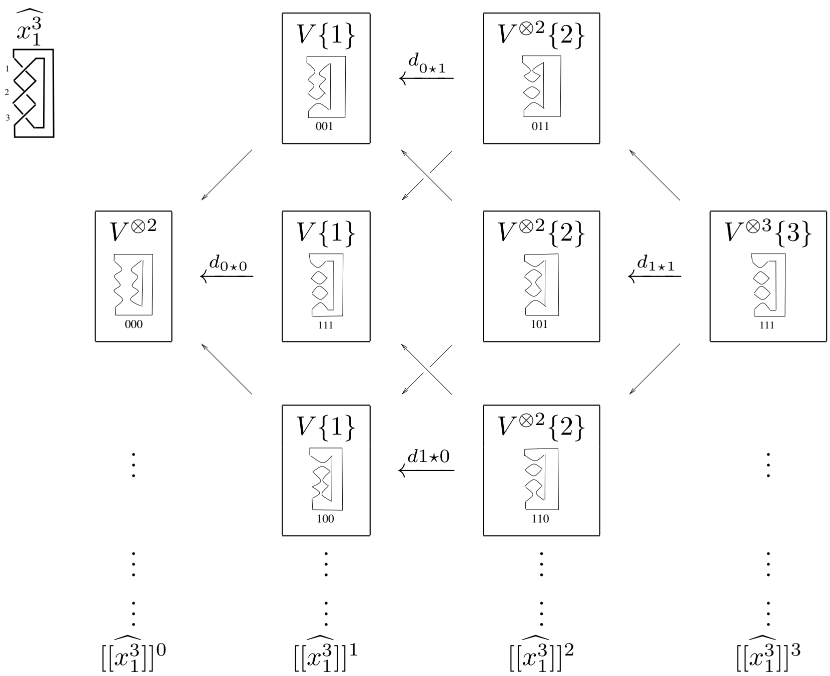
where $d^r \circ d^{r+1} = 0$ for each r.

We generally use the term "chain complex" when the differential decreases and "co-chain complex" when it increases. In order to make a chain complex of graded vector spaces $\overline{C^r}$ we need the differential maps $d^{r+1} : \overline{C}^{r+1} \to \overline{C}^r$ such that $d^r \circ d^{r+1} = 0$ for each r. For this purpose we can label the edges of the cube $\{0,1\}^\chi$ by the sequence $\xi \in \{0,1,\star\}^{\chi}$, where ξ contains only one $\star$ at a time. Here $\star$ indicates that we change a 1-smoothing to a 0-smoothing. The maps on the edges is denoted by d_ξ; the height of edges, by $|\xi|$. The direct sum of differentials in the cube along the column is

$$d^r := \sum_{|\xi|=r} (-1)^{\xi} d_\xi.$$

Now we tell the reason behind the sign $(-1)^\xi$. As we want from the differentials to satisfy $d \circ d = 0$, the maps d_ξ have to anti-commute on each of the vertex of the cube. This can be done by multiplying the edge map d_ξ by $(-1)^\xi := (-1)^{\sum_{i<j} \xi_i}$, where j is the location of the $\star$ in ξ.

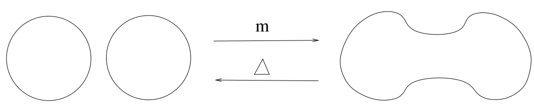
For better understanding, please see the n-cube of the trefoil knot, $\widehat{x_1^3}$.



It is useful to note that the ordered basis of V is $\langle v_+, v_- \rangle$ and the ordered basis of $V \otimes V$ is $\langle v_+ \otimes v_+, v_- \otimes v_+, v_+ \otimes v_-, v_- \otimes v_- \rangle$.

Definition 5.5. The linear map $m : V \otimes V \to V$ that merges two circles into a single circle is defined as $m(v_+ \otimes v_+) = v_+$, $m(v_+ \otimes v_-) = v_-$, $m(v_- \otimes v_+) = v_-$ and $m(v_- \otimes v_-) = 0$.

The map $\Delta : V \to V \otimes V$ that divides a circle into two circles is defined as $\Delta(v_+) = v_+ \otimes v_- + v_- \otimes v_+$ and $\Delta(v_-) = v_- \otimes v_-$.



Definition 5.6. The kernel $\ker d^r$ of the map $d^r : V^{\otimes r-1} \to V^{\otimes r}$ is the set of all elements of $V^{\otimes r-1}$ that go to the zero element of $V^{\otimes r}$. The elements of the kernel are called cycles, while the elements of $\operatorname{im} d^{r+1}$ are called boundaries.

Remark 5.7. The image of the chain complex of d^{r+1} is a subset of the kernel d^r as, in general, $d^r \circ d^{r+1} = 0$.

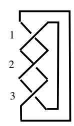
Definition 5.8. The *homology group* associated with the chain complex of a link L is defined as $\mathcal{H}^r(L) = \dfrac{\ker d^r}{\operatorname{im} d^{r+1}}$.

Definition 5.9. The *graded Poincaré polynomial* $Kh(L)$ in the variables q and t of the complex is defined as

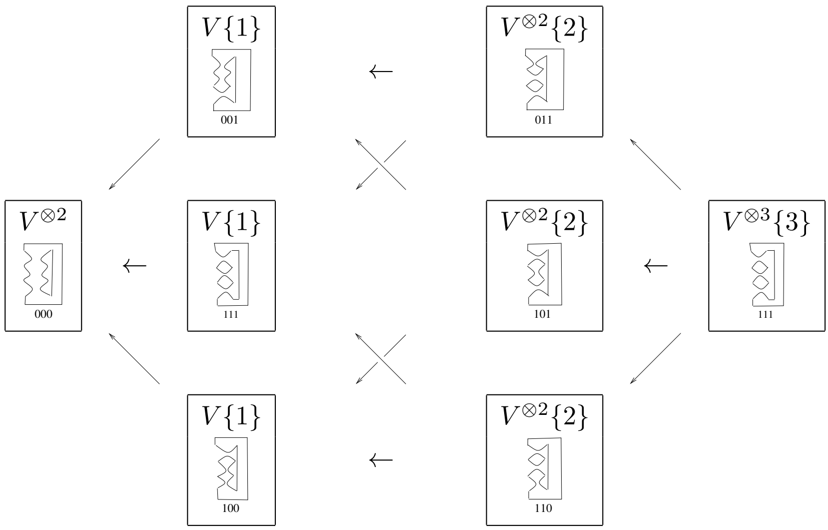
$$Kh(L) := \sum_r t^r \operatorname{qdim} \mathcal{H}^r(L).$$

Theorem 5.10 (Khovanov [8])**.** *The graded dimension of the homology groups $\mathcal{H}^r(L)$ are link invariants. The graded Poincaré polynomial $Kh(L)$ is also a link invariant and $Kh(L)\big|_{t=-1} = \hat{J}(L)$.*

Before computing the Khovanov homology, we think it is better to understand the *homological* and the *quantum* gradings. The homological grading of the Khovanov chain complex is defined as $gr(x) = c_1(v) - n_-$, where $x \in C(L)$ and $c_1(v)$ is the number of 1-smoothings in the coordinates of v. In case of chain complex, the quantum grading of the chain complex is $q(x) = -p(x) + gr(x) + n_+ - n_-$, and in case of co-chain complex it is $q(x) = p(x) + gr(x) + n_+ - n_-$. From now onwards we shall use the notation $Kh^{r,q}$ for the Khovanov homology, where the first index r indicates the homological grading and the second index q indicates the quantum grading. We need these gradings to compute the Jones polynomial from the Khovanov homology.

Example. Now we give the Khovanov homology of the link $\widehat{x_1^3}$: 

1. The n-cube:



3-cube of $\widehat{x_1^3}$

2. Chain complex: The chain complex of $\widehat{x_1^3}$ is

$$0 \xrightarrow{d^4} V^{\otimes 3} \xrightarrow{d^3} \oplus_3 V^{\otimes 2} \xrightarrow{d^2} \oplus_3 V \xrightarrow{d^1} V^{\otimes 2} \xrightarrow{d^0} 0.$$

3. Ordered basis of the chain complex: The vector spaces of the chain complex along with their ordered bases:

$$\begin{aligned} V \otimes V \otimes V = \big\langle & v_+ \otimes v_+ \otimes v_+, v_- \otimes v_+ \otimes v_+, v_+ \otimes v_- \otimes v_+, v_+ \otimes v_+ \otimes v_-, \\ & v_- \otimes v_- \otimes v_+, v_- \otimes v_+ \otimes v_-, v_+ \otimes v_- \otimes v_-, v_- \otimes v_- \otimes v_- \big\rangle, \end{aligned}$$

$$\begin{aligned} (V \otimes V) \oplus (V \otimes V) \oplus (V \otimes V) = \big\langle & (v_+ \otimes v_+, 0, 0), (0, v_+ \otimes v_+, 0), (0, 0, v_+ \otimes v_+), \\ & (v_- \otimes v_+, 0, 0), (0, v_- \otimes v_+, 0), (0, 0, v_- \otimes v_+), \\ & (v_+ \otimes v_-, 0, 0), (0, v_+ \otimes v_-, 0), (0, 0, v_+ \otimes v_-), \\ & (v_- \otimes v_-, 0, 0)(0, v_- \otimes v_-, 0), (0, 0, v_- \otimes v_-) \big\rangle, \end{aligned}$$

$$V \oplus V \oplus V = \big\langle (v_+, 0, 0), (0, v_+, 0), (0, 0, v_+), (v_-, 0, 0), (0, v_-, 0), (0, 0, v_-) \big\rangle,$$

$$V \otimes V = \big\langle v_+ \otimes v_+, v_- \otimes v_+, v_+ \otimes v_-, v_- \otimes v_- \big\rangle.$$

4. Differential maps in matrix form: The differential map $d^3\Big(v_1{\otimes}v_2{\otimes}v_3\Big) = \Big(m(v_1\otimes v_2)\otimes v_3, v_1\otimes m(v_2\otimes v_3), v_2\otimes m(v_1\otimes v_3)\Big)$ in terms of a matrix is

$$d^3 = \begin{pmatrix} 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&0&1&1&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&1&0&1&0 \end{pmatrix},$$

and the map $d^2\Big(v_1\otimes v_2, v_3\otimes v_4, v_5\otimes V_6\Big) = \Big(m(v_3\otimes v_4) - m(v_1\otimes v_2), m(v_5\otimes v_6) - m(v_1\otimes v_2), m(v_5\otimes v_6) - m(v_3\otimes v_4)\Big)$ is $d^2 = \begin{pmatrix} A & 0 & 0 & 0\\ 0 & A & A & 0\end{pmatrix}$, where $A = \begin{pmatrix} -1 & 1 & 0\\ -1 & 0 & 1\\ 0 & -1 & 1\end{pmatrix}$. Also, $d^1\Big(v_1, v_2, v_3\Big) = \Delta(v_1) - \Delta(v_2) + \Delta(v_3)$ is

$$d^1 = \begin{pmatrix} 0&0&0&0&0&0\\ 1&-1&1&0&0&0\\ 1&-1&1&0&0&0\\ 0&0&0&1&-1&1 \end{pmatrix}.$$

5. Khovanov homology: On solving $d^3.X = 0$ or

$$\begin{pmatrix} 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&1&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&1&0&1&0&0&0&0\\ 0&0&0&0&0&1&1&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&1&0&1&0 \end{pmatrix} \begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7\\ x_8 \end{pmatrix} = 0,$$

we obtain $x_1 = x_2 = x_3 = x_4 = 0$, $x_2 + x_3 = 0$, $x_3 + x_4 = 0$, $x_2 + x_4 = 0$, $x_6 + x_7 = 0$, $x_5 + x_6 = 0$, and $x_5 + x_7 = 0$. So the kernel of d^3 is

$$\left\langle \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\rangle.$$

Similarly, the image of d^3 is

$$\left\langle \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\rangle.$$

Thus,

$$\mathcal{H}^3(\widehat{x_1^3}) = \frac{\ker d^3}{\operatorname{im} d^4} = \frac{\mathbb{Z}_{(v_-\otimes v_-\otimes v_-)}}{0} = \mathbb{Z}_{(v_-\otimes v_-\otimes v_-)}.$$

To compute the homology at the next level we use a special trick: We first simply cancel out the terms which appear in both $\ker d^2$ and $\operatorname{im} d^3$. However, the last three summands of $\ker d^2$ make up all of $\mathbb{Z}^3_{(v_-\otimes v_-)}$, where the last three summands of $\operatorname{im} d^3$ span the subspace of $\mathbb{Z}^3_{(v_-\otimes v_-)}$ generated by the vectors $(0, 1, 1)$, $(1, 1, 0)$ and $(1, 0, 1)$. Since the matrix whose columns are these vectors has eigenvalues $2, 1$, and -1, we can write

$$\frac{\mathbb{Z}^3}{\langle (0,1,1),(1,1,0),(1,0,1)\rangle} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \frac{\mathbb{Z}}{\mathbb{Z}_1} \oplus \frac{\mathbb{Z}}{\mathbb{Z}_{-1}} = \mathbb{Z}_2.$$

Reducing the remaining matrices of kernel of d^2 and image of d^3 into reduced row echelon form the quotient $\dfrac{\ker d^2}{\operatorname{im} d^3}$ becomes isomorphic to $\mathbb{Z}$. Hence

$$\mathcal{H}^2(\widehat{x_1^3}) = \frac{\ker d^2}{\operatorname{im} d^3} = \mathbb{Z} \oplus \mathbb{Z}_2.$$

The range of d^2 is $\mathbb{Z}_{(v_+,v_+,0)} \oplus \mathbb{Z}_{(v_+,0,-v_+)} \oplus \mathbb{Z}_{(0,v_+,v_+)} \oplus \mathbb{Z}_{(v_-,v_-,0)} \oplus \mathbb{Z}_{(v_-,0,-v_-)} \oplus \mathbb{Z}_{(0,v_-,v_-)}$, and the kernel of d^1 is $\mathbb{Z}_{(v_+,v_+,0)} \oplus \mathbb{Z}_{(0,v_+,v_+)} \oplus \mathbb{Z}_{(v_+,0,-v_+)} \oplus \mathbb{Z}_{(v_-,v_-,0)} \oplus$

$\mathbb{Z}_{(0,v_-,v_-)} \oplus \mathbb{Z}_{(v_-,0,-v_-)}$. Since $\ker d^1 = \operatorname{im} d^2$,

$$\mathcal{H}^1(\widehat{x_1^3}) = 0.$$

It is clear from the chain complex that the kernel of d^0 is the full space $V \otimes V$.

$$\mathcal{H}^0(\widehat{x_1^3}) = \frac{\ker d^0}{\operatorname{im} d^1} = \frac{\mathbb{Z}_{(v_+\otimes v_+)} \oplus \mathbb{Z}_{(v_-\otimes v_+)} \oplus \mathbb{Z}_{(v_+\otimes v_-)} \oplus \mathbb{Z}_{(v_-\otimes v_-)}}{\mathbb{Z}_{(v_-\otimes v_+ + v_+\otimes v_-)} \oplus \mathbb{Z}_{(v_-\otimes v_-)}} = \mathbb{Z}_{(v_+\otimes v_+)} \oplus \mathbb{Z}.$$

Now

$$\begin{aligned} Kh(\widehat{x_1^3}) &= \sum_{0\le r\le 3} t^r \operatorname{qdim} \mathcal{H}^r(\widehat{x_1^3}) \\ &= t^0 \operatorname{qdim} \mathcal{H}^0(\widehat{x_1^3}) + t^1 \operatorname{qdim} \mathcal{H}^1(\widehat{x_1^3}) + t^2 \operatorname{qdim} \mathcal{H}^2(\widehat{x_1^3}) + t^3 \operatorname{qdim} \mathcal{H}^3(\widehat{x_1^3}) \\ &= t^3q^9 + t^2q^5 + t^1 0 + t^0(q^3 + q^1), \end{aligned}$$

which is actually the Poincaré polynomial of $\widehat{x_1^3}$.

Since the homological grading varies from 0 to 3 and $(n_-, n_+) = (0, 3)$, we get the following table:

		Homological grading			
Quantum grading	$Kh^{r,q}$	3	2	1	0
	1				$\mathbb{Z}_{v_+\otimes v_+}$
	3				$\mathbb{Z}$
	5		$\mathbb{Z}$		
	7		$\mathbb{Z}_2$		
	9	$\mathbb{Z}_{v_-\otimes v_-\otimes v_-}$			

6. The Jones polynomial: The unnormalized Jones polynomial of $\widehat{x_1^3}$ is

$$Kh(\widehat{x_1^3})(q)\Big|_{t=-1} = -q^9 + q^5 + q^3 + q^1.$$

6. The main theorem

Theorem 6.1 (The main theorem)**.** *a) If n is even then*

$$Kh^k(\widehat{x_1^n}) = \begin{cases} \mathbb{Z}_{v_-\otimes v_-\otimes\cdots\otimes v_-} \oplus \\ \qquad \mathbb{Z}_{v_+\otimes v_-\otimes\cdots\otimes v_- - v_-\otimes v_+\otimes\cdots\otimes v_- + \cdots - v_-\otimes v_-\otimes\cdots\otimes v_+} & \textit{if } k = n, \\ \mathbb{Z}_2 & \textit{if } k = n-1. \end{cases}$$

b) If n is odd then

$$Kh^k(\widehat{x_1^n}) = \begin{cases} \mathbb{Z}_{v_-\otimes v_-\otimes\cdots\otimes v_-} & \textit{if } k = n, \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \textit{if } k = n-1. \end{cases}$$

c) If $n \in \mathbb{Z}_+$ then

$$Kh^k(\widehat{x_1^n}) = \begin{cases} 0 & \textit{if } n-1 < k \le 1, \\ \mathbb{Z}_{v_+\otimes v_+} \oplus \mathbb{Z} & \textit{if } k = 0. \end{cases}$$

Proof. We prove it for the i-th level of the chain complex, which is

$$\ldots \to \binom{n}{i+1} V^{\otimes i+1} \xrightarrow{d^{i+1}} \binom{n}{i} V^{\otimes i} \to \ldots.$$

a) (n is even.) Since $\operatorname{im} d^{n+1} = 0$, there does not exist any space at the $(n+1)$-th level. We only need the kernel of d^n. The linear map d^n has order $n2^{n-1} \times 2^n$ and is

$$d^n = \begin{pmatrix} X_{2^1} \\ X_{2^2} \\ X_{2^3} \\ X_{2^4} \\ \vdots \\ X_{2^i} \\ \vdots \\ X_{2^{n-1}} \end{pmatrix}.$$

Here X_{2^i} is a matrix of order $n \times 2^n$:

$$X_{2^1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$X_{2^2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$X_{2^3} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$X_{2^4} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \vdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$\vdots$$

$$X_{2^i} = \begin{pmatrix} \cdots & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & \vdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 1 & 0 & \vdots & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$\vdots$$

$$X_{2^{n-1}} = \begin{pmatrix} \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The last $n+1$ columns of $X_{2^{n-1}}$ are $(1,0,\ldots,0,1)$, $(1,1,0,0,\ldots)$, $(0,1,1,0,\ldots)$, $(0,0,1,1,\ldots)$, $\ldots$, $(0,\ldots,0,1,1)$ and $(0,0,\ldots,0,0)$. Thus,

$$\ker d^n = \mathbb{Z}_{v_-\otimes v_-\otimes\cdots\otimes v_-} \oplus \mathbb{Z}_{v_+\otimes v_-\otimes\cdots\otimes v_- - v_-\otimes v_+\otimes\cdots\otimes v_- + \ldots - v_-\otimes v_-\otimes\cdots\otimes v_+}.$$

Finally, we get

$$Kh^n(\widehat{x_1^n}) = \mathbb{Z}_{v_-\otimes v_-\otimes\cdots\otimes v_-} \oplus \mathbb{Z}_{v_+\otimes v_-\otimes\cdots\otimes v_- - v_-\otimes v_+\otimes\cdots\otimes v_- + \cdots - v_-\otimes v_-\otimes\cdots\otimes v_+}.$$

Now if $i = n-1$, then the differential d^{n-1} of order $\binom{n}{2}2^{n-2} \times \binom{n}{1}2^{n-1}$ is

$$d^{n-1} = \begin{pmatrix} A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & A_1 & A_5 & A_6 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & A_2 & A_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & A_2 & A_3 & A_4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_2 & A_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Here the matrix A_i of order $\binom{n}{2} \times \binom{n}{1}$ is

$$A_1 = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ -1 & 0 & 0 & 1 & \cdots \\ 0 & -1 & 0 & 1 & \cdots \\ \vdots & \vdots & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}.$$

A_2 is generated by converting the last non zero entry of each column of A_1 to 0:

$$A_2 = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 1 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}.$$

Similarly, A_3 is generated by converting the first non zero entry of each column of A_1 into 0, A_4 is generated by converting the second non zero entry of each column of A_1 into 0, A_5 is generated by converting each non zero entry into 0 except the first 1-element of each column of A_1, and in the same way A_6 is generated by converting each non zero entry into 0 except the last 1-element of each column of A_1. Using the same trick as in the example above, we get $Kh^{n-1}(\widehat{x_1^n}) = \mathbb{Z}_2$.

b) (n is odd.) Since $\operatorname{im} d^{n+1} = 0$, there does not exist any space at the $(n+1)$-th level. We only need the kernel of d^n. The linear map d^n has order $n2^{n-1} \times 2^n$ and is

$$d^n = \begin{pmatrix} Y_{2^1} \\ Y_{2^2} \\ Y_{2^3} \\ Y_{2^4} \\ \vdots \\ Y_{2^i} \\ \vdots \\ Y_{2^{n-1}} \end{pmatrix}.$$

Here each matrix Y_{2^i} has order $n \times 2^n$:

$$Y_{2^1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$Y_{2^2} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$Y_{2^3} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$Y_{2^4} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \vdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$\vdots$$

$$Y_{2^i} = \begin{pmatrix} \cdots & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & \vdots & 0 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 1 & 0 & \vdots & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$\vdots$$

$$Y_{2^{n-1}} = \begin{pmatrix} \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

Here, the last $n+1$ columns of $Y_{2^{n-1}}$ are $(1,0,\ldots,0,1)$, $(1,1,0,0,\ldots)$, $(0,1,1,0,\ldots)$, $(0,0,1,1,\ldots)$, $\ldots$, $(0,\ldots,0,1,1)$ and $(0,0,\ldots,0,0)$. Thus, $\ker d^n = \mathbb{Z}_{v_-\otimes v_-\otimes\cdots\otimes v_-}$. Hence, $Kh^n(\widehat{x_1^n}) = \mathbb{Z}_{v_-\otimes v_-\otimes\cdots\otimes v_-}$.

If $i = n-1$, the differential d^{n-1} of order $\binom{n}{2}2^{n-2} \times \binom{n}{1}2^{n-1}$ is

$$d^{n-1} = \begin{pmatrix} B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \cdots \\ 0 & B_1 & B_3 & B_4 & 0 & 0 & 0 & 0 & 0 & & \cdots \\ 0 & 0 & B_2 & B_5 & 0 & 0 & 0 & 0 & 0 & & \cdots \\ 0 & 0 & 0 & 0 & & B_5 & B_6 & 0 & 0 & & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_6 & B_3 & B_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Here each matrix B_i has order $\binom{n}{2} \times \binom{n}{1}$:

$$B_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & -1 & 0 & \dots \\ 0 & 1 & 0 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & 0 & \dots \\ 1 & 0 & -1 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & -1 & \dots \\ 0 & 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 1 & 0 & -1 & \dots \\ 0 & 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & -1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

B_2 is generated by converting the last non zero entry of each column of B_1 to 0, B_3 is generated by converting each non zero entry into 0 except the first non zero entry of each column of B_1, B_4 is generated by converting each non zero entry into 0 except the last non zero entry of each column of B_1, B_5 is generated by converting the first non zero entry of each column of B_1 into 0, B_6 is generated by converting the second non zero entry of each column of B_1 into 0.

Using the same trick as in the example above, we finally get

$$Kh^{n-1}(\widehat{x_1^n}) = \mathbb{Z} \oplus \mathbb{Z}_2.$$

c) Note that $\ker d^0 = \mathbb{Z}_{v_+ \otimes v_+} \oplus \mathbb{Z}_{v_+ \otimes v_-} \oplus \mathbb{Z}_{v_- \otimes v_+} \oplus \mathbb{Z}_{v_- \otimes v_-}$. The differential d^1 of order $4 \times 2n$ is

$$d^1 = \left(\begin{array}{ccccc|ccccc} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 & 1 & \cdots & 1 \end{array}\right).$$

Each of the first n columns appears with alternate signs; the first column receives the positive sign. Each of the columns from $n+1$ to $2n$ appears with alternate signs. It follows that the image of d^1 is spanned by two spaces, that is, $\mathbb{Z}_{v_+ \otimes v_- - v_- \otimes v_+}$ and $\mathbb{Z}_{v_- \otimes v_-}$. So $Kh^0(\widehat{x_1^n}) = \mathbb{Z}_{v_+ \otimes v_+} \oplus \mathbb{Z}$. Now if $i = 1$, then

$$d^2 = \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & P & 0 \end{pmatrix}_{2n \times 4\binom{n}{2}},$$

where

$$P = \begin{pmatrix} P_1 & P_2 & P_3 & \cdots & P_{n-2} & P_{n-1} \end{pmatrix}_{n \times \binom{n}{2}}.$$

Here each matrix P_i has order $n-(n-i)$:

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 1 & -1 & 0 & \cdots \\ 0 & -1 & 1 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \; P_2 = \begin{pmatrix} -1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 1 & 0 & -1 & \cdots \\ 0 & -1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$P_3 = \begin{pmatrix} -1 & 0 & \cdots \\ 0 & 1 & \cdots \\ 0 & 0 & \cdots \\ 1 & 0 & \cdots \\ 0 & -1 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \; \ldots, \; P_{n-2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } P_{n-1} = \begin{pmatrix} -1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

It follows from simple computation that $\ker d^1 = \operatorname{im} d^0$. This completes the proof. □

The homology of the link $\widehat{\Delta^{2k}}$, where $\Delta = x_1x_2x_1$, is given in the following theorem.

Theorem 6.2.

$$Kh^i(\widehat{\Delta^{2k}}) = \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}^2 & \text{if } i = 6k, \\ 0 & \text{if } 3 \le i \le 6k-1, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 2, \\ 0 & \text{if } i = 1, \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } i = 0. \end{cases}$$

Proof. The cochain complex of the link $\widehat{\Delta^{2k}}$ is

$$0 \xrightarrow{d^{-1}} V^{\otimes 3} \xrightarrow{d^0} \oplus_{6k} V^{\otimes 2} \xrightarrow{d^1} \oplus_{\binom{2k}{1}\binom{4k}{1}} V^{\otimes 1} \oplus_{\binom{2k}{2}+\binom{4k}{2}} V^{\otimes 3}$$
$$\xrightarrow{d^2} \oplus_{\left[\binom{2k}{1}\binom{4k}{2}+\binom{2k}{2}\binom{4k}{1}\right]} V^{\otimes 2} \oplus_{\left[\binom{2k}{3}+\binom{4k}{3}\right]} V^{\otimes 4} \xrightarrow{d^3} \cdots$$
$$\xrightarrow{d^{6k-2}} \oplus_{\binom{4k}{1}} V^{\otimes 2k} \oplus_{\binom{2k}{1}} V^{\otimes 2k+2} \xrightarrow{d^{6k-1}} V^{\otimes 2k+1} \xrightarrow{d^{6k}} 0,$$

where the symbol $\binom{n}{c}$ is a binomial coefficient and $V^{\otimes 1}$ is actually V. If $i = 0$, then

$$Kh^0(\widehat{\Delta^{2k}}) = \frac{\ker d^0}{\operatorname{im} d^{-1}}.$$

The linear map d^0 is $d^0 = \begin{pmatrix} A & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & 0 & 0 & B & 0 & 0 & 0 \\ 0 & 0 & A & 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & A & 0 & C & B & 0 \end{pmatrix}_{24k\times 8}$. Here the matrices A, B, and C of order $6k \times 1$ are:

$$A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Since the image of d^{-1} is zero, there does not exist any space at the (-1)-th level. Thus, the homology at this level is spanned by two spaces $\mathbb{Z}_{v_-\otimes v_-\otimes v_-}$ and $\mathbb{Z}_{v_+\otimes v_-\otimes v_- - v_-\otimes v_+\otimes v_- + v_-\otimes v_-\otimes v_+}$:

$$Kh^0(\widehat{\Delta^{2k}}) = \mathbb{Z}_{v_-\otimes v_-\otimes v_-} \oplus \mathbb{Z}_{v_+\otimes v_-\otimes v_- - v_-\otimes v_+\otimes v_- + v_-\otimes v_-\otimes v_+}.$$

Now if $i = 1$,

$$d^1 = \begin{pmatrix} R_1 & 0 & 0 & 0 & \dots & 0 \\ 0 & R_1 & 0 & 0 & \dots & 0 \\ 0 & 0 & R_1 & 0 & \dots & 0 \\ 0 & 0 & 0 & R_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & R_1 \\ R_2 & 0 & 0 & 0 & \dots & 0 \\ 0 & R_2 & 0 & 0 & \dots & 0 \\ 0 & 0 & R_2 & 0 & \dots & 0 \\ 0 & 0 & 0 & R_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & R_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ R_n & 0 & 0 & 0 & \dots & 0 \\ 0 & R_n & 0 & 0 & \dots & 0 \\ 0 & 0 & R_n & 0 & \dots & 0 \\ 0 & 0 & 0 & R_n & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & R_n \end{pmatrix}_{12(6k)\times 4(6k)}.$$

Here the order of R_i is $(6k-i)\times(6k-(i-1))$:

$$R_1=\begin{pmatrix}1&1&0&0&\dots&0\\1&0&1&0&\dots&0\\1&0&0&1&\dots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\1&0&0&0&\dots&1\end{pmatrix},\quad R_2=\begin{pmatrix}0&1&1&0&0&\dots&0\\0&1&0&1&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&1&0&0&0&\dots&1\end{pmatrix},$$

$$R_3=\begin{pmatrix}0&0&1&1&0&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&1&0&0&0&\dots&1\end{pmatrix},\dots,R_n=\begin{pmatrix}\cdots&\cdots&\cdots&\cdots&1&1\end{pmatrix}.$$

It follows from a simple computation that $\ker d^1=\operatorname{im} d^0$.

If $i=2$, then the differential map d^2 of order $2^5k(9k^2-6k+1)\times 2^3k(12k-3)$ is:

$$d^2=\begin{pmatrix}S_1&S_2&S_3&S_4&S_5&S_6&S_7&S_8&S_9\\S_{10}&S_{11}&S_{12}&S_{13}&S_{14}&S_{15}&S_{16}&S_{17}&S_{18}\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\S_{n-8}&S_{n-7}&S_{n-6}&S_{n-5}&S_{n-4}&S_{n-3}&S_{n-2}&S_{n-1}&S_n\end{pmatrix}.$$

Here the order of S_i is $2^3k(9k^2-6k+1)\times 2k(12k-3)$:

$$S_1=\begin{pmatrix}0&-1&0&0&0&0&\dots&0\\1&0&0&0&0&1&\dots&0\\1&0&0&0&0&1&\dots&0\\0&0&0&0&0&0&\dots&0\\0&0&-1&0&0&0&\dots&0\\1&0&0&0&0&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&0&0&\dots&0\end{pmatrix},\ S_2=\begin{pmatrix}0&0&0&0&0&0&\dots&0\\0&0&0&0&0&0&\dots&0\\0&0&0&0&0&0&\dots&0\\0&0&0&0&0&0&\dots&1\\0&0&0&0&0&0&\dots&0\\0&0&0&0&0&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&0&0&\dots&1\end{pmatrix},$$

$$S_3=\begin{pmatrix}0&0&0&0&0&0&\dots&0\\-1&0&0&0&0&0&\dots&0\\0&0&0&0&0&0&\dots&0\\0&0&0&0&1&0&\dots&0\\0&0&0&0&0&0&\dots&0\\0&-1&0&0&0&0&\dots&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&0&1&\dots&0\end{pmatrix},\ S_4=\begin{pmatrix}0&0&0&0&0&0&\dots&0\\0&0&0&0&0&0&\dots&0\\0&0&0&0&0&0&\dots&0\\0&0&0&0&0&0&\dots&0\\0&0&0&0&0&0&\dots&0\\0&0&0&0&0&0&\dots&-1\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\0&0&0&0&0&0&\dots&0\end{pmatrix},\dots,$$

$$S_{n-2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \; S_{n-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & \dots & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -1 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$\text{and} \quad S_n = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

It follows that

$$Kh^2(\widehat{\Delta^{2k}}) = \mathbb{Z} \oplus \mathbb{Z}.$$

For $3 \leq i \leq 6k-1$ we prove for the i-th level of the cochain complex, which is

$$\cdots \xrightarrow{d^{i-1}} \oplus_{\left[\binom{2k}{1}\binom{4k}{i-1}+\binom{2k}{2}\binom{4k}{i-2}+\cdots+\binom{2k}{i-1}\binom{4k}{1}\right]} V^{\otimes i-1} \oplus_{\left[\binom{2k}{i}+\binom{4k}{i}\right]} V^{\otimes i+1}$$
$$\xrightarrow{d^{i}} \oplus_{\left[\binom{2k}{1}\binom{4k}{i}+\binom{2k}{2}\binom{4k}{i-1}+\cdots+\binom{2k}{i}\binom{4k}{1}\right]} V^{\otimes i} \oplus_{\left[\binom{2k}{i+1}+\binom{4k}{i+1}\right]} V^{\otimes i+2} \xrightarrow{d^{i+1}} \cdots .$$

The differential d^{i-1} of order $\left[\binom{2k}{1}\binom{4k}{i-1} + \binom{2k}{2}\binom{4k}{i-2} + \cdots + \binom{2k}{i-1}\binom{4k}{1}\right] 2^{i-1} \times \left[\binom{2k}{i} + \binom{4k}{i}\right] 2^i$ is

$$d^{i-1} = \begin{pmatrix} W_1 & W_2 & W_3 & W_4 & W_5 & W_6 & \dots & W_{r-1} & W_r \end{pmatrix}.$$

Here the order of W_i is $\left[\binom{2k}{1}\binom{4k}{i-1} + \binom{2k}{2}\binom{4k}{i-2} + \cdots + \binom{2k}{i-1}\binom{4k}{1}\right] 2^{i-1} \times \left[\binom{2k}{i} + \binom{4k}{i}\right] 2^{i-2}$:

$$W_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$W_2 = \begin{pmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & \dots & 0
\end{pmatrix},$$

$$W_3 = \begin{pmatrix}
1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0
\end{pmatrix},$$

$$\vdots$$

$$W_{r-1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0
\end{pmatrix},$$

$$\text{and } W_r = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&1&0&0&0&-1&\dots&0\\ 1&0&0&0&-1&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&0&0&0&1&-1&\dots&0\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ -1&0&0&0&0&-1&0&0&0&0&0&\dots&0 \end{pmatrix}.$$

The differential d^i of order $\left[\binom{2k}{1}\binom{4k}{i}+\binom{2k}{2}\binom{4k}{i-1}+\cdots+\binom{2k}{i}\binom{4k}{1}\right]\times\left[\binom{2k}{i+1}+\binom{4k}{i+1}\right]$ is

$$d^{i-1} = \begin{pmatrix} Q_1 & Q_2 & Q_3 & Q_4 & \dots & Q_{t-1} & Q_t \end{pmatrix}.$$

Here the order of Q_i is $\left[\binom{2k}{1}\binom{4k}{i-1}+\binom{2k}{2}\binom{4k}{i-2}+\cdots+\binom{2k}{i-1}\binom{4k}{1}\right]2^{i-1}\times\left[\binom{2k}{i}+\binom{4k}{i}\right]2^{i-2}$:

$$Q_1 = \begin{pmatrix} 1&0&0&0&0&0&-1&0&0&0&1&\dots&0\\ 0&-1&0&1&0&0&0&0&0&0&0&\dots&0\\ 0&-1&0&1&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&0&-1&0&0&0&\dots&0\\ 0&0&0&0&1&0&0&0&0&0&0&\dots&0\\ 0&0&-1&0&1&0&0&0&0&0&0&\dots&0\\ 1&0&-1&0&0&0&0&0&0&0&0&\dots&1\\ 1&0&0&0&0&0&0&0&0&0&0&\dots&1\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 1&-1&0&0&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&1&-1&0&0&0&\dots&0\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ 1&0&-1&0&0&0&0&0&0&0&0&\dots&0\\ -1&0&1&0&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&1&0&1&0&0&\dots&0\\ 0&0&0&0&0&0&-1&0&-1&0&0&\dots&0\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ 0&0&0&0&0&0&0&0&0&0&0&\dots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&0&0&0&0&0&1&-1&\dots&0 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 1 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \dots & 0 \end{pmatrix},$$

$$\vdots$$

$$Q_{t-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix},$$

$$\text{and } Q_t = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & \dots & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -1 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

By simple computation, we get $\ker d^i = \operatorname{im} d^{i-1}$. If $i = 6k$, then $d^{6k-1} = \begin{pmatrix} Y_1 & Y_2 & Y_3 & Y_4 & Y_5 & 0 \end{pmatrix}_{2^{2k+1} \times 3k2^{2k+2}}$. The matrices Y_i have order $2^{2k+1} \times$

2^{2k+1} and are:

$$Y_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$Y_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & \cdots \\ -1 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & \cdots \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$Y_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & \cdots \\ -1 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & \cdots \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix},$$

$$\text{and } Y_5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & \cdots & 1 \end{pmatrix}.$$

Since the kernel of d^{6k} is $V^{\otimes 2k+1}$, we get

$$Kh^{6k}(\widehat{\Delta^{2k}}) = \mathbb{Z}^2 \oplus \mathbb{Z}^2. \qquad \square$$

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