

CLASSIFICATION OF REAL SOLVABLE LIE ALGEBRAS WHOSE SIMPLY CONNECTED LIE GROUPS HAVE ONLY ZERO OR MAXIMAL DIMENSIONAL COADJOINT ORBITS

ANH VU LE, VAN HIEU HA, ANH TUAN NGUYEN,
TRAN TU HAI CAO, AND THI MONG TUYEN NGUYEN

ABSTRACT. We study a special subclass of real solvable Lie algebras having small dimensional or small codimensional derived ideals. It is well-known that the derived ideal of any Heisenberg Lie algebra is 1-dimensional and the derived ideal of the 4-dimensional real Diamond algebra is 1-codimensional. Moreover, all the coadjoint orbits of any Heisenberg Lie group as well as 4-dimensional real Diamond group are orbits of dimension zero or maximal dimension. In general, a (finite dimensional) real solvable Lie group is called an MD-group if its coadjoint orbits are zero-dimensional or maximal dimensional. The Lie algebra of an MD-group is called an MD-algebra and the class of all MD-algebras is called MD-class. Simulating the mentioned above characteristic of Heisenberg Lie algebras and 4-dimensional real Diamond algebra, we give a complete classification of MD-algebras having 1-dimensional or 1-codimensional derived ideals.

1. CLASSIFICATION OF SOLVABLE LIE ALGEBRAS: A QUICK INTRODUCTION

Classifying all Lie algebras of dimension less than 4 is an elementary exercise. However, when considering Lie algebras of dimension n ($n \geq 4$), complete classifications are much harder. As it has long been well known, there exist three different types of Lie algebras: the semisimple, the solvable, and those which are neither semi-simple nor solvable. By the Levi-Maltsev theorem [18] (1945), any finite-dimensional Lie algebra over a field of characteristic zero can be expressed as a semidirect sum of a semi-simple subalgebra and its maximal solvable ideal. It reduces the task of classifying all finite-dimensional Lie algebras to obtaining the classification of semi-simple and of solvable Lie algebras.

The problem of classifying semi-simple Lie algebras over the complex field was completely solved by E. Cartan [3] in 1894; and over the real field, by F. R. Gantmakher [9] in 1939.

2010 *Mathematics Subject Classification*. Primary 17Bxx, 22E60, Secondary 20G05.

Key words and phrases. K -orbit, MD-algebra, $\text{MD}(*, 1)$ -algebra, $\text{MD}(*, * - 1)$ -algebra.

The authors would like to thank the scientific program of University of Economics and Law, Vietnam National University, Ho Chi Minh City and the University of Physical Education and Sports, Ho Chi Minh City, Vietnam for their financial support.

Although several classifications of solvable Lie algebras of small dimension are known, the problem of the complete classification of the (real or complex) solvable Lie algebras is still open. There are at least two ways of proceeding in the classification of solvable Lie algebras: *by dimension* or *by structure*.

Firstly, we list some results about the classification of solvable Lie algebras in the dimensional approach:

- All solvable Lie algebras up to dimension 6 over the complex field \mathbb{C} and the real field \mathbb{R} were classified by G. M. Mubarakzyanov [19] in 1963 and by P. Turkowski [26] in 1990.
- All solvable Lie algebras up to dimension 4 over any perfect field were classified by J. Patera and H. Zassenhaus [20] in 1990.
- Some incomplete classifications of solvable Lie algebras in dimension 7 and nilpotent algebras up to dimension 8 were given by G. Tsagas [24] in 1999.

It seems to be very difficult to proceed by dimension in the classification of Lie algebras of dimension greater than 6. However, it is possible to proceed by structure, i.e. to classify solvable Lie algebras with some specific given properties. Now, we list some results about the classification of solvable Lie algebras in the structural approach:

- In 1973, M. A. Gauger [10] gave a complete classification of metabelian Lie algebras of dimension no more than 7 and nearly complete results for dimension 8.
- In 1995, D. Arnal, M. Cahen, and J. Ludwig [1] gave the list of all solvable Lie algebras such that the coadjoint orbits of the connected Lie groups corresponding to them are of dimension zero or two. But they have not classified them yet, up to isomorphism.
- In 1999, L. Yu. Galitski and D. A. Timashev [8] completely classified all of metabelian Lie algebras of dimension 9.
- In 2007, R. Campoamor-Stursberg [4] gave a complete classification of nine-dimensional Lie algebras with nontrivial Levi decomposition.
- In 2007, I. Kath [11] classified the class of nilpotent quadratic Lie algebras of dimension no more than 10.
- In 2010, another class of Lie algebras relating to the nilradicals has been classified by L. Šnobl [21].
- In 2012, M. T. Duong, G. Pinczon, and R. Ushirobira [7] gave a classification of solvable singular quadratic Lie algebras.
- In 2012, L. Chen [5] classified a class of solvable Lie algebras with triangular decompositions.

In an attempt to classify solvable Lie algebras by structure, we study in this paper a special subclass of real solvable Lie algebras having small dimensional or small codimensional derived ideals. This idea comes from an investigation of Kirillov's Orbit Method on the $(2m + 1)$ -dimensional Heisenberg Lie algebras ($0 < m \in \mathbb{N}$) and the 4-dimensional real Diamond Lie algebra. Recall that, in 1962, A. A. Kirillov [12] introduced the Orbit Method which quickly became the most important method in the theory of representations of Lie groups and Lie algebras.

The key of Kirillov's Orbit Method is the coadjoint orbits or K -orbits (i.e., orbits in the coadjoint representation) of Lie groups. We emphasize that any K -orbit of the $(2m+1)$ -dimensional Heisenberg Lie group and the 4-dimensional real Diamond Lie group has dimension zero or maximal. Hence, it is reasonable to consider the class of solvable Lie groups (and corresponding algebras) having the similar property. A (finite dimensional) real solvable Lie group is called an MD-group (in term of N. D. Do [6]) if its K -orbits are orbits of dimension zero or maximal dimension. The Lie algebra of an MD-group is called an MD-algebra and the class of all MD-algebras is called MD-class. In particular, if the maximal dimension of the K -orbits of some MD-group G is equal to $\dim G$ then G is called an SMD-group and its Lie algebra is called an SMD-algebra. The class of all SMD-algebras is called SMD-class. It is clear that SMD-class is a subclass of MD-class.

The investigation of MD-class was suggested for the first time by N. D. Do [6] in 1982. Now, we list the main results about MD-class:

- In 1984, H. V. Ho [27] completely classified all of SMD-algebras (of arbitrary dimension).
- In 1990, A. V. Le [13–15] gave a complete classification of all 4-dimensional MD-algebras.
- In 1995, D. Arnal, M. Cahen, and J. Ludwig [1] gave the list of all MD-algebras such that the maximal dimension of K -orbits of corresponding MD-groups is just 2. However, they have not yet classified them up to isomorphism.
- Up to 2012, A. V. Le et al. [16, 17] had classified (up to isomorphism) all of MD-algebras of dimension 5.
- In 2013, the MD-class was listed as a specific attention in classification of Lie Algebras by L. Boza, E. M. Fedriani, J. Núñez, and A. F. Tenorio [2].

The investigation of general properties of MD-class, in particular the complete classification of MD-class, is still open up to now.

As we say above, the $(2m+1)$ -dimensional real Heisenberg Lie algebra and the 4-dimensional real Diamond Lie algebra are MD-algebras. The real Lie Heisenberg algebras and their extensions are investigated by a lot of mathematicians because of their physical origin and applications. Moreover, the first derived ideal of the Heisenberg Lie algebra is 1-dimensional and the first derived ideal of the 4-dimensional Diamond Lie algebra is 1-codimensional. We will generalize these properties to consider MD-algebras having the first derived ideal of dimension 1 or codimension 1. For convenience, we shall denote by $\text{MD}(*, 1)$ or $\text{MD}(*, * - 1)$ the subclasses of MD-algebras having 1-dimensional or 1-codimensional derived ideals, respectively. If \mathcal{G} belongs to $\text{MD}(*, 1)$ or $\text{MD}(*, * - 1)$ then it is called an $\text{MD}(*, 1)$ -algebra or $\text{MD}(*, * - 1)$ -algebra, respectively. In particular, every $\text{MD}(*, 1)$ -algebra or $\text{MD}(*, * - 1)$ -algebra of dimension n is called an $\text{MD}(n, 1)$ -algebra or $\text{MD}(n, n - 1)$ -algebra, respectively. Of course, the $(2m+1)$ -dimensional Heisenberg Lie algebra belongs to $\text{MD}(2m+1, 1)$ and the 4-dimensional real Diamond Lie algebra belongs to $\text{MD}(4, 3)$. The main purpose of this paper is to completely classify, up to isomorphism, $\text{MD}(*, 1)$ -class and $\text{MD}(*, * - 1)$ -class. We

also prove that any real solvable Lie algebra having 1-dimensional derived ideal belongs to $\text{MD}(*, 1)$ and give a sufficient and necessary condition in order that a n -dimensional real solvable Lie algebra having 1-codimensional derived ideal belongs to $\text{MD}(n, n - 1)$ with $n > 4$.

The next part of the paper will be organized as follows: Section 2 gives some basic concepts, especially we recall the Lie algebra of the group of the affine transformations of the real straight line, the real Heisenberg Lie algebras and the real Diamond Lie algebras. Section 3 deals with some well-known remarkable classifications of some subclasses of MD-algebras. The main results about the complete classifications of $\text{MD}(*, 1)$ -class and $\text{MD}(*, * - 1)$ -class are given in Section 4. The last section is devoted to the discussion of some open problems.

2. SOME BASIC CONCEPTS

First of all, we recall in this section some preliminary results and notations which will be used later. For details we refer the reader to the book [12] of A. A. Kirillov and the book [6] of N. D. Do.

2.1. The coadjoint representation and coadjoint orbits. Let G be a Lie group, $\mathcal{G} = \text{Lie}(G)$ be the corresponding Lie algebra of G , and \mathcal{G}^* be the dual space of \mathcal{G} . For every $g \in G$, we denote the internal automorphism associated with g by $A_{(g)}$, and hence, $A_{(g)} : G \rightarrow G$ can be defined as follows: $A_{(g)} := g.x.g^{-1}$, $\forall x \in G$.

This automorphism induces the map $A_{(g)}^* : \mathcal{G} \rightarrow \mathcal{G}$, which is defined as follows:

$$A_{(g)}^*(X) := \frac{d}{dt} [g \cdot \exp(tX) \cdot g^{-1}]|_{t=0}, \quad \forall X \in \mathcal{G}.$$

This map is called *tangent map* of $A_{(g)}$.

Definition 2.1. The action

$$\begin{aligned} Ad : G &\rightarrow \text{Aut}(\mathcal{G}) \\ g &\mapsto A_{(g)}^* \end{aligned}$$

is called *the adjoint representation of G in \mathcal{G}* .

The coadjoint representation is the dual of the adjoint representation. Namely, we have the following definition.

Definition 2.2. *The coadjoint representation or K -representation*

$$\begin{aligned} K : G &\rightarrow \text{Aut}(\mathcal{G}^*) \\ g &\mapsto K_{(g)} \end{aligned}$$

of G in \mathcal{G}^* is defined by

$$\langle K_{(g)}F, X \rangle := \langle F, Ad(g^{-1})X \rangle, \quad (F \in \mathcal{G}^*, X \in \mathcal{G}),$$

where $\langle F, Y \rangle$ denotes the value of a linear functional $F \in \mathcal{G}^*$ on an arbitrary vector $Y \in \mathcal{G}$.

A geometrical interpretation of the coadjoint representation of G is as the action by left-translation on the space of right-invariant 1-form on G .

Definition 2.3. Each orbit of the coadjoint representation of G is called a K -orbit of G .

We denote the K -orbit containing F by Ω_F . For every $F \in \mathcal{G}^*$, the K -orbit containing F can be defined by $\Omega_F := \{K_{(g)}F \mid g \in G\}$. The dimension of every K -orbit of an arbitrary Lie group G is always even. In order to define the dimension of the K -orbits Ω_F for each F in the dual space \mathcal{G}^* of the Lie algebra $\mathcal{G} = \text{Lie}(G)$, it is useful to consider the following (skew-symmetric bilinear) Kirillov form B_F on \mathcal{G} corresponding to F : $B_F(X, Y) = \langle F, [X, Y] \rangle$ for all $X, Y \in \mathcal{G}$. Denote the stabilizer of F under the co-adjoint representation of G in \mathcal{G}^* by G_F and $\mathcal{G}_F := \text{Lie}(G_F)$.

We shall need in the sequel the following result.

Proposition 2.4 ([12, Section 15.1]). *Under the above notations, we have that $\ker B_F = \mathcal{G}_F$ and $\dim \Omega_F = \dim \mathcal{G} - \dim \mathcal{G}_F = \text{rank } B_F$.*

2.2. MD-groups and MD-algebras and some of their properties.

Definition 2.5. An n -dimensional MD-group or, for brevity, an MDn-group is an n -dimensional real solvable Lie group such that its K -orbits are orbits of dimension zero or maximal dimension. The Lie algebra of an MDn-group is called an MDn-algebra. MD-class and MDn-class are the sets of all MD-algebras (of arbitrary dimension) and MDn-algebras, respectively.

Definition 2.6. An MD(n, m)-algebra is an MDn-algebra whose first derived ideal is m -dimensional with $m, n \in \mathbb{N}$ and $0 < m < n$. MD(n, m)-class is the set of all MD(n, m)-algebras. In particular, MD($*$, 1)-class and MD($*$, $*$ - 1)-class are the sets of all MD-algebras (of arbitrary dimension) having the first derived ideal of dimension 1 and codimension 1, respectively.

Remark 2.7. Note that all the Lie algebras of dimension n ($n \leq 3$) are MD-algebras, and moreover they can be listed easily. So we only take interest in MDn-algebras for $n \geq 4$.

For any real Lie algebra \mathcal{G} , as usual, we denote the first and second derived ideals of \mathcal{G} by $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}]$ and $\mathcal{G}^2 := [\mathcal{G}^1, \mathcal{G}^1]$, respectively. Now, we introduce some well-known properties of MD-algebras.

Firstly, the following proposition gives a necessary condition for a Lie algebra belonging to MD-class.

Proposition 2.8 ([27, Theorem 4]). *Let \mathcal{G} be an MD-algebra. Then its second derived ideal \mathcal{G}^2 is commutative.*

We point out here that the converse of the above result is in general not true. In other words, the above necessary condition is not a sufficient condition.

Proposition 2.9 ([6, Proposition 2.1]). *Let \mathcal{G} be an MD-algebra. If $F \in \mathcal{G}^*$ is not vanishing perfectly in \mathcal{G}^1 , i.e. there exists $U \in \mathcal{G}^1$ such that $\langle F, U \rangle \neq 0$, then the K -orbit Ω_F has maximal dimension.*

Proposition 2.10 ([17]). *There is no MD-algebra \mathcal{G} such that its second derived ideal \mathcal{G}^2 is not trivial and $\dim \mathcal{G}^2 = \dim \mathcal{G}^1 - 1$. In other words, if $0 < \dim \mathcal{G}^2 = \dim \mathcal{G}^1 - 1$ then \mathcal{G} is not an MD-algebra.*

To illustrate and show the role of the MD-class, in the rest of this section we will introduce some typical examples and counter-examples of MD-algebras.

2.3. The Lie algebra of the group of affine transformations of the real straight line. The Lie algebra $\text{aff}(\mathbb{R})$ of the group $\text{Aff}(\mathbb{R})$ of affine transformations of the real straight line \mathbb{R} is the unique non-commutative real Lie algebra of dimension 2 and it is defined as follows:

$$\text{aff}(\mathbb{R}) := \text{Span}(X, Y); \quad [X, Y] = Y.$$

Remark 2.11. Clearly, every real Lie algebra of dimension $n \leq 3$ is an MD-algebra. In particular $\text{aff}(\mathbb{R})$ is an MD(2, 1)-algebra.

2.4. The real Heisenberg Lie algebras. The $(2m + 1)$ -dimensional real Heisenberg Lie algebra ($0 < m \in \mathbb{N}$) is the following real Lie algebra:

$$\mathfrak{h}_{2m+1} := \text{Span}(X_i, Y_i, Z \mid i = 1, 2, \dots, m), \quad [X_i, Y_i] = Z, \quad i = 1, 2, \dots, m;$$

the other Lie brackets are trivial.

Remark 2.12. The first derived ideal $\mathfrak{h}_{2m+1}^1 = \mathbb{R}.Z = \text{Span}(Z)$ of \mathfrak{h}_{2m+1} is 1-dimensional and coincides with the center of \mathfrak{h}_{2m+1} .

Let $(X_1^*, Y_1^*, \dots, X_m^*, Y_m^*, Z^*)$ be the dual basis of $(X_1, Y_1, \dots, X_m, Y_m, Z)$ in the dual space \mathfrak{h}_{2m+1}^* of \mathfrak{h}_{2m+1} , and

$$F = a_1 X_1^* + b_1 Y_1^* + \dots + a_m X_m^* + b_m Y_m^* + c Z^* \equiv (a_1, b_1, \dots, a_m, b_m, c)$$

be an arbitrary element in \mathfrak{h}_{2m+1}^* . Then the Kirillov form B_F is given by the following matrix:

$$B_F = \begin{bmatrix} 0 & c & 0 & 0 & \dots & 0 & 0 & 0 \\ -c & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & c & \dots & 0 & 0 & 0 \\ 0 & 0 & -c & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & c & 0 \\ 0 & 0 & 0 & 0 & \dots & -c & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix} = \text{diag}(\Lambda, \dots, \Lambda, 0)$$

with m blocks $\Lambda = \begin{bmatrix} 0 & c \\ -c & 0 \end{bmatrix}$.

In view of Proposition 2.4, it is a simple matter to get the following proposition.

Proposition 2.13. *\mathfrak{h}_{2m+1} is a MD(2m + 1, 1)-algebra and the maximal dimension of K -orbits in \mathfrak{h}_{2m+1}^* is $2m$. Moreover, for $F = (a_1, b_1, \dots, a_m, b_m, c) \in \mathfrak{h}_{2m+1}^*$, we have*

- (i) *K -orbits containing F is of dimension 0 if and only if $c = 0$.*
- (ii) *K -orbits containing F is of dimension $2m$ if and only if $c \neq 0$.*

2.5. The real Diamond Lie algebras. The $(2m + 2)$ -dimensional real Diamond Lie algebra $(0 < m \in \mathbb{N})$ is one semi-direct extension of the $(2m + 1)$ -dimensional Heisenberg Lie algebra by \mathbb{R} , namely it is the following real Lie algebra:

$$\mathbb{R}.\mathfrak{h}_{2m+1} := \text{Span}(X_i, Y_i, Z, T \mid i = 1, 2, \dots, m),$$

where the Lie structure is given by

$$[X_i, Y_i] = Z, \quad [T, X_i] = -X_i, \quad [T, Y_i] = Y_i, \quad i = 1, 2, \dots, m;$$

the other Lie brackets are trivial.

Remark 2.14. The $(2m + 1)$ -dimensional real Heisenberg Lie algebra is the first derived ideal of the $(2m + 2)$ -dimensional real Diamond Lie algebra. In particular, the first derived ideal of $\mathbb{R}.\mathfrak{h}_{2m+1}$ is of codimension 1.

Let $(X_1^*, Y_1^*, \dots, X_m^*, Y_m^*, Z^*, T^*)$ be the dual basis of $(X_1, Y_1, \dots, X_m, Y_m, Z, T)$ in the dual space $(\mathbb{R}.\mathfrak{h}_{2m+1})^*$ of $\mathbb{R}.\mathfrak{h}_{2m+1}$ and $F = a_1 X_1^* + b_1 Y_1^* + \dots + a_m X_m^* + b_m Y_m^* + c Z^* + d T^* \equiv (a_1, b_1, \dots, a_m, b_m, c, d)$ be an arbitrary element in $(\mathbb{R}.\mathfrak{h}_{2m+1})^*$. Then we get the Kirillov form B_F as follows:

$$B_F = \begin{bmatrix} 0 & c & 0 & 0 & \cdots & 0 & 0 & 0 & a_1 \\ -c & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -b_1 \\ 0 & 0 & 0 & c & \cdots & 0 & 0 & 0 & a_2 \\ 0 & 0 & -c & 0 & \cdots & 0 & 0 & 0 & -b_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & c & 0 & a_m \\ 0 & 0 & 0 & 0 & \cdots & -c & 0 & 0 & -b_m \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -a_1 & b_1 & -a_2 & b_2 & \cdots & -a_m & b_m & 0 & 0 \end{bmatrix}.$$

By virtue of Proposition 2.4, one can verify the following proposition.

Proposition 2.15. *The $(2m+2)$ -dimensional real Diamond Lie algebra $\mathbb{R}.\mathfrak{h}_{2m+1}$ is an MD $(2m+2, 2m+1)$ -algebra if and only if $m = 1$. That means the 4-dimensional real Diamond Lie algebra is an MD $(4, 3)$ -algebra and the $(2m + 2)$ -dimensional real Diamond Lie algebra is not an MD-algebra for every natural number $m > 1$.*

3. SOME SUBCLASSES OF MD-CLASS

In this section, we would like to introduce some well-known remarkable results of classification of MD-class. Firstly, we recall that all of the MD-algebras of dimension 4 or 5 were classified, up to isomorphism, by A. V. Le et al. [13, 15, 16]. However, to illustrate the general results which will be given in the last section of the paper, we will introduce here the classification of MD $(n, 1)$ -class and MD $(n, n - 1)$ -class for small n , namely $n = 4$ or $n = 5$.

3.1. Classification of MD(4, 1)-class and MD(4, 3)-class.

Proposition 3.1 (Classification of MD(4, 1)-algebras, see [15]). *Let \mathcal{G} be an MD(4, 1)-algebra. Then \mathcal{G} is decomposable and we can choose a suitable basis (X, Y, Z, T) of \mathcal{G} such that $\mathcal{G}^1 = \text{Span}(Z) = \mathbb{R}.Z$, and \mathcal{G} is isomorphic to one of the following Lie algebras:*

1. $\mathcal{G}_{4,1,1} := \mathfrak{h}_3 \oplus \mathbb{R}.T, [X, Y] = Z$; the other Lie brackets are trivial.
2. $\mathcal{G}_{4,1,2} := \text{aff}(\mathbb{R}) \oplus \mathbb{R}.Z \oplus \mathbb{R}.T, [X, Y] = Y$; the other Lie brackets are trivial.

Proposition 3.2 (Classification of MD(4, 3)-algebras, see [15]). *Let \mathcal{G} be an MD(4, 3)-algebra. Then \mathcal{G} must be indecomposable and we can choose a suitable basis (X, Y, Z, T) of \mathcal{G} such that \mathcal{G} is isomorphic to one of the following Lie algebras:*

1. $\mathcal{G}^1 = \text{Span}(X, Y, Z) \cong \mathbb{R}^3, ad_T \in \text{Aut}_{\mathbb{R}}(\mathcal{G}^1) \cong GL_3(\mathbb{R})$
 - 1.1. $\mathcal{G}_{4,3,1}(\lambda_1, \lambda_2): ad_T = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}.$
 - 1.2. $\mathcal{G}_{4,3,2}(\lambda): ad_T = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}; \lambda \in \mathbb{R} \setminus \{0\}.$
 - 1.3. $\mathcal{G}_{4,3,3}: ad_T = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$
 - 1.4. $\mathcal{G}_{4,3,4}(\lambda, \varphi): ad_{X_1} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & \lambda \end{bmatrix}; \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi).$
2. $\mathcal{G}^1 = \text{Span}(X, Y, Z) = \mathfrak{h}_3, ad_T \in \text{End}_{\mathbb{R}}(\mathcal{G}^1) \cong \text{Mat}_3(\mathbb{R})$
 - 2.1. $\mathcal{G}_{4,4,1}: ad_T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$
 - 2.2. $\mathcal{G}_{4,4,2} = \mathbb{R}.\mathfrak{h}_3$ (the 4-dimensional Diamond Lie algebra):

$$ad_T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

3.2. Classification of MD(5, 1)-class and MD(5, 4)-class.

Proposition 3.3 (Classification of MD(5, 1)-algebras, see [16, 17]). *Let \mathcal{G} be an MD(5, 1)-algebra, Then we can choose a suitable basis $(X_1, X_2, X_3, X_4, X_5)$ of \mathcal{G} such that $\mathcal{G}^1 = \text{Span}(X_5) = \mathbb{R}.X_5$ and \mathcal{G} is isomorphic to one of the following Lie algebras:*

1. $\mathcal{G}_{5,1,1} = \mathfrak{h}_5$ (the 5-dimensional real Heisenberg Lie algebra): $[X_1, X_2] = [X_3, X_4] = X_5$; the other Lie brackets are trivial. In this case, \mathcal{G} is indecomposable.
2. $\mathcal{G}_{5,1,2} = \text{aff}(\mathbb{R}) \oplus \mathbb{R}.X_1 \oplus \mathbb{R}.X_2 \oplus \mathbb{R}.X_3: [X_4, X_5] = X_5$; the other Lie brackets are trivial. In this case, \mathcal{G} is decomposable.

Proposition 3.4 (Classification of MD(5,4)-algebras, see [16]). *Let \mathcal{G} be an MD(5,4)-algebra. Then \mathcal{G} must be indecomposable, and \mathcal{G}^1 is commutative. Moreover, we can choose a suitable basis $(X_1, X_2, X_3, X_4, X_5)$ of \mathcal{G} such that $\mathcal{G}^1 = \text{Span}(X_2, X_3, X_4, X_5) \cong \mathbb{R}^4$, $ad_{X_1} \in \text{Aut}(\mathcal{G}^1) \cong GL_4(\mathbb{R})$, and \mathcal{G} is isomorphic to one of the following Lie algebras:*

$$4.1. \mathcal{G}_{5,4,1(\lambda_1, \lambda_2, \lambda_3)}: ad_{X_1} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \setminus \{0, 1\}, \lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_1.$$

$$4.2. \mathcal{G}_{5,4,2(\lambda_1, \lambda_2)}: ad_{X_1} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0, 1\}, \lambda_1 \neq \lambda_2.$$

$$4.3. \mathcal{G}_{5,4,3(\lambda)}: ad_{X_1} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

$$4.4. \mathcal{G}_{5,4,4(\lambda)}: ad_{X_1} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

$$4.5. \mathcal{G}_{5,4,5}: ad_{X_1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$4.6. \mathcal{G}_{5,4,6(\lambda_1, \lambda_2)}: ad_{X_1} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0, 1\}, \lambda_1 \neq \lambda_2.$$

$$4.7. \mathcal{G}_{5,4,7(\lambda)}: ad_{X_1} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

$$4.8. \mathcal{G}_{5,4,8(\lambda)}: ad_{X_1} = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

$$4.9. \mathcal{G}_{5,4,9(\lambda)}: ad_{X_1} = \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

$$4.10. \mathcal{G}_{5,4,10}: ad_{X_1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$4.11. \mathcal{G}_{5,4,11(\lambda_1, \lambda_2, \varphi)}:$$

$$ad_{X_1} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}, \lambda_1 \neq \lambda_2, \varphi \in (0, \pi).$$

$$4.12. \mathcal{G}_{5,4,12(\lambda, \varphi)}:$$

$$ad_{X_1} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi).$$

$$4.13. \mathcal{G}_{5,4,13(\lambda, \varphi)}:$$

$$ad_{X_1} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}, \lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi).$$

$$4.14. \mathcal{G}_{5,4,14(\lambda, \mu, \varphi)}:$$

$$ad_{X_1} = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 & 0 \\ \sin \varphi & \cos \varphi & 0 & 0 \\ 0 & 0 & \lambda & -\mu \\ 0 & 0 & \mu & \lambda \end{bmatrix}, \lambda, \mu \in \mathbb{R}, \mu > 0, \varphi \in (0, \pi).$$

In the next subsection, we introduce a noticeable result of D. Arnal, M. Cahen and J. Ludwig [1] in 1995.

3.3. List of MD-algebras whose simply connected MD-groups have only coadjoint orbits of dimension zero or two. In an attempt to classify solvable Lie algebras by structure, in 1995 D. Arnal, M. Cahen and J. Ludwig [1] listed, up to a direct central factor, all Lie algebras (solvable or not) such that the maximal dimension of K -orbits of corresponding connected and simply connected Lie groups is just two. However, they have not yet classified, up to isomorphism, these Lie algebras.

Proposition 3.5 ([1]). *Let G be a connected, simply connected solvable Lie group whose coadjoint orbits have dimension smaller or equal to two. Let \mathcal{G} be the Lie algebra of G . Then, up to a direct central factor, \mathcal{G} belongs to the following list of algebras:*

- (i) $\mathbb{R}.T \oplus \mathfrak{a}$, where \mathfrak{a} is an abelian ideal and $ad_T \in End(\mathfrak{a})$.
- (ii) $\mathbb{R}.T \oplus \mathfrak{h}_3$, where \mathfrak{h}_3 is the 3-dimensional Heisenberg Lie algebra spanned by (X, Y, Z) with $[X, Y] = Z$ and

- either $[T, X] = X, [T, Y] = -Y, [T, Z] = 0$ (the 4-dimensional Diamond Lie algebra),
 - or $[T, X] = Y, [T, Y] = -X, [T, Z] = 0$.
- (iii) \mathcal{G} is 5-dimensional with basis $(X_1, X_2, X_3, Y_1, Y_2)$ and the multiplicative law reads

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = Y_1, \quad [X_2, X_3] = Y_2.$$

- (iv) \mathcal{G} is 6-dimensional with basis $(X_1, X_2, X_3, Y_1, Y_2, Y_3)$ and the nonvanishing brackets are

$$[X_1, X_2] = Y_3, \quad [X_2, X_3] = Y_1, \quad [X_3, X_1] = Y_2.$$

Remark 3.6. Clearly we have the two following remarks:

- (i) There is an infinite family of non-isomorphic MD-algebras in part (i) of Proposition 3.5. Namely, part (i) of Proposition 3.5 includes all MD(4, 1)-algebras, MD(5, 1)-algebras, MD(5, 4)-algebras except the 5-dimensional Heisenberg Lie algebra. Furthermore, the last two MD(4, 3)-algebras in Proposition 3.2 coincide with two algebras in part (ii) of Proposition 3.5, but the remaining four MD(4, 3)-algebras in Proposition 3.2 are included in part (i) of Proposition 3.5.
- (ii) However, it should be noted that the indecomposable MD(2, 1)-algebras $\text{aff}(\mathbb{R})$ and \mathfrak{h}_3 , the decomposable MD(4, 1)-algebras $\mathfrak{h}_3 \oplus \mathbb{R}.T$ (in Proposition 3.1), $\text{aff}(\mathbb{R}) \oplus \mathbb{R}.Z \oplus \mathbb{R}.T$, and the decomposable MD(5, 1)-algebra $\text{aff}(\mathbb{R}) \oplus \mathbb{R}.X_1 \oplus \mathbb{R}.X_2 \oplus \mathbb{R}.X_3$ are not included in the list of Proposition 3.5, although it is obvious that all coadjoint orbits of the Lie groups corresponding to $\mathfrak{h}_3 \oplus \mathbb{R}.T$, $\text{aff}(\mathbb{R}) \oplus \mathbb{R}.Z \oplus \mathbb{R}.T$, $\text{aff}(\mathbb{R}) \oplus \mathbb{R}.X_1 \oplus \mathbb{R}.X_2 \oplus \mathbb{R}.X_3$ have dimension zero or two. So that was one shortcoming in Arnal’s list.

4. CLASSIFICATION OF MD(*, 1)-CLASS AND MD(*, * - 1)-CLASS

Now we will introduce the complete classification, up to isomorphism, of all MD-algebras (of arbitrary dimension) having the first derived ideal of dimension 1 or codimension 1. These results are generalizations of Propositions 3.1, 3.2, 3.3 and 3.4 in Section 3.

4.1. The main results.

Theorem 4.1 (The complete classification of MD(*, 1)-class). *MD(*, 1)-class coincides with the class of all real solvable Lie algebras whose first derived ideal is 1-dimensional; moreover MD(*, 1) includes only the Lie algebra of the group of affine transformations of the real straight line, the real Heisenberg Lie algebras and their direct extensions by the real commutative Lie algebras. In other words, if \mathcal{G} is an n -dimensional real solvable Lie algebra whose first derived ideal $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}]$ is 1-dimensional ($2 \leq n \in \mathbb{N}$) then \mathcal{G} is an MD(n , 1)-algebra and \mathcal{G} is isomorphic to one and only one of the following Lie algebras:*

- (i) *The Lie algebra $\text{aff}(\mathbb{R})$ of the group $\text{Aff}(\mathbb{R})$ of all affine transformations on \mathbb{R} ; $n = 2$.*

- (ii) $\text{aff}(\mathbb{R}) \oplus \mathbb{R}^{n-2}; 3 \leq n.$
- (iii) *The real Heisenberg Lie algebra $\mathfrak{h}_{2m+1}; 3 \leq n = 2m + 1.$*
- (iv) $\mathfrak{h}_{2m+1} \oplus \mathbb{R}^{n-2m-1}; 3 \leq 2m + 1 < n.$

It is clear that Theorem 4.1 can be formulated in another way in the following consequence which gives a new character of the real Heisenberg Lie algebras.

Corollary 4.2 (A new character of the real Heisenberg Lie algebras). *Let \mathcal{G} be a real Lie algebra of dimension n ($3 \leq n \in \mathbb{N}$). Then the following conditions are equivalent:*

- (i) \mathcal{G} is indecomposable and has the first derived ideal $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}] \cong \mathbb{R}.$
- (ii) \mathcal{G} is an indecomposable MD($n, 1$)-algebra.
- (iii) \mathcal{G} is the n -dimensional Heisenberg Lie algebra (in particular, n is odd).

The next theorem gives one necessary and sufficient condition to recognize one MD($n, n - 1$)-algebra ($4 \leq n \in \mathbb{N}$).

Theorem 4.3 (Necessary and sufficient conditions for MD($*, * - 1$)-class). *Let \mathcal{G} be a real solvable Lie algebra of dimension n ($3 \leq n \in \mathbb{N}$) such that its first derived ideal \mathcal{G}^1 is $(n - 1)$ -dimensional.*

- (i) *If \mathcal{G}^1 is commutative then \mathcal{G} is an MD($n, n - 1$)-algebra, moreover \mathcal{G} is indecomposable.*
- (ii) *If $n > 4$ and \mathcal{G} is an MD($n, n - 1$)-algebra then \mathcal{G}^1 is commutative.*

Remark 4.4. When $n \leq 4$, assertion (ii) is not true. Namely, if $n < 4$, all the n -dimensional Lie algebras are MD-algebras, and moreover, they can be listed easily. If $n = 4$, as previously indicated, the first derived ideal of the 4-dimensional real Diamond Lie algebra is the 3-dimensional Heisenberg Lie algebra which is non commutative and 1-codimensional. In fact, all MD4-algebras were completely classified in 1990 by A. V. Le [14, 15], and the classifications of MD(4, 1)-class and MD(4, 3)-class were recalled in Propositions 3.1 and 3.2.

The last theorem will characterize every MD($n, n - 1$)-algebra by an invertible real $(n - 1)$ -square matrix and reduces the task of classifying MD($n, n - 1$)-class to obtaining the well-known classification of equivalent of proportional similar matrices. Let \mathcal{G} be an MD($n, n - 1$)-algebra ($3 \leq n \in \mathbb{N}$) generated by a basis (X_1, X_2, \dots, X_n) such that the first derived ideal \mathcal{G}^1 is 1-codimensional and spanned by $(X_1, X_2, \dots, X_{n-1})$. It is obvious that the Lie structure of \mathcal{G} is well understood by the invertible real $(n - 1)$ -square matrix of map ad_{X_n} considered as an automorphism of \mathcal{G}^1 for the basis $(X_1, X_2, \dots, X_{n-1})$.

Theorem 4.5. *Let \mathcal{G} be a real vector space of dimension n ($3 \leq n \in \mathbb{N}$) generated by a basis (X_1, X_2, \dots, X_n) and $\mathcal{G}^1 := \text{Span}(X_1, X_2, \dots, X_{n-1})$ be the 1-codimensional subspace of \mathcal{G} . Then we have the following assertions:*

- (i) *Each invertible real $(n - 1)$ -square matrix A always defines one Lie structure on \mathcal{G} such that \mathcal{G} is an MD($n, n - 1$)-algebra whose first derived ideal is commutative, exactly equal to \mathcal{G}^1 and A is exactly the matrix of adjoint map ad_{X_n} on \mathcal{G}^1 in the chosen basis $(X_1, X_2, \dots, X_{n-1})$.*

- (ii) Two invertible real $(n - 1)$ -square matrices A, B define two Lie structures on \mathcal{G} which are isomorphic if and only if there exist a non-zero real number c and an invertible real $(n - 1)$ -square matrix C so that $cA = CBC^{-1}$.

Remark 4.6. In view of Theorem 4.5, we have the following remarks:

- (i) Two invertible real square matrices A, B of the same order are called *proportional similar* if (and only if) there exist a non-zero real number c and an invertible real square matrix C of the same order as A, B so that $cA = CBC^{-1}$. In fact, assertion (ii) of Theorem 4.5 gives the classification of MD($n, n - 1$)-algebras ($n > 4$) by using the well-known classification of invertible real matrices in proportional similar equivalent relation.
- (ii) The classification of indecomposable MD(5, 4)-algebras in Proposition 3.4 of this paper gives one concrete illustration of Theorem 4.5 when $n = 5$. In principle, it is not hard to list all non-isomorphic indecomposable MD($n, n - 1$)-algebras by applying Theorem 4.5 for n small, for example $n = 6, 7, \dots$

4.2. Proof of Theorem 4.1. In this section, we always consider \mathcal{G} as a real solvable Lie algebra of dimension $n \geq 3$ whose first derived ideal $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}]$ is 1-dimensional, in particular \mathcal{G}^1 is commutative. Without loss of generality, we can choose a suitable basis such that

$$\mathcal{G} = \text{Span}(X_1, X_2, \dots, X_n), \quad \mathcal{G}^1 = \text{Span}(X_n) = \mathbb{R}.X_n.$$

Let G be the connected, simply connected Lie group corresponding to \mathcal{G} .

When $n = 2$, it is obvious that the part (i) in Theorem 4.1 holds because $\text{aff}(\mathbb{R})$ is an MD(2, 1)-algebra (see Remark 2.11) and it is the unique non-commutative real Lie algebra of dimension 2. Therefore, in the rest of this subsection, we can suppose that $n \geq 3$. Denote

$$[X_i, X_n] = a_i X_n, \quad [X_i, X_j] = a_{ij} X_n \quad (a_i, a_{ij} \in \mathbb{R}); \quad i, j = 1, 2, \dots, n - 1.$$

Evidently, the Lie structure on \mathcal{G} is well understood by the vector

$$\mathbf{a} := (a_1, a_2, \dots, a_{n-1})$$

and the skew-symmetric real $(n - 1)$ -square matrix $A := (a_{ij})_{i,j=1,2,\dots,n-1}$. There are two cases to consider for the values of the vector \mathbf{a} : $\mathbf{a} = 0$ or $\mathbf{a} \neq 0$.

First case: $\mathbf{a} \neq 0$. Firstly, we consider the case $\mathbf{a} \neq 0$, i.e. there exists $i \in \{1, \dots, n - 1\}$ such that $a_i \neq 0$; that means X_n is not in the center $Z(\mathcal{G})$ of \mathcal{G} . Renumbering the chosen basis if necessary, we can always suppose that $a_{n-1} \neq 0$. Then $[X_{n-1}, X_n] = a_{n-1} X_n \neq 0$. In this case, we will show that \mathcal{G} is a trivial extension of the Lie algebra $\text{aff}(\mathbb{R})$. Namely, we have the following lemma.

Lemma 4.7. *If $\mathbf{a} \neq 0$ then \mathcal{G} is an MD($n, 1$)-algebra which is isomorphic to $\text{aff}(\mathbb{R}) \oplus \mathbb{R}^{n-2}$ when $n \geq 3$.*

Proof. Using the change of basis

$$Y_i = X_i - \frac{a_i}{a_{n-1}} X_{n-1}, \quad Y_{n-1} = \frac{1}{a_{n-1}} X_{n-1}, \quad Y_n = X_n; \quad i = 1, 2, \dots, n - 2$$

we get

$$[Y_i, Y_n] = 0, \quad [Y_{n-1}, Y_n] = Y_n; \quad i = 1, 2, \dots, n - 2.$$

Hence, without loss of generality, we can now assume

$$[X_i, X_n] = 0, \quad [X_{n-1}, X_n] = X_n; \quad i = 1, 2, \dots, n - 2.$$

Using the Jacobi identities for triples (X_i, X_j, X_{n-1}) , $1 \leq i < j \leq n - 2$, we get

$$\begin{aligned} [[X_i, X_j], X_{n-1}] + [[X_{n-1}, X_i], X_j] + [[X_j, X_{n-1}], X_i] &= 0 \\ \Rightarrow a_{ij}[X_n, X_{n-1}] = 0 &\Rightarrow -a_{ij}X_n = 0 \Rightarrow a_{ij} = 0 \\ \Rightarrow [X_i, X_j] = a_{ij}X_n = 0; \quad i, j = 1, 2, \dots, n - 2. \end{aligned}$$

Now, using the change of basis

$$Z_i = X_i + a_{i,n-1}X_n, \quad Z_{n-1} = X_{n-1}, \quad Z_n = X_n; \quad i = 1, 2, \dots, n - 2$$

we get $[Z_i, Z_{n-1}] = 0$, $i = 1, 2, \dots, n - 2$. So we can suppose now that

$$[X_i, X_{n-1}] = 0, \quad i = 1, 2, \dots, n - 2.$$

Hence, in this case, \mathcal{G} is isomorphic to the Lie algebra

$$\text{aff}(\mathbb{R}) \oplus \mathbb{R}^{n-2} = \text{Span}(X_1, X_2, \dots, X_n), \quad [X_{n-1}, X_n] = X_n,$$

where the other Lie brackets are trivial. Obviously, the K -orbits of G is of dimension 0 or 2. This means that \mathcal{G} is an MD($n, 1$)-algebra. \square

Second case: $\mathbf{a} = 0$. Now, we consider the case $\mathbf{a} = 0$, i.e. $[X_i, X_n] = 0$ for all $i = 1, 2, \dots, n - 1$; in particular $X_n \in Z(\mathcal{G})$. Then the Lie structure of \mathcal{G} is uniquely defined by the skew-symmetric real $(n - 1)$ -square matrix $A = (a_{ij})_{i,j=1,2,\dots,n-1}$, which is called the *structure matrix* of \mathcal{G} . Since $\mathcal{G}^1 = \text{Span}(X_n)$ is 1-dimensional, A is non-trivial and $0 < \text{rank } A$ is even. We have the following lemma.

Lemma 4.8. *If $\mathbf{a} = 0$, i.e. $[X_i, X_n] = 0$ for all $i = 1, 2, \dots, n - 1$, then the Lie algebra \mathcal{G} is an MD($n, 1$)-algebra and the maximal dimension of the K -orbits of G is the rank of the structure matrix A .*

Proof. Let $\mathcal{G}^* \equiv \mathbb{R}^n$ be the dual space of \mathcal{G} with dual basis $(X_1^*, X_2^*, \dots, X_n^*)$ and $F = f_1X_1^* + f_2X_2^* + \dots + f_nX_n^* \equiv (f_1, f_2, \dots, f_n)$ be an arbitrary element of \mathcal{G}^* . The Kirillov form B_F is given as follows:

$$B_F := (\langle F, [X_i, X_j] \rangle)_{i,j=1,\dots,n} = f_n \begin{bmatrix} a_{11} & \dots & a_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,n-1} & 0 \\ 0 & \dots & 0 & 0 \end{bmatrix} = f_n \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

and $\text{rank } B_F \in \{0, 2k\}$, where $2k = \text{rank } A$ is the rank of the structural matrix. More precisely,

- $\text{rank } B_F = 0$ if and only if $f_n = 0$, i.e. $F = (f_1, f_2, \dots, f_{n-1}, 0)$.
- $\text{rank } B_F = \text{rank } A = 2k > 0$ if and only if $f_n \neq 0$.

Hence, in view of Proposition 2.4, \mathcal{G} is an MD($n, 1$)-algebra and the maximal dimension of K -orbits of G is the rank of the structure matrix. \square

Now, we will consider whether \mathcal{G} is decomposable in the second case.

Lemma 4.9. *If $\mathbf{a} = 0$, i.e. $[X_i, X_n] = 0$ for all $i = 1, 2, \dots, n - 1$, then the Lie algebra \mathcal{G} is decomposable if and only if the dimension of the center of \mathcal{G} is greater than 1.*

Proof. Denote $Z(\mathcal{G})$ to be the center of \mathcal{G} . Obviously, X_n is in $Z(\mathcal{G})$ because $[X_i, X_n] = 0$ for all $i = 1, 2, \dots, n - 1$, i.e. $\dim Z(\mathcal{G}) > 0$.

(\implies) Suppose that \mathcal{G} is decomposable, i.e. $\mathcal{G} = \mathcal{A} \oplus \mathcal{B}$ in which \mathcal{A}, \mathcal{B} are non-trivial proper Lie subalgebras of \mathcal{G} . Put $X_n = X_a + X_b \in Z(\mathcal{G})$ with some $X_a \in \mathcal{A}$ and some $X_b \in \mathcal{B}$. Let us consider an arbitrary element $Y = Y_a + Y_b \in \mathcal{G}$ with $Y_a \in \mathcal{A}, Y_b \in \mathcal{B}$. We have

$$\begin{aligned} 0 &= [X_n, Y] = [X_a + X_b, Y_a + Y_b] = [X_a, Y_a] + [X_b, Y_b] \\ &\implies [X_a, Y_a] = [X_b, Y_b] = 0 \implies X_a, X_b \in Z(\mathcal{G}). \end{aligned}$$

- If $X_a \neq 0 \neq X_b$ then they are of course linearly independent. This means $\dim Z(\mathcal{G}) > 1$.
- If X_a or X_b is 0, then without loss of generality, we can suppose that $X_a = 0$, i.e. $X_n = X_b \in \mathcal{B}$. In particular, $\mathcal{G}^1 = \text{Span}(X_n) \subseteq \mathcal{B}$. Let $X \neq 0 \in \mathcal{A}$ be an arbitrary element. Obviously, $[X, Z] = 0$ for every $Z \in \mathcal{B}$. On the other hand, we have

$$[X, T] \in \mathcal{A} \cap \mathcal{G}^1 \subseteq \mathcal{A} \cap \mathcal{B} = 0 \implies [X, T] = 0, \forall T \in \mathcal{A}.$$

This means that X commutes with any element of $\mathcal{G} = \mathcal{A} \oplus \mathcal{B}$, i.e. $X \in Z(\mathcal{G})$. Because of $X \in \mathcal{A}, X_n \in \mathcal{B}$ so X, X_n are linearly independent and $\dim Z(\mathcal{G}) > 1$.

Hence, $\dim Z(\mathcal{G}) > 1$ in any case.

(\impliedby) Suppose $\dim Z(\mathcal{G}) > 1$. There exists $X \in Z(\mathcal{G})$ such that X, X_n are independent. We can add T_1, \dots, T_{n-2} in $\{X, X_n\}$ to get a new basis of \mathcal{G} . Note that $[T_i, T_j] \in \mathcal{G}^1 = \text{Span}(X_n) \subset \text{Span}(X_n, T_1, \dots, T_{n-2}); 1 \leq i < j \leq n - 2$. Then we have

$$\mathcal{G} = \text{Span}(X) \oplus \text{Span}(X_n, T_1, \dots, T_{n-2}).$$

Therefore \mathcal{G} is decomposable. □

Remark 4.10. The center of the Heisenberg Lie algebra is 1-dimensional, so its indecomposability is unsurprising.

Recall that each MD($n, 1$)-algebra \mathcal{G} in the second case above is always defined uniquely by an $(n - 1)$ -square (skew-symmetric) structure matrix A . Now we will consider whether two structures A and B give us isomorphic Lie algebras.

Lemma 4.11. *Let $A = (a_{ij})_{i,j=1,2,\dots,n-1}$ and $B = (b_{ij})_{i,j=1,2,\dots,n-1}$ be skew-symmetric real $(n - 1)$ -square matrices and $\mathcal{G}_A, \mathcal{G}_B$ be MD($n, 1$)-algebras which are defined by A, B respectively. Then*

$$\mathcal{G}_A \cong \mathcal{G}_B \iff \exists c \in \mathbb{R} \setminus \{0\}, \exists C \in GL_{n-1}(\mathbb{R}) \text{ such that } cA = C^T B C,$$

where C^T is the transpose of C .

Proof. (\implies) Let $f : \mathcal{G}_A \rightarrow \mathcal{G}_B$ be an isomorphism. Since $f(\mathcal{G}_A^1) = \mathcal{G}_B^1$, there is a non-zero real number c so that $f(X_n) = cX_n$. Clearly the matrix of f in the basis $(X_1, X_2, \dots, X_{n-1}, X_n)$ is given as follows:

$$M = \begin{bmatrix} c_{11} & \cdots & c_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ c_{n-1,1} & \cdots & c_{n-1,n-1} & 0 \\ c_{n1} & \cdots & c_{n,n-1} & c \end{bmatrix} = \begin{bmatrix} C & 0 \\ * & c \end{bmatrix},$$

in which $C = (c_{ij})_{i,j=1,2,\dots,n-1}$ is a real $(n - 1)$ -square matrix, and $*$ is the vector $(c_{n1}, \dots, c_{n,n-1})$. Because f is an isomorphism, M is invertible and so is C . Hence, the linear map f is a Lie isomorphism if and only if

$$\begin{aligned} f([X_i, X_j]_A) &= [f(X_i), f(X_j)]_B \\ \Leftrightarrow f(a_{ij}X_n) &= \left[\sum_{k=1}^{n-1} c_{ki}X_k + c_{ni}X_n, \sum_{l=1}^{n-1} c_{lj}X_l + c_{nj}X_n \right]_B \\ \Leftrightarrow ca_{ij}X_n &= \sum_{k=1}^{n-1} c_{ki} \cdot \sum_{l=1}^{n-1} c_{lj} [X_k, X_l]_B \\ \Leftrightarrow ca_{ij}X_n &= \left(\sum_{k,l=1}^{n-1} c_{ki} \cdot b_{kl} \cdot c_{lj} \right) X_n \\ \Leftrightarrow ca_{ij} &= \sum_{k,l=1}^{n-1} c_{ki} \cdot b_{kl} \cdot c_{lj} \\ \Leftrightarrow cA &= C^T AC, \end{aligned}$$

where $i, j = 1, 2, \dots, n - 1$.

(\impliedby) Conversely, suppose that there exist a non-zero real number c and an invertible real $(n - 1)$ -square matrix $C = (c_{ij})_{i,j=1,2,\dots,n-1}$ such that $cA = C^T BC$. Let $f : \mathcal{G}_A \rightarrow \mathcal{G}_B$ be a linear map which is defined, in the basis (X_1, X_2, \dots, X_n) , by the matrix M' as follows:

$$M' = \begin{bmatrix} c_{11} & \cdots & c_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ c_{n-1,1} & \cdots & c_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & c \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & c \end{bmatrix}.$$

Since C is invertible and $c \neq 0$, f is a linear isomorphism. Moreover, it is easy to check that f is also a Lie homomorphism. Therefore, f is a Lie isomorphism. \square

Remark 4.12. Recall that two real $(n - 1)$ -square matrices A, B are said to be *congruent* if there exists an invertible $(n - 1)$ -square matrix C such that $B = C^T AC$. Furthermore, any non-zero skew-symmetric real square matrix can be always transformed into the canonical form. More precisely, for any non-zero skew-symmetric real $(n - 1)$ -square matrix A , there exists a real orthogonal matrix C

such that

$$C^T AC = \text{diag}(\Lambda_1, \Lambda_2, \dots, \Lambda_m, 0, \dots, 0),$$

where $\Lambda_j := \begin{bmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{bmatrix}$ and $\{\pm i\lambda_1, \dots, \pm i\lambda_m\}$ (i is the imaginary unit in the complex field \mathbb{C}) is the set of all multiple eigenvalues of A . For example, the real Heisenberg Lie algebra

$$\mathfrak{h}_{2m+1} := \langle X_i, Y_i, Z : i = 1, 2, \dots, m \rangle, \quad [X_i, Y_i] = Z, \quad i = 1, 2, \dots, m$$

has the structure matrix $H = \text{diag}(I, \dots, I)$ including n blocks $I = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. This matrix H has exactly two m -multiple eigenvalues $\pm i$ and H has no eigenvalue 0.

Proof of Theorem 4.1. Recall that we need only to show parts (ii), (iii), (iv) of Theorem 4.1. Lemmas 4.7, 4.8 and 4.9 show that the considered Lie algebra \mathcal{G} belongs to MD($n, 1$)-class. Moreover, part (ii) of Theorem 4.1 is implied directly from Lemma 4.7. We only need to prove parts (iii) and (iv).

In the basis (X_1, X_2, \dots, X_n) , the structure matrix of \mathcal{G} is A . We will choose a new basis to get the standard form $B = C^T AC$. By Lemma 4.11, the Lie algebra defined by the matrix B is isomorphic to \mathcal{G} .

If B has no zero eigenvalue, i.e. $B = \text{diag}(\Lambda_1, \Lambda_2, \dots, \Lambda_m)$, $2m = n - 1$, put

$$D := \text{diag} \left(1, \frac{1}{\lambda_1}, 1, \frac{1}{\lambda_2}, \dots, 1, \frac{1}{\lambda_m} \right).$$

Then we get the structure matrix $H = D^T B D$ of the $(2m + 1)$ -dimensional real Heisenberg algebra \mathfrak{h}_{2m+1} , $2m + 1 = n$. By Lemma 4.11, \mathcal{G} is isomorphic to \mathfrak{h}_{2m+1} . So part (iii) is proved.

If B has eigenvalues 0, i.e. $0 < 2m < n - 1$, then $Z(\mathcal{G})$ is generated by the basis X_{2m+1}, \dots, X_n whose dimension is greater than 1. By Lemma 4.9, \mathcal{G} is decomposable, namely \mathcal{G} is isomorphic to $\mathfrak{h}_{2m+1} \oplus \mathbb{R}^k$, where $k = n - (2m + 1) > 0$. Actually, the direct summand \mathbb{R}^k is the commutative Lie subalgebra of \mathcal{G} generated by (X_{2m+2}, \dots, X_n) and \mathfrak{h}_{2m+1} is generated by $(X_1, X_2, \dots, X_{2m}; X_{2m+1})$. So part (iv) is proved and the proof of Theorem 4.1 is complete. \square

4.3. Proof of Theorem 4.3. In this section, we always consider \mathcal{G} as a real solvable Lie algebra of dimension $n \geq 3$ with the 1-codimensional first derived ideal $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}]$. Assume that $\dim \mathcal{G}^2 = \dim([\mathcal{G}^1, \mathcal{G}^1]) = k \leq n - 2$. Without loss of generality, we can choose a suitable basis such that

$$\begin{aligned} \mathcal{G} &= \text{Span}(X_1, X_2, \dots, X_n), \\ \mathcal{G}^1 &= [\mathcal{G}, \mathcal{G}] = \text{Span}(X_1, X_2, \dots, X_{n-1}), \\ \mathcal{G}^2 &= [\mathcal{G}^1, \mathcal{G}^1] = \text{Span}(X_1, X_2, \dots, X_k), \quad k \leq n - 2. \end{aligned}$$

Let c_{ij}^l ($1 \leq i < j \leq n, 1 \leq l \leq n$) be the structure constants of \mathcal{G} . Then the Lie brackets of \mathcal{G} are given by

$$[X_i, X_j] = \sum_{l=1}^{n-1} c_{ij}^l X_l, \quad 1 \leq i < j \leq n.$$

In view of Proposition 2.8, if \mathcal{G} is an MD-algebra then \mathcal{G}^2 is commutative and we get

$$[X_i, X_j] = \sum_{l=1}^{n-1} c_{ij}^l X_l = 0 \Leftrightarrow c_{ij}^l = 0, \quad 1 \leq i < j \leq k, \quad 1 \leq l \leq n-1.$$

In order to prove Theorem 4.3, we need some lemmas.

Lemma 4.13. *If \mathcal{G} is an MD n -algebra ($n \geq 3$) whose first derived ideal \mathcal{G}^1 is commutative and 1-codimensional then $\dim \Omega_F \in \{0, 2\}$, for every $F \in \mathcal{G}^*$.*

Proof. Because $\mathcal{G}^1 = \text{Span}(X_1, X_2, \dots, X_{n-1})$ is $(n-1)$ -dimensional commutative and \mathcal{G} is non-commutative, so $c_{ij}^l = 0, 1 \leq i < j \leq n-1, 1 \leq l \leq n-1$, and the adjoint ad_{X_n} is an isomorphism on \mathcal{G}^1 . Therefore the matrix $(-c_{jn}^i)_{i,j=1,2,\dots,n-1}$ of ad_{X_n} in the basis $(X_1, X_2, \dots, X_{n-1})$ of \mathcal{G}^1 is invertible. In particular, the structure constants $c_{jn}^{n-1} (1 \leq j \leq n-1)$ do not concomitantly vanish. Choose $F = X_{n-1}^* \in \mathcal{G}^*$. It is easily seen that the matrix of the Kirillov form B_F in the basis (X_1, X_2, \dots, X_n) is as follows:

$$B_F = \begin{bmatrix} 0 & 0 & \cdots & 0 & c_{1n}^{n-1} \\ 0 & 0 & \cdots & 0 & c_{2n}^{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & c_{n-1,n}^{n-1} \\ -c_{1n}^{n-1} & -c_{2n}^{n-1} & \cdots & -c_{n-1,n}^{n-1} & 0 \end{bmatrix}.$$

It is obvious that $\text{rank } B_F = 2$ because $c_{1n}^{n-1}, \dots, c_{n-1,n}^{n-1}$ do not concomitantly vanish. Since \mathcal{G} is an MD-algebra, we get $\dim \Omega_F = \text{rank } B_F \in \{0, 2\}$ for any $F \in \mathcal{G}^*$. □

Lemma 4.14. *The following $(n-k-1)$ -square matrix A is invertible:*

$$A = \begin{bmatrix} c_{k+1,n}^{k+1} & \cdots & c_{n-1,n}^{k+1} \\ \vdots & \ddots & \vdots \\ c_{k+1,n}^{n-1} & \cdots & c_{n-1,n}^{n-1} \end{bmatrix}.$$

Proof. Since $\mathcal{G}^1 = [\mathcal{G}, \mathcal{G}]$, there exist real numbers $\alpha_{ij} (1 \leq i < j \leq n)$ such that

$$\begin{aligned} X_{k+1} &= \sum_{1 \leq i < j \leq n} \alpha_{ij} [X_i, X_j] \\ &= \sum_{j=k+1}^{n-1} \alpha_{jn} [X_j, X_n] + \sum_{j=1}^k \alpha_{jn} [X_j, X_n] + \sum_{1 \leq i < j \leq n-1} \alpha_{ij} [X_i, X_j] \\ &= \sum_{j=k+1}^{n-1} \alpha_{jn} [X_j, X_n] + LC_1(\mathcal{G}^2) \\ &= \sum_{j=k+1}^{n-1} \alpha_{jn} \left(\sum_{l=1}^k c_{jn}^l X_l + \sum_{l=k+1}^{n-1} c_{jn}^l X_l \right) + LC_1(\mathcal{G}^2). \end{aligned}$$

Hence, we get

$$X_{k+1} = \sum_{l=k+1}^{n-1} \left(\sum_{j=k+1}^{n-1} c_{jn}^l \alpha_{jn} \right) X_l + LC_2(\mathcal{G}^2),$$

where $LC_1(\mathcal{G}^2)$ and $LC_2(\mathcal{G}^2)$ are a linear combination of the vectors in the basis (X_1, \dots, X_k) of \mathcal{G}^2 . Because of the independence of chosen basis (X_1, \dots, X_n) , these assertions imply that there exists one row-vector $Y_{k+1} \in \mathbb{R}^{n-k-1}$ such that $Y_{k+1}A = (1, 0, \dots, 0)$. Similarly, there exist $Y_{k+2}, \dots, Y_{n-1} \in \mathbb{R}^{n-k-1}$ such that

$$Y_{k+2}A = (0, 1, \dots, 0), \dots, Y_{n-1}A = (0, \dots, 0, 1).$$

So there exists a real matrix P such that $PA = I$, where I is the unit matrix of $\text{Mat}_{n-k-1}(\mathbb{R})$. Therefore A is an invertible matrix. \square

The following lemma is the well-known result of linear algebra for any skew-symmetric real 4-square matrix and it can be easily verified by simple computation.

Lemma 4.15. *For any skew-symmetric real 4-square matrix $(a_{ij})_{1 \leq i, j \leq 4}$, its determinant is zero if and only if $a_{12} \cdot a_{34} - a_{13} \cdot a_{24} + a_{14} \cdot a_{23} = 0$.* \square

Proof of Theorem 4.3. We now prove Theorem 4.3.

Proof of part (i). Let \mathcal{G} be a real solvable Lie algebra of dimension n whose first derived ideal $\mathcal{G}^1 \cong \mathbb{R}^{n-1}$ is 1-codimensional and commutative. Recall that, with the above notations, $\mathcal{G}^1 \cong \mathbb{R} \cdot X_1 \oplus \mathbb{R} \cdot X_2 \oplus \dots \oplus \mathbb{R} \cdot X_{n-1}$.

Let F be an arbitrary element in $\mathcal{G}^* \cong \mathbb{R}^n$. Put

$$\langle F, [X_i, X_n] \rangle = a_i, \quad 1 \leq i \leq n-1.$$

Then, by simple computation, we can see that the matrix B_F of the Kirillov form B_F is given as follows:

$$B_F = \begin{bmatrix} 0 & 0 & \dots & 0 & a_1 \\ 0 & 0 & \dots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & a_{n-1} \\ -a_1 & -a_2 & \dots & -a_{n-1} & 0 \end{bmatrix}.$$

It is clear that $\text{rank } B_F \in \{0, 2\}$ and, for every $F \in \mathcal{G}^*$, $\text{rank } B_F$ does not concomitantly vanish. Hence, by virtue of Proposition 2.4, \mathcal{G} is an MD($n, n-1$)-algebra.

Proof of part (ii). We will show that if \mathcal{G} is a real solvable Lie algebra of dimension $n > 4$ whose first derived ideal \mathcal{G}^1 is 1-codimensional and non-commutative then \mathcal{G} cannot be an MD-algebra.

Recall that we always choose one basis (X_1, \dots, X_n) of \mathcal{G} such that $\mathcal{G}^1 = \text{Span}(X_1, \dots, X_{n-1})$ and $\mathcal{G}^2 = \text{Span}(X_1, \dots, X_k)$, $k \leq n-2$. There are some cases which contradict each other for the values of k as follows.

First case: $k = n-2$. Then, $\dim \mathcal{G}^2 = \dim \mathcal{G}^1 - 1$. According to Proposition 2.10, \mathcal{G} is not an MD-algebra.

Second case: $k \leq n - 3$. It is sufficient to prove for just $k = n - 3$ because the proof for each k in this case is similar. That means $\mathcal{G}^2 = \text{Span}(X_1, \dots, X_{n-3})$. The Lie structure of \mathcal{G} is given by

$$\begin{aligned}
 [X_{n-1}, X_n] &= \sum_{l=1}^{n-1} c_{n-1,n}^l X_l, & [X_{n-2}, X_n] &= \sum_{l=1}^{n-1} c_{n-2,n}^l X_l, \\
 [X_{n-2}, X_{n-1}] &= \sum_{l=1}^{n-3} c_{n-2,n-1}^l X_l, \\
 [X_i, X_n] &= \sum_{l=1}^{n-3} c_{in}^l X_l, & [X_i, X_j] &= \sum_{l=1}^{n-3} c_{ij}^l X_l; \quad 1 \leq i < j \leq n - 3.
 \end{aligned}$$

According to Lemma 4.14, matrix $P = \begin{bmatrix} c_{n-2,n}^{n-2} & c_{n-1,n}^{n-2} \\ c_{n-2,n}^{n-1} & c_{n-1,n}^{n-1} \end{bmatrix}$ is invertible. Let F be an arbitrary element of \mathcal{G}^* . Put

$$\langle F, [X_{n-2}, X_{n-1}] \rangle = a, \quad \langle F, [X_{n-2}, X_n] \rangle = b, \quad \langle F, [X_{n-1}, X_n] \rangle = c.$$

Then the matrix of the Kirillov form B_F in the basis (X_1, X_2, \dots, X_n) is given by

$$B_F = \begin{bmatrix} 0 & 0 & \cdots & 0 & * & * & * \\ 0 & 0 & \cdots & 0 & * & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & * & * & * \\ * & * & \cdots & * & 0 & a & b \\ * & * & \cdots & * & -a & 0 & c \\ * & * & \cdots & * & -b & -c & 0 \end{bmatrix},$$

in which the asterisks denote the undetermined real numbers.

Let us consider the 4-square submatrices of B_F established by the elements which are on the rows and the columns of the same ordinal numbers $i, n - 2, n - 1, n$ ($i \leq n - 3$). According to Lemma 4.13, $\text{rank } B_F \in \{0, 2\}$ and this implies that the determinants of these considered 4-square submatrices are zero for any $F \in \mathcal{G}^*$. In view of Lemma 4.15, the following structure constants vanish:

$$c_{i,n-2}^l = c_{i,n-1}^l = 0, \quad 1 \leq i, l \leq n - 3.$$

This implies

$$[X_i, X_{n-2}] = [X_i, X_{n-1}] = 0, \quad 1 \leq i \leq n - 3.$$

Therefore, we get

$$\begin{aligned}
 \text{Span}(X_1, \dots, X_{n-3}) &= \mathcal{G}^2 = [\mathcal{G}^1, \mathcal{G}^1] \\
 &= \text{Span}([X_i, X_j] \mid 1 \leq i, j \leq n - 1) \\
 &= \text{Span}([X_{n-2}, X_{n-1}]).
 \end{aligned}$$

So $n - 3 = \dim \mathcal{G}^2 \leq 1$, i.e. $n \leq 4$, which conflicts with the assumption that $n > 4$. The proof is complete. □

4.4. **Proof of Theorem 4.5.** As vector spaces (without the Lie structures), we have

$$\mathcal{G} = \text{Span}(X_1, \dots, X_n) \cong \mathbb{R}^n, \quad \mathcal{G}^1 = \text{Span}(X_1, \dots, X_{n-1}) \cong \mathbb{R}^{n-1}.$$

Let $A = (a_{ij})_{i,j=1,2,\dots,n-1}$ be an invertible real $(n - 1)$ -square matrix.

Proof of part (i). We define a Lie structure on \mathcal{G} such that \mathcal{G}^1 is commutative and A is exactly the matrix of adjoint map ad_{X_n} on \mathcal{G}^1 in the chosen basis (X_1, X_2, \dots, X_n) . Namely, the Lie brackets $[\cdot, \cdot]_A$ on \mathcal{G} are given as follows:

$$[X_n, X_j]_A := \sum_{i < n} a_{ij} X_i, \quad j = 1, 2, \dots, n - 1; \text{ the others are trivial.} \tag{4.1}$$

With such Lie structure, the derived ideal of \mathcal{G} is commutative and exactly equal to \mathcal{G}^1 . Hence, \mathcal{G} is an MD($n, n - 1$)-algebra.

Conversely, suppose there is a Lie structure on \mathcal{G} whose Lie brackets $[\cdot, \cdot]$ satisfy the above property (4.1). Because the first derived ideal of \mathcal{G} is commutative and equal to \mathcal{G}^1 , one has $[X_i, X_j] = 0$ for all $i, j = 1, 2, \dots, n - 1$. On the other hand, A is the matrix of adjoint map ad_{X_n} on \mathcal{G}^1 . Therefore, we get $[X_n, X_j] = \sum_{i < n} a_{ij} X_i$ for all $j = 1, 2, \dots, n - 1$. That means $[\cdot, \cdot] \equiv [\cdot, \cdot]_A$ and part (i) is proved.

Proof of part (ii). Let $B = (b_{ij})_{i,j=1,2,\dots,n-1}$ be another invertible real $(n - 1)$ -square matrix and $[\cdot, \cdot]_B$ be the Lie brackets on \mathcal{G} which is defined by B . Then, we have $[X_n, X_j]_B = \sum_{i < n} b_{ij} X_i, j = 1, 2, \dots, n - 1$ (the other Lie brackets are trivial).

(\implies) Suppose that A and B define two Lie structures on \mathcal{G} which are isomorphic. We will show that there exist a real number $c \neq 0$ and an invertible real $(n - 1)$ -square matrix C such that $cA = CBC^{-1}$. Denote by $f : (\mathcal{G}, [\cdot, \cdot]_B) \rightarrow (\mathcal{G}, [\cdot, \cdot]_A)$ the isomorphism between two Lie structures on \mathcal{G} defined by B and A , respectively. This means that f is a linear isomorphism and f preserves the Lie brackets. Let $M = (c_{ij})_{i,j=1,2,\dots,n}$ be the invertible n -square matrix of f in the basis (X_1, X_2, \dots, X_n) , i.e. $f(X_j) = \sum_{i=1}^n c_{ij} X_i$ for all $j = 1, 2, \dots, n$. Since f is an isomorphism, $f(\mathcal{G}^1, [\cdot, \cdot]_B) = (\mathcal{G}^1, [\cdot, \cdot]_A)$, i.e. $c_{nj} = 0$ for all $j = 1, 2, \dots, n - 1$. That means $f(X_j) = \sum_{i < n} c_{ij} X_i$ for all $j = 1, 2, \dots, n - 1$. Put $C = (c_{ij})_{i,j=1,2,\dots,n-1}$ and $c_{nn} = c$, we get $f(X_n) = \sum_{i < n} c_{in} X_i + cX_n$. Then the matrix of f in the basis (X_1, X_2, \dots, X_n) is given by

$$M = \begin{bmatrix} c_{11} & \cdots & c_{1,n-1} & c_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ c_{n-1,1} & \cdots & c_{n-1,n-1} & c_{n-1,n} \\ 0 & \cdots & 0 & c \end{bmatrix} = \begin{bmatrix} C & * \\ 0 & c \end{bmatrix},$$

where the asterisk denotes the column vector $(c_{1n}, c_{2n}, \dots, c_{n,n-1})^T$. Since f is an isomorphism, $0 \neq \det M = c \det C$. Therefore, $\det C \neq 0$ and C is an invertible

$(n - 1)$ -square matrix. Furthermore, we have

$$\begin{aligned}
 f([X_n, X_j]_B) &= [f(X_n), f(X_j)]_A, \quad 1 \leq j \leq n - 1 \\
 \Leftrightarrow f\left(\sum_{i < n} b_{ij} X_i\right) &= \left[\sum_{k < n} c_{kn} X_k + cX_n, \sum_{l < n} c_{lj} X_l\right]_A, \quad 1 \leq j \leq n - 1 \\
 \Leftrightarrow \sum_{i < n} b_{ij} f(X_i) &= c \sum_{l < n} c_{lj} [X_n, X_l]_A, \quad 1 \leq j \leq n - 1 \\
 \Leftrightarrow \sum_{i < n} b_{ij} \left(\sum_{k < n} c_{ki} X_k\right) &= c \sum_{l < n} c_{lj} \left(\sum_{k < n} a_{kl} X_k\right), \quad 1 \leq j \leq n - 1 \\
 \Leftrightarrow \sum_{k < n} \left(\sum_{i < n} c_{ki} b_{ij}\right) X_k &= c \sum_{k < n} \left(\sum_{l < n} a_{kl} c_{lj}\right) X_k, \quad 1 \leq j \leq n - 1 \\
 \Leftrightarrow \sum_{k < n} \left(\sum_{i < n} c_{ki} b_{ij} - c \sum_{l < n} a_{kl} c_{lj}\right) X_k &= 0, \quad 1 \leq j \leq n - 1 \\
 \Leftrightarrow \sum_{i < n} c_{ki} b_{ij} - c \sum_{l < n} a_{kl} c_{lj} &= 0, \quad 1 \leq k, j \leq n - 1 \\
 \Leftrightarrow \sum_{i < n} c_{ki} b_{ij} &= c \sum_{l < n} a_{kl} c_{lj}, \quad 1 \leq k, j \leq n - 1 \\
 \Leftrightarrow (CB)_{kj} &= c(AC)_{kj}, \quad 1 \leq k, j \leq n - 1 \\
 \Leftrightarrow CB &= cAC \\
 \Leftrightarrow cA &= CBC^{-1}.
 \end{aligned}$$

(\Leftarrow) Conversely, assume that there exist a non-zero real number c and an invertible $(n - 1)$ -square matrix $C = (c_{ij})_{i,j=1,2,\dots,n-1}$ such that $cA = CBC^{-1}$. We will show that A, B define on \mathcal{G} two Lie structures which are isomorphic. Indeed, we denote by $f : \mathcal{G} \rightarrow \mathcal{G}$ the linear isomorphism of \mathcal{G} which is defined, in the basis (X_1, X_2, \dots, X_n) , by the invertible n -square matrix

$$M' = \begin{bmatrix} c_{11} & \cdots & c_{1,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ c_{n-1,1} & \cdots & c_{n-1,n-1} & 0 \\ 0 & \cdots & 0 & c \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & c \end{bmatrix}.$$

It can be verified that f preserves the Lie brackets of $(\mathcal{G}, [\cdot, \cdot]_B)$ and $(\mathcal{G}, [\cdot, \cdot]_A)$, i.e. f is also a Lie isomorphism. The proof is complete. \square

5. CONCLUSIONS

We close the paper with some remarkable comments on the problem of the classification of MD-class.

5.1. **MD^{2k}-class.** We emphasize that the problem of classifying all MD-algebras is still open. There are at least three ways of proceeding in the classification of MD-class as follows:

- **First way:** By fixity of the dimension of MD-algebras.
- **Second way:** By fixity of the maximal dimension of coadjoint orbits.
- **Third way:** Combining form of the above ways.

For example, the classifications of MD4-class (see [13–15]) and MD5-class (see [16, 17]), MD(*, 1)-class and MD(*, * – 1)-class in this paper are the results of the first way, while Arnal’s list is the result of the second way. To classify MD-class by the second or third ways, we give the following definitions.

Definition 5.1. Each *n*-dimensional real solvable Lie group *G* whose coadjoint orbits have dimension zero or *2k* ($0 < 2k < n$) is called an MD^{2k}*n*-group. The Lie algebra $\mathcal{G} = \text{Lie}(G)$ of *G* is called an MD^{2k}*n*-algebra. When we do not pay attention to the dimension of the considered group or algebra, we will call *G* or \mathcal{G} an MD^{2k}-group or MD^{2k}-algebra, respectively.

Definition 5.2. The set of all MD^{2k}*n*-algebras or MD^{2k}-algebras will be denoted by MD^{2k}*n*-class or MD^{2k}-class, respectively.

5.2. **Example of MD⁴-algebras.** It has long been known that the Lie algebra $\text{aff}(\mathbb{C})$ of the group $\text{Aff}(\mathbb{C})$ of the affine transformations of the complex straight line is an MD⁴4-algebra and the 5-dimensional Heisenberg Lie algebra \mathfrak{h}_5 is an MD⁴5-algebra. Now we introduce one example of indecomposable MD⁴-algebras.

Let $\mathcal{G}_{2m} = \text{Span}(X_1, X_2, \dots, X_{2m})$ be the 2*m*-dimensional real Lie algebra ($2 \leq m \in \mathbb{N}$) with Lie brackets as follows:

$$[X_1, X_k] := X_k, \quad [X_2, X_{2j-1}] := X_{2j}, \quad [X_2, X_{2j}] := -X_{2j-1}.$$

Upon simple computation, taking Proposition 2.4 into account we get the following proposition.

Proposition 5.3. \mathcal{G}_{2m} is an indecomposable 2*m*-dimensional MD⁴-algebra. □

5.3. **Some open problems.** We have at least two open problems:

- (1) Classify MD(*n*, *m*)-class and MD(*n*, *n* – *m*)-class, $2 \leq m \leq n - 2$ and $n \geq 6$.
- (2) Classify MD^{2k}*n*-class or MD^{2k}-class, $2k \geq 4$ and $n \geq 6$.

In forthcoming papers, we will discuss the classification of MD(*n*, 2)-class, MD(*n*, *n* – 2)-class and MD⁴*n*-class for $n \geq 6$.

REFERENCES

[1] D. Arnal, M. Cahen, and J. Ludwig, *Lie groups whose coadjoint orbits are of dimension smaller or equal to two*, Lett. Math. Phys. **33** (1995), 183–186. MR1316347
 [2] L. Boza, E. M. Fedrian, J. Núñez, and A. F. Tenorio, *A historical review of the classifications of Lie algebras*, Rev. Un. Mat. Argentina **54** (2013), 75–99. MR3263653
 [3] E. Cartan, *Sur la structure des groupes de transformations finis et continus*, Nony, Paris, 1894.

- [4] R. Campoamor-Stursberg, *A note on the classification of nine dimensional Lie algebras with nontrivial Levi decomposition*, Int. Math. Forum **2** (2007), 25–28. MR2333088
- [5] L. Chen, *A class of solvable Lie algebras with triangular decompositions*, Comm. Algebra **40** (2012), 2285–2300. MR2948827
- [6] Do Ngoc Diep, *Methods of Noncommutative Geometry for Group C^* -algebras*, Chapman & Hall/CRC Press, Cambridge, 2000. MR1741113
- [7] M. T. Duong, G. Pinczon, and R. Ushirobira, *A new invariant of quadratic Lie algebras*, Algebr. Represent. Theory **15** (2012), 1163–1203. MR2994021
- [8] L. Yu. Galitski and D. A. Timashev, *On classification of metabelian Lie algebras*, J. Lie Theory **9** (1999), 125–156. MR1680007
- [9] F. R. Gantmacher, *On the classification of real simple Lie groups*, Rec. Math. [Mat. Sbornik] N.S. **5** (1939), 217–250.
- [10] M. A. Gauger, *On the classification of metabelian Lie algebras*, Trans. Amer. Math. Soc. **179** (1973), 293–329. MR0325719
- [11] I. Kath, *Nilpotent metric Lie algebras of small dimension*, J. Lie Theory **17** (2007), no. 1, 41–61. MR2286880
- [12] A. A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag, New York, 1976. MR0412321
- [13] A. V. Le, *On the structure of the C^* -algebra of the foliation formed by the K -orbits of maximal dimension of the real diamond group*, J. Operator Theory **24** (1990), 227–238. MR1150619
- [14] ———, *On the foliations formed by the generic K -orbits of the MD4-groups*, Acta Math. Vietnam. **15** (1990), 39–55. MR1105395
- [15] ———, *Foliations formed by orbits of maximal dimension in a coadjoint representation of a class of solvable Lie groups*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **3** (1993), 26–29, 104–105; English transl., Moscow Univ. Math. Bull. **48** (1993), no. 3, 24–27. MR1355517
- [16] A. V. Le and K. P. Shum, *Classification of 5-dimensional MD-algebras having commutative derived ideals*, Advances in Algebra and Combinatorics, World Scientific, Singapore, 2008, pp. 353–371. MR2440125
- [17] A. V. Le, V. H. Ha, and T. H. N. Tran, *Classification of 5-dimensional MD-algebras having non-commutative derived ideals*, East-West J. Math. **13** (2011), 118–132. MR3024979
- [18] A. I. Maltsev, *On solvable Lie algebras*, Bull. Acad. Sci. URSS. Sér. Math. **9** (1945), 329–356. MR0022217
- [19] G. M. Mubarakzhanov, *Classification of real structures of Lie algebras of fifth order*, Izv. Vyssh. Uchebn. Zaved. Matematika **3** (1963), no. 34, 99–106. MR0155871
- [20] J. Patera and H. Zassenhaus, *Solvable Lie algebras of dimension ≤ 4 over perfect fields*, Linear Algebra Appl. **142** (1990), 1–17. MR1077969
- [21] L. Šnobl, *On the structure of maximal solvable extensions and of Levi extensions of nilpotent Lie algebras*, J. Phys. A **43** (2010), 505202, 17 pp. MR2740380
- [22] L. Šnobl and D. Karásek, *Classification of solvable Lie algebras with a given nilradical by means of solvable extensions of its subalgebras*, Linear Algebra Appl. **432** (2010), 1836–1850. MR2592920
- [23] L. Šnobl and P. Winternitz, *A class of solvable Lie algebras and their Casimir invariants*, J. Phys. A **38** (2005), 2687–2700. MR2132082
- [24] G. Tsagas, *Classification of nilpotent Lie algebras of dimension eight*, J. Inst. Math. Comput. Sci. Math. Ser. **12** (1999), 179–183. MR1737529

- [25] G. Tsagas, A. Kobotis, and T. Koukouvinos, *Classification of nilpotent Lie algebras of dimension nine whose maximum abelian ideal is of dimension seven*, Int. J. Comput. Math. **74** (2000), 5–28. MR1754385
- [26] P. Turkowski, *Solvable Lie algebras of dimension six*, J. Math. Phys. **31** (1990), 1344–1350. MR1054322
- [27] M. S. Vuong and H. V. Ho, *Sur la structure des C^* -algèbres d'une classe de groupes de Lie*, J. Operator Theory **11** (1984), 77–90.

Anh Vu Le 

Department of Economic Mathematics, University of Economics and Law, Vietnam National University, Ho Chi Minh City, Viet Nam
vula@uel.edu.vn

Van Hieu Ha

Department of Economic Mathematics, University of Economics and Law, Vietnam National University, Ho Chi Minh City, Viet Nam
hieuhv@uel.edu.vn

Anh Tuan Nguyen

Faculty of Political Science and Pedagogy, University of Physical Education and Sports, Ho Chi Minh City, Viet Nam
natuan@upes.edu.vn

Tran Tu Hai Cao

Le Quy Don High School for the Gifted, Ninh Thuan Province, Viet Nam
tuhai.thptlequydon@ninhthuan.edu.vn

Thi Mong Tuyen Nguyen

Faculty of Mathematics and Information, Dong Thap University, Cao Lanh city, Dong Thap Province, Viet Nam
ntmtuyen@dtthu.edu.vn

Received: May 10, 2015

Accepted: August 31, 2016