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AN INTERPOLATION THEOREM BETWEEN CALDERÓN–HARDY SPACES

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ABSTRACT. We obtain a complex interpolation theorem between weighted Calderón–Hardy spaces for weights in a Sawyer class. The technique used is based on the method obtained by J.-O. Strömberg and A. Torchinsky; however, we must overcome several technical difficulties associated with considering one-sided Calderón–Hardy spaces. Interpolation results of this type are useful in the study of weighted weak type inequalities of strongly singular integral operators.

1. INTRODUCTION

The aim of this paper is to prove an interpolation theorem between one-sided Calderón–Hardy spaces $\mathcal{H}_{\alpha}^{p,+}(\omega)$, that we will define below. Complex interpolation theory started with the so celebrated paper due to A. Calderón [1]; we also recommend to see the paper by A. Calderón and A. Torchinsky [4]. In that line, in [18] J.-O. Strömberg and A. Torchinsky obtained a complex interpolation theorem between weighted Hardy spaces and in [14] the authors obtained a generalization of that result. On the other hand, E. Gatto, J. Jiménez and C. Segovia ([6]) studied the Calderón–Hardy spaces in order to characterize the solutions of $\Delta^m F = f, m \in \mathbb{N}$ for distributions f in the Hardy spaces H^p. They proved that the operator Δ^m is a bijective application from the Calderón–Hardy spaces onto H^p. A more general weighted version of these spaces was studied in [11]. It turns out that, when α is a positive integer, the spaces $\mathcal{H}_{\alpha}^{p,+}(\omega)$ can be identified with the one-sided Hardy space $H_{+}^{p}(\omega)$ defined in [15]. Thus, it is natural to think that, in the case that the parameter α is a natural number, it will be possible to obtain an interpolation theorem between weighted Calderón–Hardy spaces. However, if the parameter α is not an integer number that identification fails (a counterexample is obtained in [11]). So, this fact motivated us to think about the question of whether it is possible to obtain an interpolation theorem between weighted Calderón–Hardy spaces even in those cases where we do not have such identification. The main result of this paper (Theorem 1.1) gives a positive answer to that question.

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In order to state the main results we introduce the following notations and definitions which will be used throughout this paper.

A weight ω is a measurable and non-negative function. If $E \subset \mathbb{R}$ is a Lebesgue measurable set, we denote its ω-measure by $\omega(E) = \int_E \omega(t)dt$. A function $f(x)$ belongs to $L^p(\omega)$, $0 < p < \infty$, if $\|f\|_{L^p(\omega)} = \left(\int_{-\infty}^{\infty} |f(x)|^p \,\omega(x)dx\right)^{1/p}$ is finite.

The classes A_s^+, $1 \le s \le \infty$, were defined by E. Sawyer in [16](see also [10]). A weight ω belongs to the class A_s^+, $1 < s < \infty$, if there exists a constant C such that

$$\left(\frac{1}{h}\int_{x-h}^{x} \omega(t)dt\right)\left(\frac{1}{h}\int_{x}^{x+h} \omega(t)^{-\frac{1}{s-1}}dt\right)^{s-1} \le C,$$

for almost all real numbers x.

In the limit case of $s = 1$ it is said that ω belongs to the class A_1^+ if $M^-\omega(x) \le C\omega(x)$ a.e. $x \in \mathbb{R}$, where $M^-f(x) = \sup_{h>0} \frac{1}{h}\int_{x-h}^{x} |f(t)|\,dt$. Also we consider $A_\infty^+ = \bigcup_{s\ge 1} A_s^+$. The properties of the one-sided weights which we will use in this paper can be seen in [8], [12] and [16].

Let us fix $\omega \in A_s^+$. Then there exist $x_{-\infty}$ and x_∞, $-\infty \le x_{-\infty} \le x_\infty \le \infty$, such that $\omega(x) = 0$ if $x < x_{-\infty}$, $\omega(x) = \infty$ if $x > x_\infty$, and $0 < w(x) < \infty$ if $x_{-\infty} < x < x_\infty$. Also w is locally integrable in $(-\infty, x_\infty)$ (see [11] for the details). We denote by $L^2_{\text{loc}}(x_{-\infty}, \infty)$ the space of complex valued functions $f(x)$ on $\mathbb{R}$ that belong locally to L^2 for compact subsets of $(x_{-\infty}, \infty)$. We endow $L^2_{\text{loc}}(x_{-\infty}, \infty)$ with the topology generated by the seminorms

$$|f|_I = \left(|I|^{-1}\int_I |f(y)|^2\,dy\right)^{1/2},$$

where $I = (a, b)$ is an interval in $(x_{-\infty}, \infty)$ and $|I| = b - a$.

Let $f \in L^2_{\text{loc}}(x_{-\infty}, \infty)$ and let α be a real positive number. We define the maximal function $n_\alpha^+(f; x)$ by

$$n_\alpha^+(f; x) = \sup_{\rho>0} \rho^{-\alpha}\,|f|_{[x,x+\rho]}\,.$$

Let $N \ge 0$ be an integer, and $\mathcal{P}_N$ the subspace of $L^2_{\text{loc}}(x_{-\infty}, \infty)$ of all polynomials of degree at most N. This subspace is of finite dimension and therefore a closed subspace of $L^2_{\text{loc}}(x_{-\infty}, \infty)$. We denote by E_N the quotient space $L^2_{\text{loc}}(x_{-\infty}, \infty)/\mathcal{P}_N$. If $F \in E_N$, we define the seminorms

$$\|F\|_I = \inf_{f\in F} \{|f|_I\}\,.$$

The family of all these seminorms induces on E_N the quotient topology.

For $F \in E_N$, we define the maximal function $N_\alpha^+(F; x)$ as

$$N_\alpha^+(F; x) = \inf_{f\in F} \left\{n_\alpha^+(f; x)\right\}.$$

This type of maximal function was introduced by Calderón in [2]. It can be shown that if $N_\alpha^+(F,x)$ is finite for some x then there exists a unique $f \in F$ such that $n_\alpha^+(f;x) < \infty$ and, therefore, $N_\alpha^+(F;x) = n_\alpha^+(f;x)$.

For $F \in E_N$, we say that F belongs to $\mathcal{H}_\alpha^{p,+}(\omega)$, $0 < p \leq 1$, if the maximal function $N_\alpha^+(F;x) \in L^p(\omega)$. That is

$$\int_{x_{-\infty}}^{\infty} N_\alpha^+(F;x)^p \omega(x)\,dx < \infty.$$

The "norm" of F in $\mathcal{H}_\alpha^{p,+}(\omega)$ is given by $\|F\|_{\mathcal{H}_\alpha^{p,+}(\omega)} = \|N_\alpha^+(F;x)\|_{L^p(\omega)}$.

Usually the parameter α will be written as $\alpha = N+\beta$, where N is a non negative integer and $0 < \beta \leq 1$. We say that the class $F \in E_N$ belongs to $\Lambda_\alpha(x_{-\infty},\infty)$, or simply $F \in \Lambda_\alpha$, if every $f \in F$ is a α Lipschitz function, that is $f \in C^N(x_{-\infty},\infty)$, and there exists a constant C such that the derivative $D^N f$ satisfies for every $x, x' \in (x_{-\infty},\infty)$ the Lipschitz condition

$$\left|D^N f(x) - D^N f(x')\right| \leq C\ |x-x'|^\beta.$$

We shall denote by Ω the following strip in the complex plane: $\Omega = \{z \in \mathbb{C} : 0 \leq \operatorname{Re}(z) \leq 1\}$. For $0 < p_0, p_1 \leq 1$ and $z \in \Omega$ we define $p(z)$ as

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}.$$

For $0 \leq u \leq 1$ we define $\mu(u) = \frac{up(u)}{p_1}$.

Let ω and $\nu \in A_\infty^+$. We shall assume that $\omega(x) \leq \tau$ and $\nu(x) \leq \tau$ for some $\tau > 0$. Also we will consider that they have the same $x_{-\infty}$ and x_∞. We define the weight

$$\mu(u)(x) = \omega(x)^{1-\mu(u)}\nu(x)^{\mu(u)}.$$

In the same way we consider $r(z)$ and $\tilde{\mu}(u)$, for $0 < r_0, r_1 \leq 1$ and $\tilde{\omega}, \tilde{\nu} \in A_\infty^+$.

Let ℓ be a fixed real number $0 < \ell < 1$; we shall write $p = p(\ell)$, $r = r(\ell)$, $\mu = \mu(\ell)$ and $\tilde{\mu} = \tilde{\mu}(\ell)$.

In order to give an interpolation theorem between Calderón–Hardy spaces $\mathcal{H}_\alpha^{p,+}(\mu)$ on parameter p, we fix the parameter α; in particular we consider α such that $\left(\alpha + \frac{1}{2}\right)p(u) \geq s > 1$ or $(\alpha + \frac{1}{2})p(u) > 1$ if $s = 1$.

With the notation and definitions given above we can state the main results of this paper.

Theorem 1.1 (An interpolation theorem between Calderón–Hardy spaces)**.** *Let $z \in \Omega$ and $T_z : \mathcal{H}_\alpha^{p_0,+}(\omega) + \mathcal{H}_\alpha^{p_1,+}(\nu) \longrightarrow \mathcal{H}_\alpha^{r_0,+}(\tilde{\omega}) + \mathcal{H}_\alpha^{r_1,+}(\tilde{\nu})$ be a family of linear operators. Let $g_x(z,.)$ be the representative of T_zF such that $N_\alpha^+(T_zF;x) = n_\alpha^+(g_x(z,\cdot);x) < \infty$. Assume that for $x \in (x_{-\infty},\infty)$ and $\rho > 0$*

$$\phi(x,\rho,z) = \frac{1}{\rho^{2\alpha+1}}\int_x^{x+\rho} (g_x(z,y))^2\,dy$$

is a measurable function with respect to the x variable, continuous and bounded for $z \in \Omega$ and analytic in the interior of Ω for each fixed (x,ρ) in $(x_{-\infty},\infty)\times(0,\infty)$, *and such that $g_x(z,y)$ takes real values when z is real. If, in addition, for some constants $c, \beta > 0$,*

$$\begin{aligned} \|T_{it}F\|_{\mathcal{H}_\alpha^{r_0,+}(\tilde{\omega})} &\le ce^{\beta|t|}\|F\|_{\mathcal{H}_\alpha^{p_0,+}(\omega)} \\ \text{and} \quad \|T_{1+it}F\|_{\mathcal{H}_\alpha^{r_1,+}(\tilde{\nu})} &\le ce^{\beta|t|}\|F\|_{\mathcal{H}_\alpha^{p_1,+}(\nu)}, \end{aligned} \tag{1.1}$$

then, for $0 < \ell < 1$, there exists a constant $C > 0$ such that for, any $F \in \mathcal{H}_\alpha^{p,+}(\mu)$,

$$\|T_\ell F\|_{\mathcal{H}_\alpha^{r,+}(\tilde{\mu})} \le C\|F\|_{\mathcal{H}_\alpha^{p,+}(\mu)}.$$

As we mentioned before, this kind of results in complex interpolation theory were introduced by A. P. Calderón in [1] and A. P. Calderón and A. Torchinsky in [4]. Applications of these results to study strongly singular operators in the context of weighted Hardy spaces can be found in [5] and [14]. Theorem 1.1 is a consequence of the following theorem.

Theorem 1.2.

(1) *Let $F(z)$ be a function of a complex variable $z \in \Omega$ which takes values in $\mathcal{H}_\alpha^{p_0,+}(\omega) + \mathcal{H}_\alpha^{p_1,+}(\nu)$, and for each $z \in \Omega$ we consider $f_x(z,\cdot)$ the representative of $F(z)$ such that $n_\alpha^+(f_x(z,\cdot);x) < \infty$. Let $\mathcal{F}(x,\rho,z) = \dfrac{1}{\rho^{2\alpha+1}}\displaystyle\int_x^{x+\rho}(f_x(z,y))^2\,dy$ be a measurable function with respect to the x variable, continuous and bounded for $z \in \Omega$ and analytic in the interior of Ω for each fixed (x,ρ) in $(x_{-\infty},\infty) \times (0,\infty)$ and such that $f_x(z,y)$ takes real values when z is real. If $F(it) \in \mathcal{H}_\alpha^{p_0,+}(\omega)$, $\sup_t \|F(it)\|_{\mathcal{H}_\alpha^{p_0,+}(\omega)} < \infty$, $F(1+it) \in \mathcal{H}_\alpha^{p_1,+}(\nu)$ and $\sup_t \|F(1+it)\|_{\mathcal{H}_\alpha^{p_1,+}(\nu)} < \infty$, then for $0 < \ell < 1$, $F(\ell) \in \mathcal{H}_\alpha^{p,+}(\mu)$, $p_0 < p < p_1$ and*

$$\|F(\ell)\|_{\mathcal{H}_\alpha^{p,+}(\mu)} \le C\left(\sup_t \|F(it)\|_{\mathcal{H}_\alpha^{p_0,+}(\omega)}\right)^{1-\ell}\left(\sup_t \|F(1+it)\|_{\mathcal{H}_\alpha^{p_1,+}(\nu)}\right)^{\ell}. \tag{1.2}$$

(2) *If $F \in \mathcal{D}$ then there exists a function $F(z)$ such that $\mathcal{F}(x,\rho,z)$ has the properties stated in (1), $F(\ell) = F$ and*

$$\|F(u+it)\|^{p(u)}_{\mathcal{H}_\alpha^{p(u),+}(\mu(u))} \le C\|F\|^p_{\mathcal{H}_\alpha^{p,+}(\mu)} \tag{1.3}$$

holds for $z = u + it \in \Omega$ and C does not depend on F. Here $\mathcal{D}$ is a dense subset of $\mathcal{H}_\alpha^{p_0,+}(\mu)$. (See Section 3 for the definition of $\mathcal{D}$.)

Remark 1.3. A version of Theorem 1.1 without assuming the weights to be bounded, can be obtained by considering $\omega_\tau(x) = \min\{\omega(x),\tau\}$ and $\nu_\tau(x) = \min\{\nu(x),\tau\}$ for given $\tau > 0$ and with the aditional hypothesis that the constants C in (1.1) do not depend on τ.

Remark 1.4. It is worth to point out that these results can be obtained in the context of Muckenhoupt weights with a simpler proof. In fact, it is not necessary to assume that the weights are bounded.

This paper is organized as follows. In Section 2 we present some auxiliary lemmas that will be needed later in Section 4. In Section 3, we give a particular atomic decomposition of the space $\mathcal{H}_\alpha^{p,+}(\mu)$, with some additional useful properties. The main result of this paper, Theorem 1.2, is proved in Section 4. This result is the principal tool to obtain Theorem 1.1.

2. Auxiliary lemmas

In the following lemma we resume several properties of the maximal function $N_\alpha^+(F,x)$ and the spaces $\mathcal{H}_\alpha^{p,+}(\omega)$.

Lemma 2.1 ([6, 11, 12])**.**

(1) Let f_1, f_2 be two representatives of $F \in E_N$ and $P = f_1 - f_2$. There exists a constant c_k such that for every x_1, x_2 and y in $(x_{-\infty}, \infty)$ the inequality

$$\left| \left(\frac{d}{dy} \right)^k P(y) \right| \leq c_k \, \left(n_\alpha^+(f_1; x_1) + n_\alpha^+(f_2; x_2) \right) \left(|x_1 - y| + |x_2 - y| \right)^{\alpha - k}$$

holds.

(2) Let $F \in E_N$. If $N_\alpha^+(F, x_0)$ is finite for some x_0 then there exists a unique $f \in F$ such that $n_\alpha^+(f; x_0) < \infty$ and, therefore, $N_\alpha^+(F; x_0) = n_\alpha^+(f; x_0)$.

(3) Let $F \in E_N$. If $N_\alpha^+(F;x)$ is finite, f is a representative of F and we denote by $P(x,y)$ the unique polynomial of degree at most N such that $n_\alpha^+(f(y) - P(x,y); x) = N_\alpha^+(F;x)$, then $f(x) = P(x,x)$ for almost every x such that $N_\alpha^+(F;x)$ is finite.

(4) Let $\{F_i\}$ be a sequence of elements in E_N, such that $\sum_i N_\alpha^+(F_i;x)$ is finite for almost every $x \in (x_{-\infty}, \infty)$. Then

   (i) *The series $\sum_i F_i$ converges in E_N to some F. Moreover $N_\alpha^+(F;x) \leq \sum_i N_\alpha^+(F_i;x)$ for all $x \in (x_{-\infty}, \infty)$.*

   (ii) *Let x_0 be such that $\sum_i N_\alpha^+(F_i; x_0) < \infty$; if f_i is the unique representative of the class F_i such that $N_\alpha^+(F_i, x_0) = n_\alpha^+(f_i, x_0)$, then the series $\sum_i f_i$ converge in $L^2_{\mathrm{loc}}(x_{-\infty}, \infty)$ to a function f that is the unique representative of F such that $N_\alpha^+(F, x_0) = n_\alpha^+(f, x_0)$.*

(5) If $\{F_i\}$ is a sequence in E_N such that converges to F in $\mathcal{H}_\alpha^{p,+}(\omega)$ then $\{F_i\}$ converges to F in E_N.

(6) Let $0 < p \leq 1$ and $w \in A_s^+$ where $\left(\alpha + \frac{1}{2}\right) p \geq s > 1$ or $(\alpha + \frac{1}{2})p > 1$ if $s = 1$. The space $\mathcal{H}_\alpha^{p,+}(\omega)$ is complete.

(7) F belongs to Λ_α if and only if there exists a finite constant C such that $N_\alpha^+(F, x) \leq C$ for all $x \in (x_{-\infty}, \infty)$.

(8) The maximal function $N_\alpha^+(F, x)$ associated with a class $F \in E_N$ is lower semicontinuous.

(9) Let $0 < p \leq 1$ and $w \in A_s^+$, where $\left(\alpha + \frac{1}{2}\right) p \geq s > 1$ or $(\alpha + \frac{1}{2})p > 1$ if $s = 1$, and let $A \in E_N$ such that there exists a representative $a \in A$ with compact support contained in an interval I and such that $N_\alpha^+(A;x) \leq \omega(I)^{-\frac{1}{p}}$ for all $x \in (x_{-\infty}, \infty)$. Then $\|A\|^p_{\mathcal{H}_\alpha^{p,+}(\omega)} \leq C_\omega$, where C_ω does not depend on A.

A proof of (1) is obtained in [11, Lemma 3.3]. The proof of (2) is similar to the one of [6, Lemma 3]. Proofs of (3), (4) and (5) can be seen in [12, Lemma 3.4.5], [12, Lemma 3.2.6] and [12, Corollary 3.2.4], respectively. The proof of (6) is similar to that of [6, Corollary 2]; see also [12, Corollary 3.2.7]. Part (7) is Lemma 3.10 in [11]. Finally, proofs of (8) and (9) are in Lemma 3.2.9 and Lemma 3.2.12 in [12], respectively.

Remark 2.2. Let $F \in \Lambda_\alpha$ and $f \in F$. Then $f \in C^N$ and $D^N f = \frac{d^N f}{dx^N} \in \Lambda_\beta$, $\alpha = N + \beta$, $0 < \beta \leq 1$. For $x \in (x_{-\infty}, \infty)$, let $P_N(x,y) = \sum_{k=0}^{N} \frac{1}{k!} \frac{d^k f}{dx^k}(x)(y-x)^k$ be the Taylor polynomial of f in x. We claim that $f(y) - P_N(x,y)$ is the representative of F that realizes the maximal function $N_\alpha^+(F;x)$ (see items (2) and (7) in Lemma 2.1). In fact, for some $0 < \theta < 1$, we have

$$\begin{aligned} |f(y) - P_N(x,y)| &= \left| f(y) - P_{N-1}(x,y) - \frac{1}{N!} \frac{d^N f}{dx^N}(x)(y-x)^N \right| \\ &= \frac{1}{N!} |y-x|^N \left| \frac{d^N f}{dx^N}(x + \theta(y-x)) - \frac{d^N f}{dx^N}(x) \right| \\ &\leq C_N |y-x|^N |y-x|^\beta \\ &= C_N |y-x|^\alpha, \end{aligned}$$

which implies $n_\alpha^+(f - P_N(x,\cdot);x) < \infty$.

To prove the first part of Theorem 1.2, we shall need the following results that can be found in [18, Chapter XII].

Lemma 2.3 ([18])**.** *Let $g : \Omega \to \mathbb{C}$ be a continuous function on Ω and analytic in the interior of Ω such that*

$$|g(z)| \leq c \exp(A(\exp(a|\operatorname{Im}(z)|)))$$

for some constants $c > 0$, $A \in \mathbb{R}$ and $0 < a < \pi$. Then, for $0 < \ell < 1$, we have

$$\ln(|g(\ell)|) \leq \int_{-\infty}^{\infty} \ln(|g(it)|) P_0(\ell,t)\, dt + \int_{-\infty}^{\infty} \ln(|g(1+it)|) P_1(\ell,t)\, dt,$$

where P_0 and P_1 are the Poisson kernel for Ω.

Corollary 2.4 ([18])**.** *Under the assumptions of Lemma 2.3 we have*

$$|g(\ell)|^{\frac{p}{2}} \leq \left(\frac{1}{1-\ell} \int_{-\infty}^{\infty} |g(it)|^{\frac{p_0}{2}} P_0(\ell,t)\, dt \right)^{1-\mu} \left(\frac{1}{\ell} \int_{-\infty}^{\infty} |g(1+it)|^{\frac{p_1}{2}} P_1(\ell,t)\, dt \right)^{\mu},$$

where $p = p(\ell)$ and $\mu = \mu(\ell)$.

For the second part of Theorem 1.2 we shall need the following three lemmas that can be found in [14] and [9].

Lemma 2.5 ([14])**.** *Let $\omega \in A_p^+$, $\nu \in A_q^+$, $0 < \delta < 1$ and $\eta > 0$. If $s = p(1-\delta) + q\delta$ then*

$$\omega(x)^{1-\delta} \nu(x)^\delta \in A_s^+,$$

and there exists a constant $C = C(\eta, \delta, \omega, \nu)$ *such that*

$$\left(\frac{1}{I^-}\int_{I^-}\omega(x)\,dx\right)^{1-\delta}\left(\frac{1}{I^-}\int_{I^-}\nu(x)\,dx\right)^{\delta} \leq C\left(\frac{1}{I^+}\int_{I^+}\omega(x)^{1-\delta}\nu(x)^{\delta}\,dx\right)$$

holds for every pair of intervals $I^- = (a,b)$ *and* $I^+ = (b,c)$ *with* $b - a = \eta(c-b)$.

Lemma 2.6 ([14])**.** *Given* $0 < p < \infty$ *and* $\eta > 0$ *there exists a constant* $C = C(p, \eta)$ *such that if* δ *is a weight on the real line then*

$$\left\|\sum \lambda_k \chi_{I_k}\right\|_{L^p(\delta)} \leq C\left\|\sum \lambda_k \chi_{E_k}\right\|_{L^p(\delta)}$$

holds for every $\lambda_k > 0$ *and for all intervals* I_k *and all* δ *measurable* E_k*,* $E_k \subset I_k$ *with* $\delta(E_k) \geq \eta\delta(I_k)$*.*

Lemma 2.7 ([9])**.** *Let* $\delta \in A^+_\infty$*.*

(1) There exists $\beta > 0$ *such that the following implication holds: given* $\lambda > 0$ *and an interval* (a,b) *such that* $\lambda \leq \frac{\delta(a,x)}{x-a}$ *for all* $x \in (a,b)$*, then*

$$|\{x \in (a,b) : \delta(x) > \beta\lambda\}| > \frac{1}{2}(b-a).$$

(2) There exists $\gamma > 0$ *such that the following implication holds: given* $\lambda > 0$ *and an interval* (a,b) *such that* $\lambda \leq \frac{\delta(x,b)}{(b-x)}$ *for all* $x \in (a,b)$*, then*

$$|\{x \in (a,b) : \delta(x) < \gamma\lambda\}| > \frac{1}{2}\delta(b-a).$$

Finally we state some properties of the A^+_s weights that are easy to prove.

Lemma 2.8. *Let* $\omega \in A^+_s$*,* $1 \leq s < \infty$ *and let* I *and* I_h *be bounded intervals such that* $I_h \subset I \subset (x_{-\infty}, \infty)$*; then there exists* $C = C(\omega, I, s)$ *not depending on* I_h *such that*

$$\frac{|I_h|^s}{\omega(I_h)} < C.$$

If, in addition, $\omega(x) \leq \tau$ *for some* $\tau > 0$*, then*

$$\frac{|I_h|^{s-1}}{\tau} \leq \frac{|I_h|^s}{\omega(I_h)}.$$

3. A particular atomic decomposition of $\mathcal{H}^{p,+}_\alpha(w)$

In this section we give an atomic decomposition of a class $F \in \mathcal{H}^{p,+}_\alpha(\omega)$ with additional properties which are used to prove (1.3).

Definition 3.1. We say that a class $A \in E_N$ is an atom in $\mathcal{H}^{p,+}_\alpha(\omega)$ if there exists a representative $a(y)$ of A and an interval I such that

(i) $\operatorname{supp}(a) \subset I \subset (x_{-\infty}, \infty)$, $\omega(I) < \infty$;
(ii) $N^+_\alpha(A; x) \leq 1$ for all $x \in (x_{-\infty}, \infty)$.

For $\omega \in A_s^+$, the condition $\omega(I) < \infty$ does not imply that I is bounded. However, from the properties of one-sided weights, I can not be of the form (b, ∞). So if I is not bounded then we have that $x_{-\infty} = -\infty$ and $I = (-\infty, b)$, $b < \infty$.

The following lemma shows how large the norm of an atom in the space $\mathcal{H}_\alpha^{p,+}(\omega)$ is. The proof is inmediate from Lemma 2.1 item (9).

Lemma 3.2. *Let $\omega \in A_s^+$, $0 < p \leq 1$ and $(\alpha + \frac{1}{2})p \geq s > 1$ or $(\alpha + \frac{1}{2})p > 1$ if $s = 1$. There exists a constant $C_\omega > 0$ such that for any atom A in $\mathcal{H}_\alpha^{p,+}(\omega)$ with associated interval I as in Definition 3.1 the following inequality holds:*

$$\|A\|^p_{\mathcal{H}_\alpha^{p,+}(w)} \leq C_\omega \omega(I). \tag{3.1}$$

With the ideas of the proof of the second part of Theorem 1 in [18, Chapter VIII], we obtain the following analogous result for Calderón–Hardy spaces.

Theorem 3.3 ([18])**.** *Let $\omega \in A_s^+$ and $0 < p \leq 1$, $(\alpha+1/2)p \geq s > 1$ or $(\alpha+1/2)p > 1$ if $s = 1$. Let $\{\lambda_j\}$ be a sequence of positive numbers and $\{A_j\}$ a sequence of atoms with associated bounded intervals $\{I_j\}$ respectively such that $\| \sum_{j\geq 1} \lambda_j \chi_{I_j} \|_{L^p(\omega)} < \infty$. Then $\sum_{j \geq 1} \lambda_j A_j$ converges unconditionally, in the $\mathcal{H}_\alpha^{p,+}(\omega)$ norm, to some $F \in \mathcal{H}_\alpha^{p,+}(\omega)$. Moreover, there exists a constant C such that*

$$\|F\|_{\mathcal{H}_\alpha^{p,+}(\omega)} \leq C \Big\| \sum_j \lambda_j \chi_{I_j} \Big\|_{L^p(\omega)}.$$

Theorem 3.4 (Decomposition into atoms, [11])**.** *Let $\omega \in A_s^+$ and $0 < p \leq 1$, such that $(\alpha + 1/2)p \geq s > 1$ or $(\alpha + 1/2)p > 1$ if $s = 1$. If $F \in \mathcal{H}_\alpha^{p,+}(\omega)$ then there exist a sequence $\{\lambda_j\}$ of positive coefficients and a sequence $\{A_j\}$ of atoms with associated bounded intervals $\{I_j\}$ respectively, such that $\sum_{j\geq 1} \lambda_j A_j$ converges unconditionally to F in E_N and in the $\mathcal{H}_\alpha^{p,+}(\omega)$ norm. Moreover there exist two absolute constants C_1 and C_2 such that*

$$C_1 \Big\| \sum_{j\geq 1} \lambda_j \chi_{I_j} \Big\|_{L^p(\omega)} \leq \|F\|_{\mathcal{H}_\alpha^{p,+}(\omega)} \leq C_2 \Big\| \sum_{j\geq 1} \lambda_j \chi_{I_j} \Big\|_{L^p(\omega)}.$$

Also, for $r > 0$,

$$\sum_{j\geq 1} \lambda_j^r \chi_{I_j}(x) \leq C_r [N_\alpha^+(F; x)]^r$$

for almost every $x \in (x_{-\infty}, \infty)$, where C_r does not depend on F.

To prove Theorem 3.4 we follow the proof of the atomic decomposition of $\mathcal{H}_\alpha^{p,+}(\omega)$ given in [11, Theorem 2.1], making the small modifications necessary to adapt it to our definition of atom.

We shall denote by $\mathcal{D}$ the set of all finite lineal combinations, with positive coefficients, of atoms A_j supported in bounded intervals I_j. As a consequence of Theorem 3.4 we have that $\mathcal{D}$ is dense in $\mathcal{H}_\alpha^{p,+}(\omega)$.

Our aim is to show that for a given $F \in \mathcal{H}_\alpha^{p,+}(\omega)$ there exists an atomic decomposition as stated in Theorem 3.4 such that for each atom A_h with associated interval I_h as in Definition 3.1, there is another atom A_j in the decomposition with associated interval I_j following I_h and such that $|I_j| \leq |I_h| \leq 4|I_j|$. We say that an interval J follows the interval I if $I = [c, d]$ and $J = [d, e]$. We proceed as in [14]. The next lemma is Lemma 7 in [14].

Lemma 3.5 ([14]). *Let $r > 0$. There exists a sequence $\{\eta_h\}_{h=-\infty}^{\infty}$ of C_0^∞ functions such that*

(1) $0 \leq \eta_h \leq 1$ *and* $\displaystyle\sum_{h=-\infty}^{\infty} \eta_h(x) = \chi_{(-\infty,r)}(x)$.
(2) $\operatorname{supp}(\eta_h) \subset I_h = [r - 2^{-h}r, r - 2^{-h-2}r]$.
(3) *If we denote* $r_h = \frac{r}{2^h}$ *and* $x \in I_h$ *then* $\frac{1}{4}r_h \leq r - x \leq r_h$.
(4) *Each x belongs to at most three intervals I_h.*
(5) *For every non negative integer i there exists a positive constant c_i such that* $|D^i \eta_h(x)| \leq c_i r_h^{-i}$.

For more details see also [15, Theorem 2.1]. This lemma allows us to break up an atom as it is done in [14, Lemma 7].

Theorem 3.6. *Let A be an atom in $\mathcal{H}_\alpha^{p,+}(\omega)$ with associated bounded interval I. Then there exists a sequence $\{A_h\}$ of atoms with associated bounded intervals $\{I_h\}$ such that*

$$A = C \sum_{h \geq -1} A_h \quad \text{in } \mathcal{H}_\alpha^{p,+}(\omega),$$

where C does not depend on A. In addition, $I = \bigcup_{h \geq -1} I_h$, and no point $x \in I$ belongs to more than three intervals I_h. Moreover, for every h the interval I_{h+2} follows I_h and $|I_{h+2}| \leq |I_h| \leq 4|I_{h+2}|$.

Proof. Without loss of generality we supose that $I = [0, r]$, $r > 0$. Let $a(y)$ be the representative of A supported in I that fulfills the definition of atom.

Let us consider the sequence of functions $\{\eta_h\}_{h=-\infty}^{\infty}$ given in Lemma 3.5 associated with the interval $(-\infty, r)$, such that $\operatorname{supp}(\eta_h) \subset I_h = [r - 2^{-h}r, r - 2^{-h-2}r]$. Then, by Lemma 3.5 part (1), we have

$$a(x) = \sum_{h=-\infty}^{\infty} a(x)\eta_h(x) = \sum_{h=-\infty}^{\infty} \theta_h(x). \tag{3.2}$$

For each h, we denote by Θ_h the class in E_N of $\theta_h(x) = \eta_h(x)a(x)$. Considering arguments similar to those given in [11, Lemma 4.5] and using Lemma 3.5 we can prove that there exist a constant C such that

$$N_\alpha^+(\Theta_h; x) \leq C \tag{3.3}$$

for each $h \in \mathbb{Z}$ and for all $x \in (x_{-\infty}, \infty)$.

For each $h \geq -1$ we define

$$a_h(y) = C^{-1}\theta_h(y),$$

where C is the constant in (3.3), and denote by A_h the class of $a_h(y)$ in E_N. Then by (3.3) we have

$$N_\alpha^+(A_h; x) = C^{-1} N_\alpha^+(\Theta_h; x) \leq 1.$$

Since $\operatorname{supp}(a_h) \subset I_h = [r - 2^{-h}r, r - 2^{-h-2}r]$, the class A_h is an atom in $\mathcal{H}_\alpha^{p,+}(\omega)$ with associated interval I_h.

We claim that

$$A = C \sum_{h \geq -1} A_h \quad \text{in } E_N.$$

In order to prove this, first we will see that $C \sum_{h \geq -1} A_h$ converges in $\mathcal{H}_\alpha^{p,+}(\omega)$. In fact, by (3.1), we have

$$\|A_h\|^p_{\mathcal{H}_\alpha^{p,+}(\omega)} \leq C_\omega \omega(I_h),$$

where C_ω does not depend on h. Then, using item (4) in Lemma 3.5, we have that

$$\sum_{h \geq -1} \|A_h\|^p_{\mathcal{H}_\alpha^{p,+}(\omega)} \leq C_\omega \sum_{h \geq -1} \omega(I_h) < \infty,$$

and, since $\mathcal{H}_\alpha^{p,+}(\omega)$ is complete (see Lemma 2.1 item (6)), we have that $C \sum_{h \geq -1} A_h$ converges in $\mathcal{H}_\alpha^{p,+}(\omega)$ to some F. Also, by Lemma 2.1 item (5), the series converges to F in E_N.

Next we shall see that $F = A$. If we prove that

$$\sum_{h \geq -1} N_\alpha^+(\Theta_h; x) < \infty, \quad \text{for almost every } x \in (x_{-\infty}, \infty), \tag{3.4}$$

then, taking into account that $N_\alpha^+(\Theta_h; r) = n_\alpha^+(\theta_h(y); r) = 0$, we have by Lemma 2.1 part (4) (ii) that the series $\sum_{h \geq -1} \theta_h$ converges in $L^2_{\mathrm{loc}}(x_{-\infty}, \infty)$ to some $f \in F$. But, by (3.2), $f(x) = a(x)$, which implies that $F = A$.

Finally we will prove (3.4). For $x \geq r$, $\sum_{h \geq -1} N_\alpha^+(\theta_h; x) = 0$. Let $x \in (x_{-\infty}, r)$. We choose h_0 such that x is at the left of I_{h_0}, that is $x < x_{h_0} = r - 2^{-h_0}r$. For each $h > h_0 + 1$, we shall estimate

$$\frac{1}{\rho^{2\alpha+1}} \int_x^{x+\rho} |\theta_h(y)|^2 \, dy.$$

We consider $\rho > |I_{h_0}| = \frac{3}{4}2^{-h_0}r$, otherwise the integral above vanishes. Denoting by $P(x_h,\cdot)$ the N-th order Taylor's polynomial of a at x_h, we can estimate

$$
\begin{aligned}
&\left(\frac{1}{\rho}\right)^{\alpha+\frac{1}{2}}\left(\int_x^{x+\rho}|\theta_h(y)|^2dy\right)^{\frac{1}{2}}\\
&\quad\le\left(4\frac{2^{h_0}}{3r}\right)^{\alpha+\frac{1}{2}}\left(\int_{x_h}^{x_h+|I_h|}|a(y)|^2dy\right)^{\frac{1}{2}}\\
&\quad\le\left(\frac{2^{h_0}}{2^h}\right)^{\alpha+\frac{1}{2}}\left(\frac{1}{|I_h|}\right)^{\alpha+\frac{1}{2}}\left(\int_{x_h}^{x_h+|I_h|}|a(y)-P(x_h,y)|^2dy\right)^{\frac{1}{2}}\\
&\qquad+\left(\frac{C_{h_0}}{r}\right)^{\alpha+\frac{1}{2}}\left(\int_{x_h}^{x_h+|I_h|}|P(x_h,y)|^2dy\right)^{\frac{1}{2}}\\
&\quad=(I)+(II).
\end{aligned}
$$

By Remark 2.2 and Definition 3.1, we have

$$
(I)\le\left(\frac{2^{h_0}}{2^h}\right)^{\alpha+\frac{1}{2}}n_\alpha^+(a-P(x_h,\cdot);x_h)=\left(\frac{2^{h_0}}{2^h}\right)^{\alpha+\frac{1}{2}}N_\alpha^+(A;x_h)\le\frac{C_{h_0,\alpha}}{2^{h(\alpha+\frac{1}{2})}}. \tag{3.5}
$$

Let $k=0$, $f_1=a(\cdot)-P(x_h,\cdot)$, $f_2=a(\cdot)-P(r,\cdot)$, $x_1=x_h$ and $x_2=r$. Then by Lemma 2.1 part (1) we have

$$
\begin{aligned}
|P(x_h,y)|&=|P(x_h,y)-P(r,y)|\\
&\le C(n_\alpha^+(a-P(x_h,\cdot);x_h)+n_\alpha^+(a-P(r,\cdot);r))(|x_h-y|+|r-y|)^\alpha\\
&\le C(2^{-h}r)^\alpha.
\end{aligned}
$$

Then we can estimate,

$$
(II)\le C\left(\frac{C_{h_0}}{r}\right)^{\alpha+\frac{1}{2}}\left(\int_{x_h}^{x_h+|I_h|}(2^{-h}r)^{2\alpha}dy\right)^{\frac{1}{2}}\le\frac{C_{h_0,\alpha}}{2^{h(\alpha+\frac{1}{2})}}. \tag{3.6}
$$

From (3.5) and (3.6) we obtain

$$
\left(\frac{1}{\rho}\right)^{\alpha+\frac{1}{2}}\left(\int_x^{x+\rho}|\theta_h(y)|^2dy\right)^{\frac{1}{2}}\le C_{h_0,\alpha}2^{-h(\alpha+\frac{1}{2})}.
$$

Thus, taking supremum over $\rho>0$ and using Lemma 2.1 item (2), we get

$$
N_\alpha^+(\Theta_h;x)=n_\alpha^+(\theta_h;x)\le C_{h_0,\alpha}2^{-h(\alpha+\frac{1}{2})},
$$

for $h>h_0+1$. Finally, using this estimation and taking into account that $\alpha+\frac{1}{2}>0$, we obtain (3.4) as follows:

$$
\begin{aligned}
\sum_{h\ge-1}N_\alpha^+(\Theta_h;x)&=\sum_{h=-1}^{h_0+1}N_\alpha^+(\Theta_h;x)+\sum_{h>h_0+1}N_\alpha^+(\Theta_h;x)\\
&\le D_{h_0,\alpha}+C_{h_0,\alpha}\sum_{h>h_0+1}2^{-h(\alpha+\frac{1}{2})}<\infty.
\end{aligned} \tag{3.7}
$$

□

Remark 3.7. From the proof of Theorem 3.6, and with the same notation, it follows that $C\sum_{h\geq -1} a_h$ is a representative of $C\sum_{h\geq -1} A_h = A$.

From Theorem 3.4 and Theorem 3.6 we have the following atomic decomposition in $\mathcal{H}^{p,+}_\alpha(\omega)$.

Theorem 3.8. *Let $\omega \in A^+_s$ and $0 < p \leq 1$, such that $(\alpha+1/2)p \geq s > 1$ or $(\alpha+1/2)p > 1$ if $s=1$. If $F \in \mathcal{H}^{p,+}_\alpha(\omega)$ then there exist a sequence $\{\lambda_j\}$ of positive coefficients and a sequence $\{A_{j,h}\}$ of atoms with associated bounded intervals $\{I_{j,h}\}$ respectively, such that $I_{j,h+2}$ follows $I_{j,h}$, $|I_{j,h+2}| \leq |I_{j,h}| \leq 4|I_{j,h+2}|$ and $\sum_{j\geq 1, h\geq -1} \lambda_j A_{j,h}$ converges unconditionally to F in E_N and in the $\mathcal{H}^{p,+}_\alpha(\omega)$ norm. Moreover there exist two constants C_1 and C_2, both not depending on F, such that*

$$C_1 \Big\| \sum_{j\geq 1, h\geq -1} \lambda_j \chi_{I_{j,h}} \Big\|_{L^p(\omega)} \leq \|F\|_{\mathcal{H}^{p,+}_\alpha(\omega)} \leq C_2 \Big\| \sum_{j\geq 1, h\geq -1} \lambda_j \chi_{I_{j,h}} \Big\|_{L^p(\omega)}.$$

Also, for $r > 0$,

$$\sum_{j\geq 1, h\geq -1} \lambda_j^r \chi_{I_{j,h}}(x) \leq C[N^+_\alpha(F;x)]^r$$

for almost every $x \in (x_{-\infty}, \infty)$, with C not depending on F.

4. Proof of Theorem 1.2

Proof of the first part of Theorem 1.2. For each $z \in \Omega$, $F(z) \in \mathcal{H}^{p_0,+}_\alpha(\omega) + \mathcal{H}^{p_1,+}_\alpha(\nu)$, then $F(z) = F_0(z) + F_1(z)$, where $F_k(z) \in \mathcal{H}^{p_k,+}_\alpha(\mu(k))$ and so $N^+_\alpha(F_k(z);x) \in L^{p_k}(\mu(k))$ for $k=0,1$. Then $N^+_\alpha(F_k(z);x) < \infty$ for almost every $x \in (x_{-\infty}, \infty)$ with respect to the measure $\mu(k)$, $k=0,1$. Moreover, since $\mu(k)(x) > 0$ for $x \in (x_{-\infty}, \infty)$ and $k=0,1$, $N^+_\alpha(F_k(z);x) < \infty$ almost everywhere with respect to the Lebesgue measure. Also, as $N^+_\alpha(F(z);x) \leq N^+_\alpha(F_0(z);x) + N^+_\alpha(F_1(z);x)$ then $N^+_\alpha(F(z);x) < \infty$ for almost every $x \in (x_{-\infty}, \infty)$. Then, by Lemma 2.1 part (2), there exists a unique $f_x(z,\cdot) \in F(z)$ such that $n^+_\alpha(f_x(z);x) < \infty$ and

$$N^+_\alpha(F(z);x) = n^+_\alpha(f_x(z,\cdot);x) = \sup_{\rho>0} \frac{1}{\rho^\alpha}\left(\frac{1}{\rho}\int_x^{x+\rho} |f_x(z,y)|^2\, dy\right)^{\frac{1}{2}}.$$

Given $0 < \ell < 1$ and $\rho(x)$ a positive and measurable function, we consider

$$\mathcal{F}(x,\rho(x),\ell) = \frac{1}{\rho(x)^{2\alpha+1}} \int_x^{x+\rho(x)} (f_x(\ell,y))^2\, dy.$$

Then applying Corollary 2.4, Hölder's inequality and Fubini's theorem, we have

$$\begin{aligned}\int_{x_{-\infty}}^{\infty} & (\mathcal{F}(x,\rho(x),\ell))^{\frac{p}{2}}\,\mu(x)\,dx \\ &\leq \left(\frac{1}{1-\ell}\int_{-\infty}^{\infty}\left(\int_{x_{-\infty}}^{\infty}|\mathcal{F}(x,\rho(x),it)|^{\frac{p_0}{2}}\,\omega(x)dx\right)P_0(\ell,t)\,dt\right)^{1-\mu} \\ &\quad\times\left(\frac{1}{\ell}\int_{-\infty}^{\infty}\left(\int_{x_{-\infty}}^{\infty}|\mathcal{F}(x,\rho(x),1+it)|^{\frac{p_1}{2}}\,\nu(x)dx\right)P_1(\ell,t)\,dt\right)^{\mu}.\end{aligned} \tag{4.1}$$

If $k=0,1$ we have

$$\begin{aligned}|\mathcal{F}(x,\rho(x),k+it)|^{\frac{1}{2}} &\leq \frac{1}{\rho(x)^{\alpha}}\left(\frac{1}{\rho(x)}\int_{x}^{x+\rho(x)}|f_x((k+it),y)|^2dy\right)^{\frac{1}{2}} \\ &\leq n_{\alpha}^{+}(f_x(k+it);x)=N_{\alpha}^{+}(F(k+it);x).\end{aligned}$$

Taking into account that $\int_{-\infty}^{\infty}P_0(\ell,t)\,dt=1-\ell$, $\int_{-\infty}^{\infty}P_1(\ell,t)\,dt=\ell$, $\mu p_1=\ell p$ and $p_0(1-\mu)=(1-\ell)p$, the last term in inequality (4.1) can be estimated by

$$\begin{aligned}&\left(\frac{1}{1-\ell}\int_{-\infty}^{\infty}\|F(it)\|^{p_0}_{\mathcal{H}^{p_0,+}_{\alpha}(\omega)}P_0(\ell,t)\,dt\right)^{1-\mu}\left(\frac{1}{\ell}\int_{-\infty}^{\infty}\|F(1+it)\|^{p_1}_{\mathcal{H}^{p_1,+}_{\alpha}(\nu)}P_1(\ell,t)\,dt\right)^{\mu} \\ &\qquad\leq\left(\sup_t\|F(it)\|_{\mathcal{H}^{p_0,+}_{\alpha}(\omega)}\right)^{(1-\ell)p}\left(\sup_t\|F(1+it)\|_{\mathcal{H}^{p_1,+}_{\alpha}(\nu)}\right)^{\ell p}.\end{aligned}$$

So, for $0<\ell<1$ and $\rho(x)$ a positive and measurable function, we have proved that

$$\|\,(\mathcal{F}(\cdot,\rho(\cdot),\ell))^{\frac{1}{2}}\|^{p}_{L^p(\mu)}\leq\left(\sup_t\|F(it)\|_{\mathcal{H}^{p_0,+}_{\alpha}(\omega)}\right)^{(1-\ell)p}\left(\sup_t\|F(1+it)\|_{\mathcal{H}^{p_1,+}_{\alpha}(\nu)}\right)^{\ell p}.$$

Then the first part of Theorem 1.2 follows at once from the fact that given $\epsilon>0$, there exists a positive measurable function $\rho(x)$ defined on $(x_{-\infty},\infty)$ such that, for all $0<\ell<1$,

$$\|N_{\alpha}^{+}(F(\ell);\cdot)\|^{p}_{L^p(\mu)}\leq\|\,(\mathcal{F}(\cdot,\rho(\cdot),\ell))^{\frac{1}{2}}\|^{p}_{L^p(\mu)}+\epsilon.$$

Proof of the second part of Theorem 1.2. Let $F\in\mathcal{D}$, that is

$$F=\sum_{j\in J}\lambda_jA_j,$$

where J is a finite set, $\lambda_j\geq 0$, and A_j are atoms. Let $a_j\in A_j$ and let I_j be a bounded interval such that $\operatorname{supp}(a_j)\subset I_j$ as in Definition 3.1. We decompose each A_j as in Theorem 3.6 to obtain a decomposition of F as in Theorem 3.8, that is

$$F=C\sum_{j\in J}\lambda_j\sum_{h\geq -1}A_{j,h}\quad\text{in}\ \ \mathcal{H}^{p,+}_{\alpha}(\mu),$$

where $A_j=C\sum\limits_{h\geq -1}A_{j,h}$, with C not depending on A_j. Let $a_{j,h}\in A_{j,h}$ and $I_{j,h}$ as in the proof of Theorem 3.6. Thus $\operatorname{supp}(a_{j,h})\subset I_{j,h}\subset I_j$ and for each $x\in I_j$,

x belongs to at most three $I_{j,h}$. Also $I_{j,h+2}$ follows $I_{j,h}$ and $|I_{j,h+2}| \leq |I_{j,h}| \leq 4|I_{j,h+2}|$.

We shall define $F(z)$ in the same fashion as in [18, Theorem 3, Chapter XII]. In order to do that we consider $\displaystyle\sum_{j\in J}\sum_{h\geq -1} \lambda_{j,h}(z)A_{j,h}$, where

$$\lambda_{j,h}(z) = \lambda_j{}^{\frac{p}{p(z)}} \left(\frac{\omega(I_{j,h+2})}{\nu(I_{j,h+2})}\right)^{\frac{(z-\ell)p}{p_0p_1}}.$$

Using the special properties of the atomic decomposition and following the arguments given in [14] to prove an analogous inequality for one-sided Hardy spaces, we can obtain

$$\Big\|\sum_{j\in J}\sum_{h\geq -1} |\lambda_{j,h}(u+it)|\chi_{I_{j,h}}\Big\|^{p(u)}_{L^{p(u)}(\mu(u))} \leq C\|F\|^p_{\mathcal{H}^{p,+}_\alpha(\mu)}. \tag{4.2}$$

Then, by Theorem 3.3, $\displaystyle\sum_{j\in J}\sum_{h\geq -1} |\lambda_{j,h}(u+it)|B_{j,h}$ converges in $\mathcal{H}^{p(u),+}_\alpha(\mu(u))$ for all sequences of atoms $\{B_{j,h}\}$ in $\mathcal{H}^{p(u),+}_\alpha(\mu(u))$ with associated bounded intervals $\{I_{j,h}\}$ respectively. In particular, taking $B_{j,h} = e^{i\theta_{j,h}}A_{j,h}$, where $\theta_{j,h} = Arg(\lambda_{j,h})$, we can define

$$F(z) = \sum_{j\in J}\sum_{h\geq -1} \lambda_{j,h}(z)A_{j,h} \quad \text{in } \mathcal{H}^{p(u),+}_\alpha(\mu(u)).$$

Also by Theorem 3.3 we have that

$$\|F(u+it)\|^{p(u)}_{\mathcal{H}^{p(u),+}_\alpha(\mu(u))} \leq C^{p(u)}\Big\|\sum_{j\in J}\sum_{h\geq -1} |\lambda_{j,h}(u+it)|\chi_{I_{j,h}}\Big\|^{p(u)}_{L^{p(u)}(\mu(u))}. \tag{4.3}$$

Finally, combining inequalities (4.2) and (4.3), we get

$$\|F(u+it)\|^{p(u)}_{\mathcal{H}^{p(u),+}_\alpha(\mu(u))} \leq C^{p(u)}\|F\|^p_{\mathcal{H}^{p,+}_\alpha(\mu)}.$$

Next we shall prove that $F(z)$ and $\mathcal{F}(x,\rho,z)$ satisfy the properties stated in Theorem 1.2.

To see that $F(z) \in \mathcal{H}^{p_0,+}_\alpha(\omega) + \mathcal{H}^{p_1,+}_\alpha(\nu)$ we use the fact that at least $|\lambda_{j,h}(z)| \leq \lambda_{j,h}(0)$ or $|\lambda_{j,h}(z)| \leq \lambda_{j,h}(1)$ hold, and write $F(z) = F_0(z) + F_1(z)$, where

$$F_0(z) = \sum_{j\in J}\sum_{\substack{h\geq -1\\ |\lambda_{j,h}(z)|\leq\lambda_{j,h}(0)}} \lambda_{j,h}(z)A_{j,h}.$$

Then we can prove that $F_k(z) \in \mathcal{H}^{p_k,+}_\alpha(\mu(k))$, $k = 0, 1$, as it is done in the proof of the second part of [18, Theorem 3, Chapter XII] for weighted Hardy spaces.

We claim that, for almost every $x \in (x_{-\infty}, \infty)$, there exists a unique $f_x(z,\cdot) \in F(z)$ such that

$$N^+_\alpha(F(z);x) = n^+_\alpha(f_x(z,\cdot);x) < \infty, \tag{4.4}$$

where

$$f_x(z,y) = \sum_{j\in J}\sum_{h\geq -1} \lambda_{j,h}(z)\left(a_{j,h}(y) - P_{j,h}(x,y)\right),$$

with $P_{j,h}(x,\cdot)$ the N-th order Taylor's polynomial of $a_{j,h}$ at x. In fact, by the same argument that led to (3.7), we have

$$\begin{aligned}\sum_{j\in J}\sum_{h\geq -1} N_\alpha^+(\lambda_{j,h}(z)A_{j,h};x) &= \sum_{j\in J}\sum_{h=-1}^{h_0+1} N_\alpha^+(\lambda_{j,h}(z)A_{j,h};x)\\ &\quad+\sum_{j\in J}\sum_{h>h_0+1} N_\alpha^+(\lambda_{j,h}(z)A_{j,h};x)\\ &\leq D_{h_0,\alpha,J}+C_{h_0,\alpha,J}\sum_{j\in J}\sum_{h>h_0+1}|\lambda_{j,h}(z)|2^{-h(\alpha+\frac{1}{2})}.\end{aligned}$$

By Lemma 2.8, we have that $|\lambda_{j,h}(z)|\leq C\lambda^{\frac{p}{p(u)}}2^{h(s-1)\frac{|u-\ell|p}{p_0p_1}}$, and taking into account that $(s-1)\frac{|u-\ell|p}{p_0p_1}-(\alpha+\frac{1}{2})<0$, we see that the last series converges. Also, by Remark 2.2, we have that $\lambda_{j,h}(z)\,(a_{j,h}(y)-P_{j,h}(x,y))$ is the unique representative in the class $\lambda_{j,h}(z)A_{j,h}$ such that $N_\alpha^+(\lambda_{j,h}(z)A_{j,h},x)=n_\alpha^+(\lambda_{j,h}(z)(a_{j,h}(y)-P_{j,h}(x,y),x))$. Then, (4.4) follows from Lemma 2.1 part (4) (ii).

For any non-negative integer M we consider

$$F_M(z)=\sum_{j\in J}\sum_{h=-1}^{M}\lambda_{j,h}(z)A_{j,h},$$

$$f_{x,M}(z,y)=\sum_{j\in J}\sum_{h=-1}^{M}\lambda_{j,h}(z)\,(a_{j,h}(y)-P_{j,h}(x,y))$$

and

$$\mathcal{F}_M(x,\rho,z)=\frac{1}{\rho^{2\alpha+1}}\int_x^{x+\rho}(f_{x,M}(z,y))^2\,dy.$$

It is not difficult to prove that $\mathcal{F}_M(x,\rho,z)$ is a measurable function of x, and that for each (x,ρ) it is a bounded and continuous function in Ω, and analytic in the interior of Ω. To prove that $\mathcal{F}(x,\rho,z)$ has also these properties it will be enough to show that, for every (x,ρ), $\mathcal{F}_M(x,\rho,z)$ converges uniformly to $\mathcal{F}(x,\rho,z)$ in Ω. In order to see that, we decompose $F(z)-F_M(z)=F_0^M(z)+F_1^M(z)$, where

$$F_0^M(z)=\sum_{j\in J}\sum_{\substack{h\geq M+1\\ |\lambda_{j,h}(z)|\leq\lambda_{j,h}(0)}}\lambda_{j,h}(z)A_{j,h},$$

and $F_1^M(z)=F(z)-F_M(z)-F_0^M(z)$. Using inequalities (4.3) and (4.2) with $F(z)-F_M(z)$ instead of $F(z)$, it can be shown that given $\epsilon>0$ there exists M_0 not depending on $z\in\Omega$ such that for $M\geq M_0$,

$$\|F_k^M(z)\|_{\mathcal{H}_\alpha^{p_k,+}(\mu(k))}<\epsilon, \tag{4.5}$$

for $k=0,1$.

Also, if

$$f_{0,x}^M(z,y)=\sum_{j\in J}\sum_{\substack{h\geq M+1\\ |\lambda_{j,h}(z)|\leq\lambda_{j,h}(0)}}\lambda_{j,h}(z)\,(a_{j,h}(y)-P_{j,h}(x,y)),$$

we have that $f_x(z,\cdot)-f_{x,M}(z,\cdot)=f^M_{0,x}(z,\cdot)+f^M_{1,x}(z,\cdot)$, where $f^M_{k,x}(z,\cdot)$ is the unique representative of $F^M_k(z)$ such that $n^+_\alpha(f^M_{k,x}(z,\cdot);x)<\infty$ for $k=0,1$. The proof of the last assertion is analogous to the one given to prove (4.4). Then,

$$\begin{aligned}
&|\mathcal{F}(x,\rho,z)-\mathcal{F}_M(x,\rho,z)|\\
&\qquad\le \frac{1}{\rho^{2\alpha+1}}\int_x^{x+\rho}|f_x(z,y)-f_{x,M}(z,y)|\,|(f_x(z,y))+f_{x,M}(z,y)|\,dy\\
&\qquad\le \frac{1}{\rho^{2\alpha+1}}\left(\int_x^{x+\rho}|f_x(z,y)-f_{x,M}(z,y)|^2dy\right)^{\frac12}\\
&\qquad\quad\times\left(\int_x^{x+\rho}|f_x(z,y)+f_{x,M}(z,y)|^2dy\right)^{\frac12}\\
&\qquad\le\left(\underbrace{\frac{1}{\rho^\alpha}\left(\frac1\rho\int_x^{x+\rho}|f^M_{0,x}(z,y)|^2dy\right)^{\frac12}}_{\mathcal{T}_0}+\underbrace{\frac{1}{\rho^\alpha}\left(\frac1\rho\int_x^{x+\rho}|f^M_{1,x}(z,y)|^2dy\right)^{\frac12}}_{\mathcal{T}_1}\right)\\
&\qquad\quad\times\underbrace{\frac{1}{\rho^\alpha}\left(\frac1\rho\int_x^{x+\rho}|f_x(z,y)+f_{x,M}(z,y)|^2dy\right)^{\frac12}}_{\mathcal{T}_2}
\end{aligned}$$

In order to estimate $\mathcal{T}_0$, we will use how the atoms are supported in the atomic decomposition. For the sake of simplicity we shall assume that F is just one atom with associated interval $I=[0,r]$ and, so, the j subindex shall not be written. Since for $x\ge r$, $\mathcal{T}_0=0$, we consider $x<r$ and choose $\delta>0$ such that $x+\delta<r$ and $M^*=M^*(x,\delta)$ so that for every $h>M^*$ the support of a_h and $[x,x+\delta]$ are disjoint. Note that, since M^* does not depend on ϵ, we can choose it first, and then take M_0 as in (4.5) such that $M_0\ge M^*$. Also note that, for $h>M_0$ and $\xi\in[x,x+\delta]$, $P_h(\xi,y)=0$ and thus

$$f^M_{0,\xi}(z,y)=\sum_{\substack{h\ge M+1\\|\lambda_h(z)|\le\lambda_h(0)}}\lambda_h(z)a_h(y).$$

Then, we have that

$$\mathcal{T}_0\le\frac{1}{\rho^\alpha}\left(\frac1\rho\int_\xi^{\xi+\rho}\left|f^M_{0,\xi}(z,y)\right|^2dy\right)^{\frac12}\le n^+_\alpha(f^M_{0,\xi}(z,\cdot);\xi).$$

Therefore,

$$\left(\int_x^{x+\delta}\mathcal{T}_0^{p_0}\omega(\xi)d\xi\right)^{\frac{1}{p_0}}\le\left(\int_{x-\infty}^{\infty}\left(N^+_\alpha(F^M_0(z);\xi)\right)^{p_0}\omega(\xi)d\xi\right)^{\frac{1}{p_0}}$$

and, by (4.5),

$$\mathcal{T}_0\le\omega([x,x+\delta])^{-1/p_0}\,\epsilon.$$

A similar argument gives

$$\mathcal{T}_1 \le \nu([x, x+\delta])^{-1/p_1}\,\epsilon.$$

Now we shall estimate $\mathcal{T}_2$.

$$\begin{aligned}
\mathcal{T}_2 &= \frac{1}{\rho^\alpha}\Bigg(\frac{1}{\rho}\int_x^{x+\rho} \big|f_{x,M}(z,y) - f_x(z,y) + 2(f_x(z,y) \\
&\qquad - f_{x,M^*}(z,y) + f_{x,M^*}(z,y))\big|^2\,dy\Bigg)^{\frac12} \\
&\le \underbrace{\frac{1}{\rho^\alpha}\left(\frac{1}{\rho}\int_x^{x+\rho} |f_{x,M}(z,y) - f_x(z,y)|^2\,dy\right)^{\frac12}}_{\mathcal{T}_3} \\
&\quad + \underbrace{\frac{2}{\rho^\alpha}\left(\frac{1}{\rho}\int_x^{x+\rho} |f_x(z,y) - f_{x,M^*}(z,y)|^2\,dy\right)^{\frac12}}_{\mathcal{T}_4} \\
&\quad + \underbrace{\frac{2}{\rho^\alpha}\left(\frac{1}{\rho}\int_x^{x+\rho} |f_{x,M^*}(z,y)|^2\,dy\right)^{\frac12}}_{\mathcal{T}_5}.
\end{aligned}$$

We proceed as above to get

$$\mathcal{T}_3 \le \epsilon\,(\omega([x, x+\delta])^{-\frac{1}{p_0}} + \nu([x, x+\delta])^{-\frac{1}{p_1}}).$$

In the same way we can estimate $\mathcal{T}_4$, but instead of using (4.5) we use that $\|F_k^{M^*}(z)\|_{\mathcal{H}_\alpha^{p_k,+}(\mu(k))} < C$, where C does not depend on z, to obtain

$$\mathcal{T}_4 \le C\,(\omega([x, x+\delta])^{-\frac{1}{p_0}} + \nu([x, x+\delta])^{-\frac{1}{p_1}}).$$

Finally, by Lemma 2.8,

$$|\lambda_h(z)| = \left|\lambda^{\frac{p}{p(z)}}\left(\frac{\omega(I_{h+2})}{\nu(I_{h+2})}\right)^{\frac{(z-\ell)p}{p_0 p_1}}\right| \le C_{p,s,\tau,I}\,\lambda^{\frac{p}{p(u)}} 2^{h(s-1)\frac{|u-\ell|p}{p_0 p_1}} \le C_{h,s,p,\tau,I},$$

and thus,

$$\mathcal{T}_5 \le \left(\frac{2}{\rho^{2\alpha+1}}\int_x^{x+\rho}\left(\sum_{h=-1}^{M^*} C_{h,s,p,\tau,I}\big|(a_h(y) - P_h(x,y))\big|\right)^2 dy\right)^{\frac12} \le C,$$

where C does not depend on z.

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