REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 58, No. 1, 2017, Pages 37–52
Published online: October 31, 2016

AFFINE SZABÓ CONNECTIONS ON SMOOTH MANIFOLDS

ABDOUL SALAM DIALLO AND FORTUNÉ MASSAMBA

Abstract. We introduce a new structure, called affine Szabó connection. We prove that, on 2-dimensional affine manifolds, the affine Szabó structure is equivalent to one of the cyclic parallelisms of the Ricci tensor. A characterization for locally homogeneous affine Szabó surfaces is obtained. Examples of two- and three-dimensional affine Szabó manifolds are also given.

1. Introduction

The theory of connections is a classical topic in differential geometry. Initially developed to solve pure geometrical problems, it provides an extremely important tool to study geometrical structures on manifolds and, as such, has been applied with great success in many different settings. For affine connections, a survey of the development of the theory can be found in [13] and references therein. In [14], Opozda classified locally homogeneous torsion-free affine connections on 2-manifolds. Arias-Marco and Kowalski in [1] classified locally homogeneous connections with arbitrary torsion on 2-manifolds. In [7], García-Río et al. introduced the notion of affine Osserman connections. Affine Osserman connections are well-understood in dimension two. For instance, in [5] and [7], the authors proved in a different way that an affine connection is Osserman if and only if its Ricci tensor is skew-symmetric. The situation is however more involved in higher dimensions where the skew-symmetry of the Ricci tensor is a necessary (but not a sufficient) condition for an affine connection to be Osserman.

A (pseudo-) Riemannian manifold (M,g) is said to be *Szabó* if the eigenvalues of the Szabó operator given by

$$\mathcal{S}(X) : Y \to (\nabla_X \mathcal{R})(Y,X)X$$

are constant on the unit (pseudo-) sphere bundle [9]. The Szabó operator is a self-adjoint operator with $\mathcal{S}(X)X = 0$. It plays an important role in the study of totally isotropic manifolds [10]. Szabó in [18] used techniques from algebraic topology to show, in the Riemannian setting, that any such metric is locally symmetric. He used this observation to give a simple proof that any two point homogeneous space is either flat or is a rank one symmetric space. Subsequently Gilkey and

---

2010 *Mathematics Subject Classification.* Primary 53B05; Secondary 53B20.
Key words and phrases. Affine connection; Szabó connection; Cyclic parallel Ricci tensor.
A. S. Diallo would like to thank the University of KwaZulu-Natal for financial support.

Stavrov [11] extended his results to show that any Szabó Lorentzian manifold has constant sectional curvature. However, for metrics of higher signature the situation is different. Indeed in [10] it was shown the existence of Szabó pseudo-Riemannian manifolds with metrics of signature (p, q) with $p \geq 2$ and $q \geq 2$ which are not locally symmetric.

The aim of this paper is to extend the definition of (pseudo-) Riemannian Szabó manifold to the affine case by introducing a new concept called *affine Szabó manifold*. We investigate the torsion-free affine connections to be Szabó. We shall call a connection with such a condition an *affine Szabó connection*.

The paper is organized as follows. In Section 2, we recall some basic definitions and geometric objects, namely, torsion, curvature and Ricci tensor on an affine manifold. In Section 3, we study the cyclic parallelism of the Ricci tensor for a particular case of affine connections in two and three dimensional affine manifolds. We establish geometric configurations of affine manifolds admitting a cyclic parallel Ricci tensor (Propositions 3.2 and 3.3). We introduce in Section 4 a new concept of affine Szabó manifold. We prove that, on a two-dimensional smooth affine manifold, the affine Szabó structure coincides with the cyclic parallelism of the Ricci tensor (Theorem 4.4). We end the section by giving some examples of affine Szabó connections in two and three dimensional affine manifolds. In Section 5, a characterization of locally homogeneous affine Szabó surfaces is given. We investigate in Section 6 the twisted Riemannian extension of an affine Szabó connection on a two-dimensional affine manifold (M, ∇). We show that the twisted Riemannian extension of an affine Szabó manifold is a pseudo-Riemannian nilpotent Szabó manifold of neutral signature and the degree of nilpotency of the Szabó operators depends on the direction of the unit vectors.

2. Preliminaries

Let M be an n-dimensional smooth manifold and ∇ be an affine connection on M. Let us consider a system of coordinates $(u_1, u_2, \ldots, u_n)$ in a neighborhood $\mathcal{U}$ of a point p in M. In $\mathcal{U}$, the connection is given by

$$\nabla_{\partial_i}\partial_j = f_{ij}^k \partial_k, \tag{2.1}$$

where $\{\partial_i = \frac{\partial}{\partial u_i}\}_{1 \leq i \leq n}$ is a basis of the tangent space T_pM and the functions $f_{ij}^k (i, j, k = 1, 2, 3, \ldots, n)$ are called the *coefficients* of the affine connection. The pair (M, ∇) shall be called *affine manifold*.

Next, we define a few tensor fields associated with a given affine connection ∇. The *torsion tensor field* T^∇ is defined by

$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

for any vector fields X and Y on M. The components of the torsion tensor T^∇ in local coordinates are

$$T_{ij}^k = f_{ij}^k - f_{ji}^k.$$

If the torsion tensor of a given affine connection ∇ vanishes, we say that ∇ is torsion-free.

The *curvature tensor field* $\mathcal{R}^\nabla$ is defined by

$$\mathcal{R}^\nabla(X,Y)Z := \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

for any vector fields X, Y and Z on M. The components in local coordinates are

$$\mathcal{R}^\nabla(\partial_k,\partial_l)\partial_j = \sum_i R^i_{jkl}\partial_i.$$

We shall assume that ∇ is torsion-free. If $\mathcal{R}^\nabla = 0$ on M, we say that ∇ is a *flat affine connection*. It is know that ∇ is flat if and only if around each point there exists a local coordinate system such that $f^k_{ij} = 0$, for all i, j and k.

We define the *Ricci tensor* Ric^∇ by

$$\mathrm{Ric}^\nabla(X,Y) = \mathrm{trace}\{Z \mapsto \mathcal{R}^\nabla(Z,X)Y\}.$$

The components in local coordinates are given by

$$\mathrm{Ric}^\nabla(\partial_j,\partial_k) = \sum_i R^i_{kij}.$$

It is known in Riemannian geometry that the Levi-Civita connection of a Riemannian metric has symmetric Ricci tensor, that is, $\mathrm{Ric}(Y,Z) = \mathrm{Ric}(Z,Y)$. But this property is not true for an arbitrary affine connection with torsion-free. In fact, the property is closely related to the concept of parallel volume element (see [13] for more details).

In a 2-dimensional manifold, the curvature tensor $\mathcal{R}^\nabla$ and the Ricci tensor Ric^∇ are related by

$$\mathcal{R}^\nabla(X,Y)Z = \mathrm{Ric}^\nabla(Y,Z)X - \mathrm{Ric}^\nabla(X,Z)Y. \tag{2.2}$$

The covariant derivative of the curvature tensor $\mathcal{R}^\nabla$ is given by

$$(\nabla_X\mathcal{R}^\nabla)(Y,Z)W = (\nabla_X\mathrm{Ric}^\nabla)(Z,W)Y - (\nabla_X\mathrm{Ric}^\nabla)(Y,W)Z,$$

where the covariant derivative of the Ricci tensor Ric^∇ is defined as

$$(\nabla_X\mathrm{Ric}^\nabla)(Z,W) = X(\mathrm{Ric}^\nabla(Z,W)) - \mathrm{Ric}^\nabla(\nabla_X Z,W) - \mathrm{Ric}^\nabla(Z,\nabla_X W).$$

For $X \in \Gamma(T_pM)$, we define the affine Szabó operator $\mathcal{S}^\nabla(X)$ with respect to X by $\mathcal{S}^\nabla(X) : T_pM \longrightarrow T_pM$ such that

$$\mathcal{S}^\nabla(X)Y = (\nabla_X\mathcal{R}^\nabla)(Y,X)X,$$

for any vector field Y. The affine Szabó operator satisfies $\mathcal{S}^\nabla(X)X = 0$ and $\mathcal{S}^\nabla(\beta X) = \beta^3\mathcal{S}^\nabla(X)$, for $\beta \in \mathbb{R}\setminus\{0\}$ and $X \in T_pM$. If $Y = \partial_m$, for $m = 1, 2, \dots, n$ and $X = \sum_i \alpha_i\partial_i$, one gets

$$\mathcal{S}^\nabla(X)\partial_m = \sum_{i,j,k=1}^{n} \alpha_i\alpha_j\alpha_k(\nabla_i\mathcal{R}^\nabla)(\partial_m,\partial_j)\partial_k,$$

where $\nabla_i = \nabla_{\partial_i}$.

Let $A = (a_{ij})$ be the $(n\times n)$-matrix associated with the affine Szabó operator $\mathcal{S}^\nabla(X)$. Then its characteristic polynomial $P_\lambda[\mathcal{S}^\nabla(X)]$ is given by

$$P_\lambda[\mathcal{S}^\nabla(X)] = \lambda^n - \sigma_1\lambda^{n-1} + \sigma_2\lambda^{n-2} - \cdots + (-1)^n\sigma_n,$$

where the coefficients $\sigma_1, \ldots, \sigma_n$ are given by

$$\sigma_1 = \sum_{i=1}^{n} a_{ii} = \operatorname{trace} A,$$

$$\sigma_2 = \sum_{i<j} \begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix},$$

$$\sigma_3 = \sum_{i<j<k} \begin{vmatrix} a_{ii} & a_{ij} & a_{ik} \\ a_{ji} & a_{jj} & a_{jk} \\ a_{ki} & a_{kj} & a_{kk} \end{vmatrix},$$

$$\vdots$$

$$\sigma_n = \det A.$$

3. Affine connections with cyclic parallel Ricci tensor

In this section, we investigate affine connections whose Ricci tensors are cyclic parallel. Two cases of dimensions (two and three) of smooth manifolds will be considered with specific affine connections.

We start with the following formal definition.

Definition 3.1 ([12])**.** An affine manifold (M, ∇) is said to be an L_3-space if its Ricci tensor Ric^∇ is cyclic parallel, that is

$$(\nabla_X \mathrm{Ric}^\nabla)(X, X) = 0, \tag{3.1}$$

for any vector field X tangent to M or, equivalently, if

$$\mathfrak{S}_{X,Y,Z}(\nabla_X \mathrm{Ric})(Y, Z) = 0,$$

for any vector fields X, Y, Z tangent to M, where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum with respect to X, Y and Z.

Locally, the equation (3.1) takes the form

$$(\nabla_{(i}\mathrm{Ric}^\nabla)_{jk)} = 0,$$

or, written out without the symmetrizing brackets,

$$(\nabla_i \mathrm{Ric}^\nabla)_{jk} + (\nabla_j \mathrm{Ric}^\nabla)_{ki} + (\nabla_k \mathrm{Ric}^\nabla)_{ij} = 0.$$

For $X = \sum_i \alpha_i \partial_i$, it is easy to show that

$$(\nabla_X \mathrm{Ric}^\nabla)(X, X) = \sum_{i,j,k} \alpha_i \alpha_j \alpha_k (\nabla_i \mathrm{Ric}^\nabla)_{jk}. \tag{3.2}$$

Now, we are going to present the two cases of affine connections in which we investigate the cyclic parallelism of the Ricci tensor.

Case 1: Let M be a two-dimensional smooth manifold and ∇ be an affine torsion-free connection. By (2.1), we have

$$\nabla_{\partial_i} \partial_j = f_{ij}^k \partial_k, \quad \text{for } i, j, k = 1, 2, \tag{3.3}$$

where $f_{ij}^k = f_{ij}^k(u_1, u_2)$. The components of the curvature tensor $\mathcal{R}^\nabla$ are given by

$$\mathcal{R}^\nabla(\partial_1, \partial_2)\partial_1 = a\partial_1 + b\partial_2, \qquad \mathcal{R}^\nabla(\partial_1, \partial_2)\partial_2 = c\partial_1 + d\partial_2,$$

where a, b, c and d are given by

$$\begin{aligned} a &= \partial_1 f_{12}^1 - \partial_2 f_{11}^1 + f_{12}^1 f_{12}^2 - f_{11}^2 f_{22}^1, \\ b &= \partial_1 f_{12}^2 - \partial_2 f_{11}^2 + f_{11}^2 f_{12}^1 + f_{12}^2 f_{12}^2 - f_{11}^1 f_{12}^2 - f_{11}^2 f_{22}^2, \\ c &= \partial_1 f_{22}^1 - \partial_2 f_{12}^1 + f_{11}^1 f_{22}^1 + f_{12}^1 f_{22}^2 - f_{12}^1 f_{12}^1 - f_{12}^2 f_{22}^1, \\ d &= \partial_1 f_{22}^2 - \partial_2 f_{12}^2 + f_{11}^2 f_{22}^1 - f_{12}^1 f_{12}^2. \end{aligned} \tag{3.4}$$

From (2.2), the components of the Ricci tensor are given by

$$\begin{aligned} \mathrm{Ric}^\nabla(\partial_1, \partial_1) &= -b, \qquad & \mathrm{Ric}^\nabla(\partial_1, \partial_2) &= -d, \\ \mathrm{Ric}^\nabla(\partial_2, \partial_1) &= a, & \mathrm{Ric}^\nabla(\partial_2, \partial_2) &= c. \end{aligned}$$

Proposition 3.2. *The affine connection ∇ defined in* (3.3) *satisfies* (3.1) *if the functions f_{ij}^k, for $i, j, k = 1, 2$, satisfy the following partial differential equations:*

$$\begin{aligned} &\partial_1 b - 2b f_{11}^1 - (d-a) f_{11}^2 = 0, \\ &\partial_2 c - 2c f_{22}^2 + (d-a) f_{22}^1 = 0, \\ &\partial_1 a - \partial_2 b - \partial_1 d + 4b f_{12}^1 - 2c f_{11}^2 + (d-a)(f_{11}^1 + 2f_{12}^2) = 0, \\ &\partial_2 a + \partial_1 c - \partial_2 d + 2b f_{22}^1 - 4c f_{12}^2 + (d-a)(2f_{12}^1 + f_{22}^2) = 0. \end{aligned}$$

Proof. Using (3.1) and (3.2), one obtains

$$\begin{aligned} 0 = {} & \alpha_1^3 (\nabla_{\partial_1} \mathrm{Ric}^\nabla)(\partial_1, \partial_1) + \alpha_2^3 (\nabla_{\partial_2} \mathrm{Ric}^\nabla)(\partial_2, \partial_2) \\ & + \alpha_1^2 \alpha_2 \Big[(\nabla_{\partial_1} \mathrm{Ric}^\nabla)(\partial_1, \partial_2) + (\nabla_{\partial_1} \mathrm{Ric}^\nabla)(\partial_2, \partial_1) + (\nabla_{\partial_2} \mathrm{Ric}^\nabla)(\partial_1, \partial_1) \Big] \\ & + \alpha_1 \alpha_2^2 \Big[(\nabla_{\partial_1} \mathrm{Ric}^\nabla)(\partial_2, \partial_2) + (\nabla_{\partial_2} \mathrm{Ric}^\nabla)(\partial_1, \partial_2) + (\nabla_{\partial_2} \mathrm{Ric}^\nabla)(\partial_2, \partial_1) \Big]. \end{aligned}$$

From a straightforward calculation using (2), the components of the covariant derivative of the Ricci tensor Ric^∇ are given by

$$\begin{aligned} (\nabla_{\partial_1} \mathrm{Ric}^\nabla)(\partial_1, \partial_1) &= -\partial_1 b + 2b f_{11}^1 + f_{11}^2 (d-a); \\ (\nabla_{\partial_1} \mathrm{Ric}^\nabla)(\partial_1, \partial_2) &= -\partial_1 d + d(f_{11}^1 + f_{12}^2) + b f_{12}^1 - c f_{11}^2; \\ (\nabla_{\partial_1} \mathrm{Ric}^\nabla)(\partial_2, \partial_1) &= \partial_1 a - a(f_{11}^1 + f_{12}^2) + b f_{12}^1 - c f_{11}^2; \\ (\nabla_{\partial_1} \mathrm{Ric}^\nabla)(\partial_2, \partial_2) &= \partial_1 c + (d-a) f_{12}^1 - 2c f_{12}^2; \\ (\nabla_{\partial_2} \mathrm{Ric}^\nabla)(\partial_1, \partial_1) &= -\partial_2 b + 2b f_{12}^1 + f_{12}^2 (d-a); \\ (\nabla_{\partial_2} \mathrm{Ric}^\nabla)(\partial_1, \partial_2) &= -\partial_2 d + d(f_{12}^1 + f_{22}^2) + b f_{22}^1 - c f_{12}^2; \\ (\nabla_{\partial_2} \mathrm{Ric}^\nabla)(\partial_2, \partial_1) &= \partial_2 a - a(f_{12}^1 + f_{22}^2) + b f_{22}^1 - c f_{12}^2; \\ (\nabla_{\partial_2} \mathrm{Ric}^\nabla)(\partial_2, \partial_2) &= \partial_2 c + (d-a) f_{22}^2 - 2c f_{22}^2. \end{aligned}$$

This completes the proof. ☐

Case 2: Let M be a three-dimensional smooth manifold and ∇ be an affine torsion-free connection. Suppose that the action of the affine connection ∇ on the basis of the tangent space $\{\partial_i\}_{1\leq i\leq 3}$ is given by

$$\nabla_{\partial_i}\partial_i = f_i\partial_i, \quad \text{for } i = 1, 2, 3, \tag{3.5}$$

where $f_i = f_i(u_1, u_2, u_3)$ are smooth functions. Then the non-zero components of the curvature tensor $\mathcal{R}^\nabla$ of the connection ∇ defined in (3.5) are given by

$$\mathcal{R}^\nabla(\partial_i, \partial_j)\partial_i = -\partial_j f_i \partial_i \quad \text{and} \quad \mathcal{R}^\nabla(\partial_i, \partial_j)\partial_j = \partial_i f_j \partial_j,$$

for $i \neq j$, $i, j = 1, 2, 3$. The non-zero components of the Ricci tensor of the connection (3.5) are given by

$$\mathrm{Ric}^\nabla(\partial_i, \partial_j) = -\partial_i f_j, \quad \text{for } i \neq j,\ i, j = 1, 2, 3. \tag{3.6}$$

The non-zero components of the covariant derivative of the Ricci tensor are given by

$$\begin{aligned}
(\nabla_{\partial_i}\mathrm{Ric}^\nabla)(\partial_j, \partial_k) &= -\partial_i\partial_j f_k,\\
(\nabla_{\partial_i}\mathrm{Ric}^\nabla)(\partial_i, \partial_j) &= -\partial_i^2 f_j + f_i\partial_i f_j,\\
(\nabla_{\partial_i}\mathrm{Ric}^\nabla)(\partial_j, \partial_i) &= -\partial_i\partial_j f_i + f_i\partial_j f_i,
\end{aligned}$$

for $i \neq j \neq k$, $i, j, k = 1, 2, 3$. In this case, the relation (3.2) is explicitly given by

$$\begin{aligned}
(\nabla_X\mathrm{Ric}^\nabla)(X, X) &= \alpha_1^2\alpha_2\{-\partial_1^2 f_2 + f_1\partial_1 f_2 - \partial_1\partial_2 f_1 + f_1\partial_2 f_1\}\\
&+ \alpha_1^2\alpha_3\{-\partial_1^2 f_3 + f_1\partial_1 f_3 - \partial_1\partial_3 f_1 + f_1\partial_3 f_1\}\\
&+ \alpha_1\alpha_2^2\{-\partial_2^2 f_1 - \partial_2\partial_1 f_2 + f_2\partial_1 f_2 + f_2\partial_2 f_1\}\\
&+ \alpha_1\alpha_3^2\{-\partial_3^2 f_1 + f_3\partial_3 f_1 - \partial_3\partial_1 f_3 + f_3\partial_1 f_3\}\\
&+ \alpha_2^2\alpha_3\{-\partial_2^2 f_3 + f_2\partial_2 f_3 - \partial_2\partial_3 f_2 + f_2\partial_3 f_2\}\\
&+ \alpha_2\alpha_3^2\{-\partial_3^2 f_2 + f_3\partial_3 f_2 - \partial_3\partial_2 f_3 + f_3\partial_2 f_3\}\\
&- 2\alpha_1\alpha_2\alpha_3\{\partial_1\partial_2 f_3 + \partial_1\partial_3 f_2 + \partial_2\partial_3 f_1\},
\end{aligned}$$

for $X = \sum_{i=1}^{3}\alpha_i\partial_i$. Therefore, we have the following result.

Proposition 3.3. *The affine connection ∇ defined in* (3.5) *satisfies* (3.1) *if the functions f_i, for $i = 1, 2, 3$, satisfy the following partial differential equations:*

$$\partial_{(i}\partial_j f_{k)} = 0 \quad \textit{and} \quad \partial_i^2 f_j + \partial_i\partial_j f_i - f_i(\partial_i f_j + \partial_j f_i) = 0,$$

for $i \neq j \neq k$, $i, j, k = 1, 2, 3$.

The manifolds with cyclic parallel Ricci tensor, known as L_3-spaces, are well-developed in Riemannian geometry (see [12] and [16], and references therein). The cyclic parallelism of the Ricci tensor is sometimes called "the first Ledger condition" [16]. In [19], for instance, the author proved that a smooth Riemannian manifold satisfying the first Ledger condition is real analytic. Tod in [20] used the same condition to characterize the four-dimensional Kähler manifolds which are not Einstein.

4. The affine Szabó manifolds

In this section we adapt the definition of pseudo-Riemannian Szabó manifold given by Fiedler and Gilkey in [6] to the affine case. We shall prove that, on a smooth affine surface, the affine Szabó condition is closely related to the cyclic parallelism of the Ricci tensor.

Definition 4.1. Let (M, ∇) be a smooth affine manifold and $p \in M$.

(i) (M, ∇) is called affine Szabó at $p \in M$ if the affine Szabó operator $\mathcal{S}^\nabla(X)$ has the same characteristic polynomial for every vector field X on M.
(ii) (M, ∇) is called affine Szabó if (M, ∇) is affine Szabó at each $p \in M$.

Theorem 4.2. *Let (M, ∇) be an n-dimensional affine manifold and $p \in M$. Then (M, ∇) is affine Szabó at $p \in M$ if and only if the characteristic polynomial of the affine Szabó operator $\mathcal{S}^\nabla(X)$ is $P_\lambda(\mathcal{S}^\nabla(X)) = \lambda^n$, for every $X \in T_pM$.*

Proof. If the characteristic polynomial of the affine Szabó operator at p is given by $P_\lambda(\mathcal{S}^\nabla(X)) = \lambda^n$, then the affine manifold (M, ∇) is obviously affine Szabó. Assume that (M, ∇) is affine Szabó, then for $X \in T_pM$, the characteristic polynomial of the affine Szabó operator $\mathcal{S}^\nabla(X)$ is given by $P_\lambda[\mathcal{S}^\nabla(X)] = \lambda^n - \sigma_1\lambda^{n-1} + \sigma_2\lambda^{n-2} - \cdots + (-1)^n\sigma_n$. Then for $\beta \in \mathbb{R}$, $\beta \neq 0$, the characteristic polynomial of the affine Szabó operator $\mathcal{S}^\nabla(\beta X)$ is given by $P_\lambda[\mathcal{S}^\nabla(\beta X)] = \lambda^n - \sigma_1\beta^3\lambda^{n-1} + \sigma_2\lambda^{n-2} - \cdots + (-1)^n\beta^{3n}\sigma_n$. Since (M, ∇) is affine Szabó, that is $P_\lambda[\mathcal{S}^\nabla(X)] = P_\lambda[\mathcal{S}^\nabla(\beta X)]$, it follows that $\sigma_1 = \cdots = \sigma_n = 0$, which completes the proof. □

Corollary 4.3. *If (M, ∇) is affine Szabó at $p \in M$, then the Ricci tensor of (M, ∇) is cyclic parallel.*

Now, we give a complete description of affine Szabó surfaces. We shall prove the following result.

Theorem 4.4. *Let (M, ∇) be a two-dimensional smooth affine manifold. Then (M, ∇) is affine Szabó at $p \in M$ if and only if the Ricci tensor of (M, ∇) is cyclic parallel at $p \in M$.*

Proof. Suppose that (M, ∇) is affine Szabó. Let $X = \alpha_i\partial_i$, $i = 1, 2$, be a vector on M, then, using the connection (3.3), the affine Szabó operator is given by

$$(\nabla_X\mathcal{R}^\nabla)(\partial_1, X)X = A\partial_1 + B\partial_2, \qquad (\nabla_X\mathcal{R}^\nabla)(\partial_2, X)X = C\partial_1 + D\partial_2,$$

where the coefficients A, B, C and D are given by

$$
\begin{aligned}
A &= \alpha_1^2\alpha_2[\partial_1 a - a(f_{11}^1 + f_{12}^2) + bf_{12}^1 - cf_{11}^2] \\
&\quad + \alpha_1\alpha_2^2[\partial_2 a + \partial_1 c - a(f_{12}^1 + f_{22}^2) + (d-a)f_{12}^1 + bf_{22}^1 - 3cf_{12}^2] \\
&\quad + \alpha_2^3[\partial_2 c - 2cf_{22}^2 + (d-a)f_{22}^1], \\
B &= \alpha_1^2\alpha_2[\partial_1 b - 2bf_{11}^1 - (d-a)f_{11}^2] \\
&\quad + \alpha_1\alpha_2^2[\partial_2 b + \partial_1 d - 3bf_{12}^1 + cf_{11}^2 - (d-a)f_{12}^2 - d(f_{11}^1 + f_{12}^2)] \\
&\quad + \alpha_2^3[\partial_2 d - bf_{22}^1 + cf_{12}^2 - d(f_{12}^1 + f_{22}^2)], \\
C &= \alpha_1^3[-\partial_1 a + a(f_{11}^1 + f_{12}^2) - bf_{12}^1] \\
&\quad + \alpha_1^2\alpha_2[-\partial_2 a - \partial_1 c + a(f_{12}^1 + f_{22}^2) - bf_{22}^1 + 3cf_{12}^2 - (d-a)f_{12}^1] \\
&\quad + \alpha_1\alpha_2^2[-\partial_2 c + 2cf_{22}^2 - (d-a)f_{22}^1], \\
D &= \alpha_1^3[-\partial_1 b + 2bf_{11}^1 + (d-a)f_{11}^2] \\
&\quad + \alpha_1^2\alpha_2[-\partial_2 b - \partial_1 d + 3bf_{12}^1 - cf_{11}^2 + d(f_{11}^1 + f_{12}^2) + (d-a)f_{12}^2] \\
&\quad + \alpha_1\alpha_2^2[-\partial_2 d + bf_{22}^1 - cf_{12}^2 + d(f_{12}^1 + f_{22}^2)].
\end{aligned}
$$

The matrix associated with $\mathcal{S}^\nabla(X)$ with respect to the basis $\{\partial_1, \partial_2\}$ is given by

$$(\mathcal{S}^\nabla(X)) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Its characteristic polynomial is given by $P_\lambda[\mathcal{S}^\nabla(X)] = \lambda^2 - \lambda(A+D) + (AD - BC)$. Since (M, ∇) is affine Szabó, by Theorem 4.2, 0 is the only eigenvalue of the affine Szabó operator $\mathcal{S}^\nabla(X)$. Therefore, $\det(\mathcal{S}^\nabla(X)) = AD - BC = 0$ and $\operatorname{trace}(\mathcal{S}^\nabla(X)) = A + D = 0$. The latter implies that

$$
\begin{aligned}
&\partial_2 c - 2cf_{22}^2 + (d-a)f_{22}^1 = 0, \\
&-\partial_1 b + 2bf_{11}^1 + (d-a)f_{11}^2 = 0, \\
&\partial_1 a - \partial_2 b - \partial_1 d + 4bf_{12}^1 - 2cf_{11}^2 + (d-a)(f_{11}^1 + 2f_{12}^2) = 0, \\
&\partial_2 a + \partial_1 c - \partial_2 d + 2bf_{22}^1 - 4cf_{12}^2 + (d-a)(2f_{12}^1 + f_{22}^2) = 0.
\end{aligned}
$$

The converse is obvious. □

Corollary 4.5. *Let ∇ be the affine connection on $\mathbb{R}^2$ defined by $\nabla_{\partial_1}\partial_1 = f_{11}^1\partial_1$, $\nabla_{\partial_1}\partial_2 = 0$, $\nabla_{\partial_2}\partial_2 = f_{22}^2\partial_2$. Then ∇ is affine Szabó if and only if the functions $f_{11}^1 = f_{11}^1(u_1, u_2)$ and $f_{22}^2 = f_{22}^2(u_1, u_2)$ satisfy the partial differential equations $\partial_1 a - \partial_1 d + (d-a)f_{11}^1 = 0$ and $\partial_2 a - \partial_2 d + (d-a)f_{22}^2 = 0$, where a and d are defined in* (3.4).

To support this, we have the following example. Consider on $\mathbb{R}^2$ the torsion-free connection ∇ with the only non-zero coefficient functions given by

$$\nabla_{\partial_1}\partial_1 = (u_1 + u_2)\partial_1 \quad \text{and} \quad \nabla_{\partial_2}\partial_2 = (u_1 + u_2 + 1)\partial_2.$$

It is easy to check that $(\mathbb{R}^2, \nabla)$ is an affine Szabó manifold.

Corollary 4.6. *Let ∇ be the affine connection on $\mathbb{R}^2$ defined by $\nabla_{\partial_1}\partial_1 = 0$, $\nabla_{\partial_1}\partial_2 = f_{12}^1\partial_1$, $\nabla_{\partial_2}\partial_2 = f_{22}^1\partial_1$. Then ∇ is affine Szabó if and only if the functions $f_{12}^1 = f_{12}^1(u_1, u_2)$ and $f_{22}^1 = f_{22}^1(u_1, u_2)$ satisfy the partial differential equations $\partial_1 a = 0$, $\partial_2 c - af_{22}^1 = 0$ and $\partial_2 a + \partial_1 c - 2af_{12}^1 = 0$, where a and c are defined in* (3.4).

Let us consider on $\mathbb{R}^2$ the torsion-free connection ∇ with the only non-zero coefficient functions given by $\nabla_{\partial_1}\partial_2 = u_2\partial_1$ and $\nabla_{\partial_2}\partial_2 = u_1(1 + u_2)\partial_1$. It is easy to check that $(\mathbb{R}^2, \nabla)$ is an affine Szabó manifold.

Now we give an example of a family of affine Szabó connections on a 3-dimensional manifold. Let us consider the affine connection defined in (3.5), i.e.,

$$\nabla_{\partial_i}\partial_i = f_i\partial_i, \quad \text{for } i = 1, 2, 3,$$

where $f_i = f_i(u_1, u_2, u_3)$ are smooth functions. For $X = \sum_{i=1}^3 \alpha_i\partial_i$, the affine Szabó operator is given by

$$(\nabla_X \mathcal{R}^\nabla)(\partial_i, X)X = \sum_{j=1}^3 A_{ji}\partial_j,$$

where

$$\begin{aligned}
A_{11} &= \alpha_1^2\alpha_2(-\partial_1\partial_2 f_1 + f_1\partial_2 f_1) + \alpha_1^2\alpha_3(-\partial_1\partial_3 f_1 + f_1\partial_3 f_1) \\
&\quad + \alpha_1\alpha_2^2(-\partial_2^2 f_1 + f_2\partial_2 f_1) + \alpha_1\alpha_3^2(-\partial_3^2 f_1 + f_3\partial_3 f_1) \\
&\quad + \alpha_1\alpha_2\alpha_3(-2\partial_2\partial_3 f_1), \\
A_{21} &= \alpha_2^3(\partial_2\partial_1 f_2 - f_2\partial_1 f_2) + \alpha_1\alpha_2^2(\partial_1^2 f_2 - f_1\partial_1 f_2) + \alpha_2^2\alpha_3(\partial_3\partial_1 f_2), \\
A_{31} &= \alpha_3^3(\partial_3\partial_1 f_3 - f_3\partial_1 f_3) + \alpha_1\alpha_3^2(\partial_1^2 f_3 - f_1\partial_1 f_3) + \alpha_2\alpha_3^2(\partial_2\partial_1 f_3), \\
A_{12} &= \alpha_1^3(\partial_1\partial_2 f_1 - f_1\partial_2 f_1) + \alpha_1^2\alpha_2(\partial_2^2 f_1 - f_2\partial_2 f_1) + \alpha_1^2\alpha_3(\partial_3\partial_2 f_1), \\
A_{22} &= \alpha_1^2\alpha_2(-\partial_1^2 f_2 + f_1\partial_1 f_2) + \alpha_1\alpha_2^2(-\partial_2\partial_1 f_2 + f_2\partial_1 f_2) \\
&\quad + \alpha_2^2\alpha_3(-\partial_2\partial_3 f_2 + f_2\partial_3 f_2) + \alpha_2\alpha_3^2(-\partial_3^2 f_2 + f_3\partial_3 f_2) \\
&\quad + \alpha_1\alpha_2\alpha_3(-2\partial_1\partial_3 f_2), \\
A_{32} &= \alpha_3^3(\partial_3\partial_2 f_3 - f_3\partial_2 f_3) + \alpha_2\alpha_3^2(\partial_2^2 f_3 - f_2\partial_2 f_3) + \alpha_1\alpha_3^2(\partial_1\partial_2 f_3), \\
A_{13} &= \alpha_1^3(\partial_1\partial_3 f_1 - f_1\partial_3 f_1) + \alpha_1^2\alpha_3(\partial_3^2 f_1 - f_3\partial_3 f_1) + \alpha_1^2\alpha_2(\partial_2\partial_3 f_1), \\
A_{23} &= \alpha_2^3(\partial_2\partial_3 f_2 - f_2\partial_3 f_2) + \alpha_2^2\alpha_3(\partial_3^2 f_2 - f_3\partial_3 f_2) + \alpha_1\alpha_2^2(\partial_1\partial_3 f_2), \\
A_{33} &= \alpha_1^2\alpha_3(-\partial_1^2 f_3 + f_1\partial_1 f_3) + \alpha_1\alpha_3^2(-\partial_3\partial_1 f_3 + f_3\partial_1 f_3) \\
&\quad + \alpha_2^2\alpha_3(-\partial_2^2 f_3 + f_2\partial_2 f_3) + \alpha_2\alpha_3^2(-\partial_3\partial_2 f_3 + f_3\partial_2 f_3) \\
&\quad + \alpha_1\alpha_2\alpha_3(-2\partial_1\partial_2 f_3).
\end{aligned}$$

For specific functions f_i, we have the following result.

Theorem 4.7. *Let $M = \mathbb{R}^3$ and let ∇ be the torsion-free connection whose non-zero coefficients are given by $f_1 = \frac{1}{2}u_1u_2^2$, $f_2 = -\frac{1}{2}u_1^2u_2$ and $f_3 = u_3$. Then (M, ∇) is an affine Szabó manifold.*

We have also the following family of examples of affine Szabó connections.

Theorem 4.8. *Let us consider a torsion free connection on* $\mathbb{R}^3$ *given by* $\nabla_{\partial_i}\partial_k = \frac{1}{u_i}\partial_k$, $\nabla_{\partial_j}\partial_k = \frac{1}{u_j}\partial_k$, $\nabla_{\partial_i}\partial_j = \frac{u_k}{u_i u_j}\partial_k$, *with* $i \neq j \neq k$, $i,j,k = 1,2,3$ *and* $u_i \neq 0$, $u_j \neq 0$, $u_k \neq 0$. *Then* (M, ∇) *is affine Szabó.*

Proof. It is easy to see that the curvature tensor of the affine connections is flat. □

Example 4.9. The affine connections on $\mathbb{R}^3$ given by

(1) $\nabla_{\partial_1}\partial_2 = \frac{1}{u_2}\partial_1$, $\nabla_{\partial_1}\partial_3 = \frac{1}{u_3}\partial_1$, $\nabla_{\partial_2}\partial_3 = \frac{u_1}{u_2 u_3}\partial_1$;
(2) $\nabla_{\partial_1}\partial_2 = \frac{1}{u_1}\partial_2$, $\nabla_{\partial_1}\partial_3 = \frac{u_2}{u_1 u_3}\partial_2$, $\nabla_{\partial_2}\partial_3 = \frac{1}{u_3}\partial_2$;
(3) $\nabla_{\partial_1}\partial_2 = \frac{u_3}{u_1 u_2}\partial_3, \nabla_{\partial_1}\partial_3 = \frac{1}{u_1}\partial_3, \nabla_{\partial_2}\partial_3 = \frac{1}{u_2}\partial_3$

are affine Szabó.

5. A classification of locally homogeneous affine Szabó manifolds in dimension two

Homogeneity is one of the fundamental notions in differential geometry. In this section we consider the homogeneity of manifolds with affine connections in dimension two. This homogeneity means that for each two points of a manifold there is an affine transformation which sends one point into another. We characterize locally homogeneous connections which are Szabó in a two dimensional smooth manifold. Note that locally homogeneous Riemannian structures were first studied by Singer in [17].

A smooth connection ∇ on M is *locally homogeneous* [14] if and only if it admits, in neighborhoods of each point $p \in M$, at least two linearly independent affine Killing vector fields. An affine Killing vector field X is characterized by the equation

$$[X, \nabla_Y Z] - \nabla_Y [X, Z] - \nabla_{[X,Y]} Z = 0 \tag{5.1}$$

for any arbitrary vector fields Y and Z on M. Let us express the vector field X on M in the form

$$X = F(u_1, u_2)\partial_1 + G(u_1, u_2)\partial_2.$$

Writing the formula (5.1) in local coordinates, we find that any affine Killing vector field X must satisfy six basics equations. We shall write these equations in the simplified notation

$$\begin{aligned}
\partial_{11}F + f_{11}^1\partial_1 F + \partial_1 f_{11}^1 F - f_{11}^2\partial_2 F + \partial_2 f_{11}^1 G + 2f_{12}^1\partial_1 G = 0,\\
\partial_{11}G + 2f_{11}^2\partial_1 F + (2f_{12}^2 - f_{11}^1)\partial_1 G - f_{11}^2\partial_2 G + \partial_1 f_{11}^2 F + \partial_2 f_{11}^2 G = 0,\\
\partial_{12}F + (f_{11}^1 - f_{12}^2)\partial_2 F + f_{22}^1\partial_1 G + f_{12}^1\partial_2 G + \partial_1 f_{12}^1 F + \partial_2 f_{12}^1 G = 0,\\
\partial_{12}G + f_{12}^2\partial_1 F + f_{11}^2\partial_2 F + (f_{22}^2 - f_{11}^2)\partial_1 G + \partial_1 f_{12}^2 F + \partial_2 f_{12}^2 G = 0,\\
\partial_{22}F - f_{22}^1\partial_1 F + (2f_{12}^1 - f_{22}^2)\partial_2 F + 2f_{22}^1\partial_2 G + \partial_1 f_{22}^1 F + \partial_2 f_{22}^1 G = 0,\\
\partial_{22}G + 2f_{12}^2\partial_2 F - f_{22}^1\partial_1 G + f_{22}^2)\partial_2 G\partial_1 f_{22}^2 F + \partial_2 f_{22}^2 G = 0.
\end{aligned}$$

The following result is the first classification of torsion free homogeneous connections on two dimensional manifolds.

Theorem 5.1 ([14]). *Let ∇ be a locally homogeneous torsion free affine connection on a two-dimensional manifold M. Then, in a neighborhood $\mathcal{U}$ of each point $u \in M$, either ∇ is the Levi-Civita connection of the standard metric of the unit sphere or there is a system (u_1, u_2) of local coordinates and constants a, b, c, d, e, f such that ∇ is expressed in $\mathcal{U}$ by one of the following formulas:*

(1) *Type $\mathcal{A}$:*

$$\nabla_{\partial_1}\partial_1 = a\partial_1 + b\partial_2, \quad \nabla_{\partial_1}\partial_2 = c\partial_1 + d\partial_2, \quad \nabla_{\partial_2}\partial_2 = e\partial_1 + f\partial_2.$$

(2) *Type $\mathcal{B}$:*

$$\nabla_{\partial_1}\partial_1 = \frac{1}{u_1}(a\partial_1 + b\partial_2), \quad \nabla_{\partial_1}\partial_2 = \frac{1}{u_1}(c\partial_1 + d\partial_2), \quad \nabla_{\partial_2}\partial_2 = \frac{1}{u_1}(e\partial_1 + f\partial_2).$$

Next, we characterize all affine connections given in Theorem 5.1 which are *affine Szabó*.

Theorem 5.2. *The affine manifolds of type $\mathcal{A}$ are affine Szabó if and only if they have parallel Ricci tensor.*

Proof. The components of the Ricci tensor are given by $\mathrm{Ric}(\partial_1, \partial_1) = (ad - d^2 + bf - bc)$, $\mathrm{Ric}(\partial_1, \partial_2) = (cd - be)$, $\mathrm{Ric}(\partial_2, \partial_1) = (cd - be)$, $\mathrm{Ric}(\partial_2, \partial_2) = (ae - de + cf - c^2)$. The Ricci tensor is symmetric. Then, the covariant derivatives of the Ricci tensor are given by

$$\begin{aligned}
(\nabla_{\partial_1}\mathrm{Ric})(\partial_1, \partial_1) &= 2(abc + ad^2 - a^2d - abf + b^2e - bcd),\\
(\nabla_{\partial_1}\mathrm{Ric})(\partial_1, \partial_2) &= 2(bc^2 + bde - acd - bcf),\\
(\nabla_{\partial_1}\mathrm{Ric})(\partial_2, \partial_2) &= 2(bce - ade - cdf + d^2e),\\
(\nabla_{\partial_2}\mathrm{Ric})(\partial_1, \partial_1) &= 2(bc^2 + bde - acd - bcf),\\
(\nabla_{\partial_2}\mathrm{Ric})(\partial_1, \partial_2) &= 2(bce - ade - cdf + d^2e),\\
(\nabla_{\partial_2}\mathrm{Ric})(\partial_2, \partial_2) &= 2(be^2 + c^2f - cf^2 - aef - cde + def).
\end{aligned}$$

From Theoreom 4.4, the proof is complete. □

Theorem 5.3. *The affine manifolds of type $\mathcal{B}$ are affine Szabó if and only if the coefficients a, b, c, d, e and f satisfy*

$$\begin{aligned}
2abc + 3bc - d - 2ad - a^2d - bcd + d^2 + ad^2 + b^2e &= 0,\\
2c + ac + 4bc^2 - 2cd - 3acd + 3be + 3bde + 2bce &= 0,\\
3c^2 + 3c^2d + e - ae + 3bce + 2de - 3ade + 3d^2e &= 0,\\
-2c^3 + ace - 2cde + be^2 &= 0.
\end{aligned}$$

Proof. The components of the Ricci tensor are given by

$$\mathrm{Ric}(\partial_1, \partial_1) = \frac{1}{u_1^2}[d + d(a - d) + b(f - c)], \quad \mathrm{Ric}(\partial_1, \partial_2) = \frac{1}{u_1^2}(f + cd - be),$$
$$\mathrm{Ric}(\partial_2, \partial_1) = \frac{1}{u_1^2}(-c + cd - be), \quad \mathrm{Ric}(\partial_2, \partial_2) = \frac{1}{u_1^2}[-e + e(a - d) + c(f - c)],$$

and it is symmetric if and only if $f = -c$ holds. So we set $f = -c$. Then, the covariant derivatives of the Ricci tensor are given by

$$
\begin{aligned}
(\nabla_{\partial_1}\mathrm{Ric})(\partial_1, \partial_1) &= \frac{2}{u_1^3}(2abc + 3bc - d - 2ad - a^2 d - bcd + d^2 + ad^2 + b^2 e),\\
(\nabla_{\partial_1}\mathrm{Ric})(\partial_1, \partial_2) &= \frac{1}{u_1^3}(2c + ac + 2bc^2 - 2cd - 2acd + 3be + 2bde + 2bce),\\
(\nabla_{\partial_1}\mathrm{Ric})(\partial_2, \partial_2) &= \frac{2}{u_1^3}(3c^2 + c^2 d + e - ae + bce + 2de - ade + d^2 e),\\
(\nabla_{\partial_2}\mathrm{Ric})(\partial_1, \partial_1) &= \frac{2}{u_1^3}(2bc^2 - acd + bde),\\
(\nabla_{\partial_2}\mathrm{Ric})(\partial_1, \partial_2) &= \frac{2}{u_1^3}(c^2 d + bce - ade + d^2 e),\\
(\nabla_{\partial_2}\mathrm{Ric})(\partial_2, \partial_2) &= \frac{2}{u_1^3}(-2c^3 + ace - 2cde + be^2).
\end{aligned}
$$

A straightforward calculation using Theorem 4.4 completes the proof. □

As stated by Brozos et al. in [3], the surfaces of Type $\mathcal{A}$ and Type $\mathcal{B}$ can have quite different geometric properties. For instance, the Ricci tensor of any Type $\mathcal{A}$ surface is symmetric while this property can fail for a Type $\mathcal{B}$ surface. Thus the geometry of a Type $\mathcal{B}$ surface is not as rigid as that of a Type $\mathcal{A}$ surface. This is closely related to the existence of non-flat affine Osserman structures (see [7] and references therein). This difference in terms of geometric properties is also remarkable when those surfaces satisfy the Szabó condition (Theorems 5.2 and 5.3).

In the paper [2], the authors determined the moduli space of Type $\mathcal{A}$ affine geometries. Depending on the signature it is either a smooth 2-dimensional surface or a smooth 2-dimensional surface with a single cusp point (signature (2, 0)). They also wrote down complete sets of invariants that determine the local isomorphism type depending on the rank of the Ricci tensor.

Clearly the condition that the Szabó operator is nilpotent is gauge invariant and therefore depends only on the Christoffel symbols modulo the action of the gauge group. This has opened up some perspectives which are under investigation in order to have a more invariant formulation using recent classification results of Brozos-Vázquez et al. (see [2, 3] for more details). Note that the classification of locally homogeneous affine connections in two dimensions is a nontrivial problem (see [1] and [14] for more details).

6. The twisted Riemannian extensions of an affine Szabó manifold

Affine Szabó connections are of interest not only in affine geometry, but also in the study of pseudo-Riemannian Szabó metrics, since they provide some nice examples without Riemannian analogue by means of the Riemannian extensions and the twisted Riemannian extensions.

A pseudo-Riemannian manifold (M, g) is said to be Szabó if the Szabó operator $(\nabla_X R)(\cdot, X)X$ has constant eigenvalues on the unit pseudo-sphere bundles

$S^{\pm}(TM)$ ([9]). Any Szabó manifold is locally symmetric in the Riemannian [18] and the Lorentzian [11] setting but the higher signature case supports examples with nilpotent Szabó operators (cf. [9] and the references therein). Next we will use the twisted Riemannian construction to exhibit a four-dimensional Szabó metric where the degree of nilpotency of the associated Szabó operators changes at each point depending on the direction.

Let (M, ∇) be an affine manifold of dimension n. The *Riemannian extension* is the pseudo-Riemannian metric g_∇ on T^*M of neutral signature (n, n), which is given in local coordinates relative to the frame $\{\partial_{u_1}, \dots, \partial_{u_n}, \partial_{u_{1'}}, \dots, \partial_{u_{n'}}\}$ by

$$g_\nabla = 2du_i \circ du_{i'} - 2u_{k'}\Gamma_{ij}^k du_i \circ du_j,$$

where Γ_{ij}^k give the Christoffel symbols of the affine connection ∇. Riemannian extensions were originally defined by Patterson and Walker [15] and further investigated in relating pseudo-Riemannian properties of N with the affine structure of the base manifold (M, ∇). Moreover, Riemannian extensions were also considered in [12] in relation with L_3-spaces. One has:

Theorem 6.1. *Let (M, ∇) be a two-dimensional smooth torsion-free affine manifold. Then the following assertions are equivalent:*

(1) *(M, ∇) is an affine Szabó manifold.*
(2) *The Riemannian extension (T^*M, g_∇) of (M, ∇) is a pseudo-Riemannian nilpotent Szabó manifold of neutral signature.*

We also have the following theorem.

Theorem 6.2 ([12])**.** *Let (M, ∇) be a smooth torsion-free affine manifold of dimension $n \geq 3$. Then the following assertions hold:*

(1) *If (M, ∇) is an affine Szabó manifold, then its Riemannian extension (T^*M, g_∇) is a pseudo-Riemannian Szabó manifold.*
(2) *If the Ricci tensor of (M, ∇) is symmetric and the Riemannian extension (T^*M, g_∇) of (M, ∇) is a pseudo-Riemannian Szabó manifold, then (M, ∇) is an affine Szabó manifold.*

More generally, if Φ is a symmetric $(0, 2)$-tensor field on M, then the *twisted Riemannian extension* $g_{\nabla,\Phi}$ is the metric of neutral signature on T^*M given by

$$g_{\nabla,\Phi} = \begin{pmatrix} \Phi_{ij}(\vec{u}) - 2u_{k'}\Gamma_{ij}^k & Id_n \\ Id_n & 0 \end{pmatrix}. \tag{6.1}$$

Thus in particular, if ∇ is flat, the Szabó operators of $g_{\nabla,\Phi}$ are nilpotent and the couple $(N, g_{\nabla,\Phi})$ is a Szabó pseudo-Riemannian manifold [4]. Here, we consider the twisted Riemannian of a not flat affine connection and we will prove the following result.

Theorem 6.3. *Let $M = \mathbb{R}^2$ and let ∇ be the torsion-free connection whose non-zero Christoffel symbols are given by $\nabla_{\partial_1}\partial_1 = (u_1 + u_2)\partial_1$ and $\nabla_{\partial_2}\partial_2 = (u_1 + u_2 + 1)\partial_2$. Let $\overline{g} := g_{\nabla,\Phi}$ on T^*M. Then $\overline{g}$ is a Szabó metric of signature $(2, 2)$. Moreover the degree of nilpotency of the Szabó operators $(\nabla_X R)(\,\cdot\,, X)X$ depends on the direction X at each point.*

Proof. Let (M, ∇) be a 2-dimensional affine manifold. The twisted Riemannian extension of the connection $\nabla_{\partial_1}\partial_1 = (u_1 + u_2)\partial_1$ and $\nabla_{\partial_2}\partial_2 = (u_1 + u_2 + 1)\partial_2$ is the pseudo-Riemannian metric $\overline{g}$ on the cotangent bundle T^*M of neutral signature $(2, 2)$ defined by

$$\begin{aligned}\overline{g} = &\Big[\Phi_{11}(u_1, u_2) - 2(u_1 + u_2)u_3\Big] du_1 \otimes du_1 + 2\Phi_{12}(u_1, u_2) du_1 \otimes du_2 \\ &+ 2du_1 \otimes du_3 + \Big[\Phi_{22}(u_1, u_2) - 2(u_1 + u_2 + 1)u_4\Big] du_2 \otimes du_2 + 2du_2 \otimes du_4.\end{aligned}$$

The Levi-Civita connection is determined by the Christoffel symbols as follows:

$$\begin{aligned}
&\Gamma_{11}^1 = u_1 + u_2, \quad \Gamma_{22}^2 = (u_1 + u_2 + 1),\\
&\Gamma_{13}^3 = -(u_1 + u_2), \quad \Gamma_{24}^4 = -(u_1 + u_2 + 1),\\
&\Gamma_{11}^3 = \frac{1}{2}\partial_1\Phi_{11}(u_1, u_2) - (u_1 + u_2)[\Phi_{11}(u_1, u_2) - 2(u_1 + u_2)u_3] - u_3,\\
&\Gamma_{11}^4 = \partial_1\Phi_{12}(u_1, u_2) - \frac{1}{2}\partial_2\Phi_{11}(u_1, u_2) - (u_1 + u_2)\Phi_{12}(u_1, u_2) + u_3,\\
&\Gamma_{12}^3 = \frac{1}{2}\partial_2\Phi_{11}(u_1, u_2) - u_3, \quad \Gamma_{12}^4 = \frac{1}{2}\partial_1\Phi_{22}(u_1, u_2) - u_4,\\
&\Gamma_{22}^3 = \partial_2\Phi_{12}(u_1, u_2) - \frac{1}{2}\partial_1\Phi_{22}(u_1, u_2) - (u_1 + u_2 + 1)\Phi_{12}(u_1, u_2) + u_4,\\
&\Gamma_{22}^4 = \frac{1}{2}\partial_2\Phi_{22}(u_1, u_2) - (u_1 + u_2 + 1)[\Phi_{22}(u_1, u_2) - 2(u_1 + u_2 + 1)u_4] - u_4.
\end{aligned}$$

A straightforward calculation from the Christoffel symbols shows that the non zero components curvature tensor are given by

$$\begin{aligned}
R(\partial_1, \partial_2)\partial_1 = &-\partial_1 + \Big[\Phi_{11} - 2(u_1 + u_2)u_3\Big]\partial_3\\
&+ \Big[\frac{1}{2}\partial_1^2\Phi_{22} - \partial_1\partial_2\Phi_{12} + \frac{1}{2}\partial_2^2\Phi_{11} + \Phi_{12}\\
&+ (u_1 + u_2 + 1)\Big(\partial_1\Phi_{12} - \frac{1}{2}\partial_2\Phi_{11}\Big)\\
&+ (u_1 + u_2)\Big(\partial_2\Phi_{12} - \frac{1}{2}\partial_1\Phi_{22}\Big) - (u_1 + u_2)(u_1 + u_2 + 1)\Phi_{12}\\
&+ (u_1 + u_2 + 1)u_3 + (u_1 + u_2)u_4\Big]\partial_4,
\end{aligned}$$

$$\begin{aligned}
R(\partial_1, \partial_2)\partial_2 = &\ \partial_2 - \Big[\frac{1}{2}\partial_1^2\Phi_{22} - \partial_1\partial_2\Phi_{12} + \frac{1}{2}\partial_2^2\Phi_{11} + \Phi_{12}\\
&+ (u_1 + u_2 + 1)\Big(\partial_1\Phi_{12} - \frac{1}{2}\partial_2\Phi_{11}\Big)\\
&+ (u_1 + u_2)\Big(\partial_2\Phi_{12} - \frac{1}{2}\partial_1\Phi_{22}\Big) - (u_1 + u_2)(u_1 + u_2 + 1)\Phi_{12}\\
&+ (u_1 + u_2 + 1)u_3 + (u_1 + u_2)u_4\Big]\partial_3\\
&- \Big[\Phi_{22} - 2(u_1 + u_2 + 1)u_4\Big]\partial_4,
\end{aligned}$$

$$R(\partial_1,\partial_2)\partial_3 = \partial_3, \quad R(\partial_1,\partial_2)\partial_4 = -\partial_4, \quad R(\partial_1,\partial_3)\partial_1 = -\partial_4, \quad R(\partial_1,\partial_3)\partial_2 = \partial_3.$$

Let $X = \sum_{i=1}^{4} \alpha_i \partial_i$ be a non-null vector, where $\{\partial_i\}$ denotes the coordinate basis. The associated Szabó operator $(\nabla_X R)(\cdot, X)X$ can be expressed with respect to the coordinate basis $\{\partial_i\}$ as follows:

$$\mathcal{S}(X) = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & 0 \end{pmatrix},$$

with

$$a_{11} = f_1(u_1,u_2,u_3,u_4), \quad a_{21} = f_2(u_1,u_2,u_3,u_4), \quad a_{31} = f_3(u_1,u_2,u_3,u_4);$$
$$a_{41} = f_4(u_1,u_2,u_3,u_4), \quad a_{21} = f_1(u_1,u_2,u_3,u_4), \quad a_{22} = f_2(u_1,u_2,u_3,u_4);$$
$$a_{32} = f_3(u_1,u_2,u_3,u_4), \quad a_{42} = f_4(u_1,u_2,u_3,u_4);$$
$$a_{33} = [\alpha_1^2\alpha_2(u_1+u_2) + \alpha_1\alpha_2^2(u_1+u_2+1);$$
$$a_{43} = -[\alpha_1^3(u_1+u_2) + \alpha_1^2\alpha_2(u_1+u_2+1).$$

For the particular choice of the unit vectors $X_1 = \partial_1 + \partial_3$ and $X_2 = \partial_2 + \partial_4$, respectively, it is easy to show that $\mathcal{S}(X_1)$ is three-step nilpotent while $\mathcal{S}(X_2)$ is two-step nilpotent. □

Acknowledgments

We would like to thank Professor P. Gilkey (University of Oregon, USA) for reading the manuscript and for his valuable comments. We also thank the referee for his/her valuable suggestions and comments, and also for bringing references [2, 3] to our attention.

References

[1] T. Arias-Marco and O. Kowalski, Classification of locally homogeneous affine connections with arbitrary torsion on 2-dimensional manifolds, Monatsh. Math. 153 (2008), 1–18. MR 2366132.

[2] M. Brozos-Vázquez, E. García-Río and P. Gilkey, Homogeneous affine surfaces: Affine Killing vector fields and gradient Ricci-solitons. Preprint, 2015. arXiv:1512.05515 [math.DG].

[3] M. Brozos-Vázquez, E. García-Río and P. Gilkey, Homogeneous affine surfaces: Moduli spaces, J. Math. Anal. Appl. 444 (2016), 1155–1184. MR 3535753.

[4] M. Brozos-Vázquez, E. García-Río, P. Gilkey, S. Nikčević and R. Vázquez-Lorenzo, The geometry of Walker manifolds, Synthesis Lectures on Mathematics and Statistics, 5. Morgan and Claypool Publishers, Williston, VT, 2009. MR 2656431.

[5] A. S. Diallo, Affine Osserman connections on 2-dimensional manifolds, Afr. Diaspora J. Math. 11 (2011), 103–109. MR 2792213.

[6] B. Fiedler and P. Gilkey, Nilpotent Szabó, Osserman and Ivanov-Petrova pseudo-Riemannian manifolds. In: Recent advances in Riemannian and Lorentzian geometries (Baltimore, MD, 2003), 53–63, Contemp. Math., 337, Amer. Math. Soc., Providence, RI, 2003. MR 2040468.

[7] E. García-Río, D. N. Kupeli, M. E. Vázquez-Abal and R. Vázquez-Lorenzo, Affine Osserman connections and their Riemann extensions, Differential Geom. Appl. 11 (1999), 145–153. MR 1712127.

[8] P. B. Gilkey, Geometric properties of natural operators defined by the Riemann curvature tensor, World Scientific Publishing, River Edge, NJ, 2001. MR 1877530.
[9] P. B. Gilkey, R. Ivanova and I. Stavrov, Jordan Szabó algebraic covariant derivative curvature tensors. In: Recent advances in Riemannian and Lorentzian geometries (Baltimore, MD, 2003), 65–75, Contemp. Math., 337, Amer. Math. Soc., Providence, RI, 2003. MR 2040469.
[10] P. B. Gilkey, R. Ivanova and T. Zhang, Szabó Osserman IP pseudo-Riemannian manifolds, Publ. Math. Debrecen 62 (2003), 387–401. MR 2008103.
[11] P. Gilkey and I. Stavrov, Curvature tensors whose Jacobi or Szabó operator is nilpotent on null vectors, Bull. London Math. Soc. 34 (2002), 650–658. MR 1924351.
[12] O. Kowalski and M. Sekizawa, The Riemann extensions with cyclic parallel Ricci tensor, Math. Nachr. 287 (2014), 955–961. MR 3219223.
[13] K. Nomizu and T. Sasaki, Affine Differential Geometry. Geometry of Affine Immersions. Cambridge Tracts in Mathematics, 111, Cambridge University Press, Cambridge, 1994. MR 1311248.
[14] B. Opozda, A classification of locally homogeneous connections on 2-dimensional manifolds, Differential Geom. Appl. 21 (2004), 173–198. MR 2073824.
[15] E. M. Patterson and A. G. Walker, Riemann extensions, Quart. J. Math., Oxford Ser. (2) 3 (1952), 19–28. MR 0048131.
[16] H. Pedersen and P. Tod, The Ledger curvature conditions and D'Atri geometry, Differential Geom. Appl. 11 (1999), 155–162. MR 1712123.
[17] I. M. Singer, Infinitesimally homogeneous spaces, Comm. Pure Appl. Math. 13 (1960), 685–697. MR 0131248.
[18] Z. I. Szabó, A short topological proof for the symmetry of 2 point homogeneous spaces, Invent. Math. 106 (1991), 61–64. MR 1123372.
[19] Z. I. Szabó, Spectral theory for operator families on Riemannian manifolds. In: Differential geometry: Riemannian geometry (Los Angeles, CA, 1990), 615–665, Proc. Sympos. Pure Math., 54, Part 3, Amer. Math. Soc., Providence, RI, 1993. MR 1216651.
[20] K. P. Tod, Four-dimensional D'Atri Einstein spaces are locally symmetric. Differential Geom. Appl. 11 (1999), 55–67. MR 1702467.

A. S. Diallo
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal
Private Bag X01, Scottsville 3209 South Africa, and
Université Alioune Diop de Bambey, UFR SATIC, Département de Mathématiques
B. P. 30, Bambey, Sénégal
Diallo@ukzn.ac.za, abdoulsalam.diallo@uadb.edu.sn

F. Massamba ✉
School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal
Private Bag X01, Scottsville 3209 South Africa
massfort@yahoo.fr, Massamba@ukzn.ac.za

Received: August 22, 2015
Accepted: June 27, 2016