REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Vol. 58, No. 1, 2017, Pages 53–75
Published online: November 2, 2016

CONNECTEDNESS OF THE ALGEBRAIC SET OF VECTORS GENERATING PLANAR NORMAL SECTIONS OF HOMOGENEOUS ISOPARAMETRIC HYPERSURFACES

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Abstract. Let $M \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be a homogeneous isoparametric hypersurface and consider the algebraic set of unit tangent vectors generating planar normal sections at a point $E \in M$ (denoted by $\widehat{X}_E[M] \subset T_E(M)$). The present paper is devoted to prove that $\widehat{X}_E[M]$ is *connected by arcs.* This in turn proves that its projective image $X[M] \subset \mathbb{RP}(T_E(M))$ also has this property.

1. Introduction

Table 1, below, includes *all the homogeneous isoparametric hypersurfaces in spheres.* There are many other isoparametric hypersurfaces of spheres which *are not homogeneous* but we shall not consider them here.

Our objective is to present a result concerning the manifolds in Table 1. This property concerns their *algebraic sets* of unit tangent vectors generating planar normal sections at a point E of M (denoted by $\widehat{X}_E[M] \subset \mathbb{S}(T_E(M))$). Hence its projective image $X[M] \subset \mathbb{RP}(T_E(M))$ of $\widehat{X}_E[M]$ also has this property.

Theorem 1.1. *For all the homogeneous isoparametric hypersurfaces $M^n \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ (those in Table 1), the algebraic set $\widehat{X}_E[M] \subset \mathbb{S}(T_E(M))$ is connected by arcs.*

The paper is organized as follows. In the next section we recall basic information concerning the algebraic set $\widehat{X}_E[M]$ and its projective image $X[M]$.

In Section 3 we indicate, for each M in Table 1, the polynomials that define $\widehat{X}_E[M]$. We include the necessary notations to understand their meaning but avoid the computations required to get them. Those computations are contained in [11]. Section 3 has three natural subsections where the spaces M with the same g are placed together. In Section 4 we indicate how to construct some subsets of $\widehat{X}_E[M]$ which are required in the proof of Theorem 1.1. In Section 5 we mention the subsets that may be constructed, in $\widehat{X}_E[M]$, for each of the corresponding manifolds. The properties of these subsets are used in the proof of Theorem 1.1

2010 *Mathematics Subject Classification.* 53C30, 53C42.

Partial support from Argentina's CONICET is gratefully acknowledged.

g	M	dim	m_1, m_2
1	$M_1 = S^n$	n	n
2	$M_2 = S^k \times S^{n-k}$	n	$k, (n-k)$
3	$M_{\mathbb{R}} = \mathrm{SO}(3) / (Z_2 \times Z_2)$	3	$1, 1$
3	$M_{\mathbb{C}} = \mathrm{SU}(3) / T^2$	6	$2, 2$
3	$M_{\mathbb{H}} = Sp(3) / (Sp(1))^3$	12	$4, 4$
3	$M_{\mathbb{O}} = F_4 / \mathrm{Spin}(8)$	24	$8, 8$
4	$W_{\mathbb{R}} = \mathrm{SO}(5) / T^2$	8	$2, 2$
4	$W_{\mathbb{C}} = U(5) / (\mathrm{SU}(2)^2 \times T^1)$	18	$4, 5$
4	$N_{\mathbb{R}} = \mathrm{SO}(m) \times \mathrm{SO}(2) / (\mathrm{SO}(m-2) \times Z_2)$	$2m-2$	$1, m-2$
4	$N_{\mathbb{C}} = S(U(m) \times U(2)) / \left(\mathrm{SU}(m-2) \times T^2\right)$	$4m-2$	$2, 2m-3$
4	$N_{\mathbb{H}} = Sp(m) \times Sp(2) / (Sp(m-2) \times (Sp(1))^2)$	$8m-2$	$4, 4m-5$
4	$N_{(9,6)} = \mathrm{Spin}(10) \cdot T / (\mathrm{SU}(4) \cdot T)$	30	$6, 9$
6	$M_{\mathbb{B}} = G_2 / T^2$	12	$2, 2$
6	$M_{\mathbb{S}} = \mathrm{SO}(4) / Z_2 \times Z_2$	6	$1, 1$

Table 1. Homogeneous isoparametric hypersurfaces in spheres

given in Section 6. Using a nice result from [4] we obtain in Section 7 an interesting consequence of Theorem 1.1.

For these manifolds, g indicates the number of distinct constant principal curvatures, dim is the corresponding dimension, and m_1, m_2 are their multiplicities.

2. The algebraic set of planar normal sections

Here we use M to indicate any of the hypersurfaces in Table 1. They are orbits of a point E ($\|E\| = 1$) in the tangent linear representation of some symmetric space where the indicated group is contained in the isotropy.

By definition, *normal sections* are the curves obtained by cutting a submanifold M^n of $\mathbb{R}^{n+2}$ with the affine subspace generated by a unit tangent vector $X \in T_E(M)$ and the normal space $T_E^{\perp}(M)$, at the given point E of M. Any unit tangent vector $X \in T_E(M)$ defines a normal section. This curve can be given a C^∞ parametrization around E which is *regular* and can therefore be locally parametrized by arc-length. Let us recall the following definition.

Definition 2.1. A curve $\gamma(s)$ parametrized by arc-length in $\mathbb{R}^{n+k}$ such that $E = \gamma(0)$ is said to be *planar at E* if its first three derivatives $\gamma'(0)$, $\gamma''(0)$, $\gamma'''(0)$ are linearly dependent in $T_E\left(\mathbb{R}^{n+k}\right)$.

It is known that the unit vectors *defining planar normal sections* at the point $E \in M$ are characterized by the following condition (see [9]).

Condition 2.2. *The normal section of M defined by the unit vector $X \in T_E(M)$ is planar at E if and only if $\left(\overline{\nabla}_X \alpha\right)(X, X) = 0$.* □

Here α indicates the second fundamental form of M in $\mathbb{R}^{n+2}$ at E and $\left(\overline{\nabla}\alpha\right)$ its usual covariant derivative. As in [9] we denote by

$$\widehat{X}_E[M] = \left\{X \in T_E(M) : \|X\| = 1, \left(\overline{\nabla}_X \alpha\right)(X, X) = 0\right\} \tag{2.1}$$

the algebraic set of unit vectors generating planar normal sections at E.

For isoparametric hypersurfaces in the sphere (the case considered here) this algebraic set is determined by a *single* polynomial of degree three defined on $T_E(M)$ but restricted to the unit sphere $\mathbb{S}(T_E(M))$. This polynomial is $P(X) = \left\langle \left(\overline{\nabla}_X \alpha\right)(X, X), H_2 \right\rangle$, where $\{E, H_2\}$ is an orthonormal basis of $T_E(M)^{\perp}$, because $\left(\overline{\nabla}_X \alpha\right)(X, X)$ is orthogonal to E. We call $P(X)$ the *polynomial of normal sections of M*. The algebraic set $\widehat{X}_E[M]$ is then defined by

$$\widehat{X}_E[M] = \{X \in T_E(M) : \|X\| = 1,\ P(X) = 0\}.$$

Since $X \in \widehat{X}_E[M]$ implies $(-X) \in \widehat{X}_E[M]$ (the antipodal map of $\mathbb{S}(T_E(M))$ preserves $\widehat{X}_E[M]$) we may consider the quotient of $\widehat{X}_E[M]$ by the antipodal map of $\mathbb{S}(T_E(M))$ and obtain an algebraic set $X[M] \subset \mathbb{RP}(T_E(M))$.

It is necessary to describe the polynomials defining $\widehat{X}_E[M]$ for each M. In the next section we indicate them and the notation required. As is clear from their definition, the polynomials are homogeneous, have degree 3, and the variables in each monomial have degree 1 ([9]). They are constructed in [11].

Since our objective is to prove that $\widehat{X}_E[M]$ is connected by arcs, it is enough to prove that $\Lambda\widehat{X}_E[M]$ (*the cone over $\widehat{X}_E[M]$, without the vertex*) *is connected by arcs.* This set $\Lambda\widehat{X}_E[M] \subset (T_E(M) - \{0\})$ is defined by

$$\Lambda\widehat{X}_E[M] = \{X \in T_E(M) : X \neq 0,\ P(X) = 0\}.$$

3. The polynomials

We indicate the corresponding polynomials that define $\widehat{X}_E[M]$ for each manifold in Table 1, following the order and the notation of the table.

Remark 3.1. *The spaces corresponding to $g = 1$ and $g = 2$ are symmetric R-spaces and by a well known result of D. Ferus* [2] *have parallel second fundamental form in their corresponding ambient Euclidean spaces. So, for each of them, (if $n = \dim(M)$) we have $\widehat{X}_E[M] = \mathbb{S}^{(n-1)}$ and $X[M] = \mathbb{RP}^{(n-1)}$. Therefore we do not need to consider them in the proof of Theorem 1.1.*

So we start with

3.1. **Spaces with $g = 3$.** These are the so called *Cartan isoparametric hypersurfaces* $M_{\mathbb{R}}$, $M_{\mathbb{C}}$, $M_{\mathbb{H}}$, and $M_{\mathbb{O}}$. We indicate only required facts to understand the notation; see [11] for details. Let $F = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, or $\mathbb{O}$, and denote by $M_3(F)$ the 3×3 matrices with entries in F. Let $H_3(F) = \left\{u \in M_3(F) : \overline{u}^t = u\right\}$, where $x \mapsto \overline{x}$ denotes conjugation in F. An element $u \in H_3(F)$ is of the form

$$u = \begin{bmatrix} \xi_1 & x_3 & \overline{x_2} \\ \overline{x_3} & \xi_2 & x_1 \\ x_2 & \overline{x_1} & \xi_3 \end{bmatrix}, \quad \xi_j \in \mathbb{R},\ x_j \in F. \tag{3.1}$$

The $H_3(F)$ are real Jordan algebras with the product $u \circ v = \frac{1}{2}(uv + vu)$. The compact groups $\mathrm{SO}(3) \subset \mathrm{SU}(3) \subset Sp(3) \subset F_4$ act as groups of automorphisms of the corresponding algebras. Their actions preserve the function $\mathrm{tr}(u)$.

Let us consider the subspaces $U(F) = \{u \in H_3(F) : \mathrm{tr}(u) = 0\}$ ($F = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$) which are invariant by the corresponding groups.

Let us take the point $E = \mathrm{diag}(-1, 0, 1) \in U(F)$, $\forall F$ and consider the orbits M_F of E by the mentioned groups. Let us take in each $U(F)$ the inner product $\langle u, v\rangle = \frac{1}{2}\mathrm{tr}(u \circ v)$. The subspaces U with these inner products are our ambient Euclidean spaces for the manifolds $M_{\mathbb{R}}$, $M_{\mathbb{C}}$, $M_{\mathbb{H}}$, and $M_{\mathbb{O}}$. Note that $\|E\| = 1$. Let us consider in $U(F)$ the subspace

$$\mathfrak{a} = \{\mathrm{diag}(\xi_1, \xi_2, \xi_3) : \xi_1 + \xi_2 + \xi_3 = 0\}. \tag{3.2}$$

The normal space to M_F at E is *the same* for all F, namely $T_E(M_F)^{\perp} = \mathfrak{a}$. We may identify the tangent space at E with the subspace of U

$$T_E(M_F) = \left\{ \begin{bmatrix} 0 & x_3 & \overline{x_2} \\ \overline{x_3} & 0 & x_1 \\ x_2 & \overline{x_1} & 0 \end{bmatrix}, \ x_j \in F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O} \right\}.$$

The polynomials *determining the algebraic sets* $\widehat{X}_E[M_F]$ for M_F are defined on $T_E(M_F)$ ($F = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$):

$$P_F(X) = \mathrm{Re}\left((x_1x_2)x_3\right), \quad x_j \in F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}. \tag{3.3}$$

In all cases, the trilinear function $\mathrm{Re}((x_1x_2)x_3)$ is invariant by cyclic permutation and satisfies $\mathrm{Re}((ab)c) = \mathrm{Re}(a(bc))$.

3.2. **Spaces with $g = 4$.** We have to divide these spaces in several groups.

3.2.1. *Spaces* $W_{\mathbb{R}}$ *and* $W_{\mathbb{C}}$. The polynomials for $W_{\mathbb{R}}$ and $W_{\mathbb{C}}$ may be simultaneously described. We follow [8, p. 27] and reproduce the necessary notation. Let us take the vector space $\mathfrak{p}$ over the field F ($F = \mathbb{R}$ or $\mathbb{C}$) of skew symmetric, 5×5 matrices over F, that is, $\mathfrak{p} = \{Z \in M_5(F) : Z^t = -Z\}$. We use the notation

$$Z = \begin{bmatrix} 0 & -z_1 & -z_3 & -z_5 & -z_7 \\ z_1 & 0 & -z_4 & -z_6 & -z_8 \\ z_3 & z_4 & 0 & -z_2 & -z_9 \\ z_5 & z_6 & z_2 & 0 & -z_{10} \\ z_7 & z_8 & z_9 & z_{10} & 0 \end{bmatrix} \in \mathfrak{p}, \qquad z_j = x_j + iy_j, \quad j = 1, \ldots, 10.$$

The real case ($F=\mathbb{R}$) is given by the condition $y_j=0$, $j=1,\dots,10$. In $\mathfrak{p}$ we consider the inner product defined by

$$\langle Z,W\rangle=-\frac{1}{2}\operatorname{Re}\left(\operatorname{tr}\left(Z\left(\overline{W}\right)\right)\right)=\operatorname{Re}\sum_{j=1}^{10}z_j\overline{w_j}$$

and the subspace $\mathfrak{a}=\{H(\xi_1,\xi_2):\xi_j\in\mathbb{R}\}\subset\mathfrak{p}$, where

$$H(\xi_1,\xi_2)=\xi_1\left(E_{2,1}-E_{1,2}\right)+\xi_2\left(E_{4,3}-E_{3,4}\right),\quad \xi_j\in\mathbb{R}.$$

Then (ξ_1,ξ_2) *is an orthonormal coordinate system for* $\mathfrak{a}$.

We take the basic vector E defined by

$$E=H(t_1,t_2)=H\left(\cos\left(\frac{\pi}{8}\right),\sin\left(\frac{\pi}{8}\right)\right),\quad \|E\|=1.$$

Our manifold W_F ($F=\mathbb{R}$ or $\mathbb{C}$) is the orbit of E by the adjoint action of the corresponding group ($\mathrm{SO}(5)$ or $U(5)$) on $\mathfrak{p}$.

The normal and tangent spaces at E are

$$\left.\begin{aligned}T_E\left(W_F\right)^{\perp}&=\mathfrak{a}\\ T_E\left(W_F\right)&=\{Z\in\mathfrak{p}:x_1=x_2=0\}\end{aligned}\right\}F=\mathbb{R},\mathbb{C}$$

We have that $\dim_{\mathbb{R}}\left(T_E\left(W_{\mathbb{R}}\right)\right)=8$, while $\dim_{\mathbb{R}}\left(T_E\left(W_{\mathbb{C}}\right)\right)=18$.

For $F=\mathbb{R}$, we may write a tangent vector to $W_{\mathbb{R}}$ at E as $X=(0,0,x_3,\dots,x_{10})$ and the polynomial of normal sections is

$$\begin{aligned}P_{\mathbb{R}}(X)=\;&t_1\left(x_7x_9x_4+x_7x_{10}x_6-x_8x_3x_9-x_8x_5x_{10}\right)\\&+t_2\left(-x_7x_9x_5-x_8x_9x_6+x_{10}x_3x_7+x_{10}x_4x_8\right).\end{aligned}$$

On the other hand, on the vector $Z=(0,0,x_3,\dots,x_{10},y_1,\dots,y_{10})$ tangent to $W_{\mathbb{C}}$ at E the polynomial is

$$P_{\mathbb{C}}(Z)=t_1C+t_2D,$$

with

$$\begin{aligned}C=\;&\left(-y_2x_3y_6-y_2y_3x_6+y_2x_5y_4+y_2y_5x_4\right)\\&+\left(x_4x_7x_9+x_4y_7y_9+y_4x_7y_9-y_4y_7x_9\right)\\&+\left(-x_3x_8x_9-x_3y_8y_9-y_3x_8y_9+y_3y_8x_9\right)\\&+\left(x_6x_7x_{10}+x_6y_7y_{10}+y_6x_7y_{10}-y_6y_7x_{10}\right)\\&+\left(-x_5x_8x_{10}-x_5y_8y_{10}-y_5x_8y_{10}+y_5y_8x_{10}\right),\end{aligned}$$

$$\begin{aligned}D=\;&\left(-y_1x_3y_6-y_1y_3x_6+y_1x_5y_4+y_1y_5x_4\right)\\&+\left(-x_5x_9x_7-x_5y_9y_7-y_5x_9y_7+y_5y_9x_7\right)\\&+\left(x_3x_{10}x_7+x_3y_{10}y_7+y_3x_{10}y_7-y_3y_{10}x_7\right)\\&+\left(-x_6x_9x_8-x_6y_9y_8-y_6x_9y_8+y_6y_9x_8\right)\\&+\left(x_4x_{10}x_8+x_4y_{10}y_8+y_4x_{10}y_8-y_4y_{10}x_8\right).\end{aligned}$$

Clearly $P_{\mathbb{C}}(Z)$ reduces to $P_{\mathbb{R}}(X)$ when the imaginary parts y_j, ($j=1,\dots,10$) vanish.

3.2.2. *Spaces $N_{\mathbb{R}}$, $N_{\mathbb{C}}$ and $N_{\mathbb{H}}$.* These submanifolds are defined via Clifford systems. The reader interested in the construction of these Clifford systems should consult [3]. Since our objective are the polynomials, we shall limit ourselves to indicate the manifolds. We have three infinite families. Note that here $n \geq 3$.

The spaces where the Clifford systems act are respectively $\mathbb{R}^{2n}$, $\mathbb{R}^{4n}$, $\mathbb{R}^{8n}$. But since $\mathbb{C}^{2n} \simeq \mathbb{R}^{4n}$, $\mathbb{H}^{2n} \simeq \mathbb{R}^{8n}$, we may think that our system is defined on $F^n \oplus F^n = F^{2n}$ ($F = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$). Then *we shall consider the largest case* $N_{\mathbb{H}}$ in $\mathbb{H}^{2n} = \mathbb{H}^n \oplus \mathbb{H}^n$ and explain the required reductions to get the other ones.

We write the elements of $\mathbb{H}^{2n} = \mathbb{H}^n \oplus \mathbb{H}^n$ as

$$((u_1, u_2 \ldots, u_n), (v_1, \ldots v_{n-1}, v_n)) \in \mathbb{H}^{2n} \quad (u_j, v_k \in \mathbb{H}).$$

The inner product on $\mathbb{H}^{2n}$ is

$$\begin{aligned}\left\langle ((u_1, u_2 \ldots, u_n), (v_1, \ldots v_{n-1}, v_n)), ((u_1', u_2' \ldots, u_n'), (v_1', \ldots v_{n-1}', v_n'))\right\rangle \\ = \sum_{j=1}^{n} \left\langle u_j, u_j' \right\rangle + \left\langle v_j, v_j' \right\rangle,\end{aligned}$$

where $\left\langle u_j, u_j' \right\rangle$ is *the inner product of quaternions.* The manifolds are the orbits, by the corresponding groups, of $E \in \mathbb{H}^{2n}$ given by

$$E = ((t_1, 0, \ldots, 0), (0, \ldots, 0, t_2)), \qquad t_1 = \cos\left(\frac{\pi}{8}\right), \ t_2 = \sin\left(\frac{\pi}{8}\right).$$

We take the unit vector $\Omega = ((t_2, 0, \ldots, 0), (0, \ldots, 0, -t_1))$ orthogonal to E. The normal space at E is $T_E(M)^{\perp} = \mathbb{R}E \oplus \mathbb{R}\Omega$ and the tangent space to $N_{\mathbb{H}}$ at E is

$$\begin{aligned}T_E(N_{\mathbb{H}}) = \{((\alpha, u_2 \ldots, u_n), (v_1, \ldots v_{n-1}, \delta)) \in \mathbb{H}^{2n} : u_j, v_j \in \mathbb{H}, \\ \alpha, \delta \text{ pure quaternions}\}\end{aligned} \tag{3.4}$$

To write down our polynomial we introduce the following notation:

$$\begin{aligned}
\alpha &= a_1 i + a_2 j + a_3 k, \\
\delta &= d_1 i + d_2 j + d_3 k, \\
u_r &= b_{r,o} + b_{r,1} i + b_{r,2} j + b_{r,3} k \quad (2 \leq r \leq n-1), \\
v_r &= c_{r,o} + c_{r,1} i + c_{r,2} j + c_{r,3} k \quad (2 \leq r \leq n-1), \\
u_n &= b_{n,o} + b_{n,1} i + b_{n,2} j + b_{n,3} k, \\
v_1 &= c_{1,o} + c_{1,1} i + c_{1,2} j + c_{1,3} k.
\end{aligned} \tag{3.5}$$

Then we may write the polynomial defining $\widehat{X}_E\left[N_{\mathbb{H}}\right]$:

$$\begin{aligned}
Q_{\mathbb{H}}(X) = & \left(t_1 c_{1,o} + t_2 b_{n,o}\right)\left(a_1 c_{1,1} + a_2 c_{1,2} + a_3 c_{1,3} + d_1 b_{n,1} + d_2 b_{n,2} + d_3 b_{n,3}\right) \\
& + \left(t_1 c_{1,o} + t_2 b_{n,o}\right) \sum_{r=2}^{n-1} \left(b_{r,o} c_{r,o} + b_{r,1} c_{r,1} + b_{r,2} c_{r,2} + b_{r,3} c_{r,3}\right) \\
& + \left(-t_1 c_{1,1} + t_2 b_{n,1}\right)\left(c_{1,o} a_1 - c_{1,3} a_2 + c_{1,2} a_3 - d_1 b_{n,o} - d_3 b_{n,2} + d_2 b_{n,3}\right) \\
& + \left(-t_1 c_{1,1} + t_2 b_{n,1}\right) \sum_{r=2}^{n-1} \left(-c_{r,1} b_{r,o} + c_{r,o} b_{r,1} - c_{r,3} b_{r,2} + c_{r,2} b_{r,3}\right) \\
& + \left(-t_1 c_{1,2} + t_2 b_{n,2}\right)\left(c_{1,3} a_1 + c_{1,o} a_2 - c_{1,1} a_3 - d_2 b_{n,o} + d_3 b_{n,1} - d_1 b_{n,3}\right) \\
& + \left(-t_1 c_{1,2} + t_2 b_{n,2}\right) \sum_{r=2}^{n-1} \left(-c_{r,2} b_{r,o} + c_{r,3} b_{r,1} + c_{r,o} b_{r,2} - c_{r,1} b_{r,3}\right) \\
& + \left(-t_1 c_{1,3} + t_2 b_{n,3}\right)\left(-c_{1,2} a_1 + c_{1,1} a_2 + c_{1,o} a_3 + d_1 b_{n,2} - d_2 b_{n,1} - d_3 b_{n,o}\right) \\
& + \left(-t_1 c_{1,3} + t_2 b_{n,3}\right) \sum_{r=2}^{n-1} \left(-c_{r,3} b_{r,o} - c_{r,2} b_{r,1} + c_{r,1} b_{r,2} + c_{r,o} b_{r,3}\right).
\end{aligned}$$

For the other two spaces $N_{\mathbb{R}}$ and $N_{\mathbb{C}}$, we notice that, for $F = \mathbb{R}$, we have $\alpha = \delta = 0$, $u_s = b_{s,o}$, $v_s = c_{s,o} \in \mathbb{R}$, while for $F = \mathbb{C}$, $\alpha = a_1 i$ and $\delta = d_1 i$ are pure imaginary and $u_r = b_{r,o} + b_{r,1} i$, $v_r = c_{r,o} + c_{r,1} i \in \mathbb{C}$. Then we have for $N_{\mathbb{R}}$:

$$Q_{\mathbb{R}}(X) = \left(t_1 c_{1,o} + t_2 b_{n,o}\right) \sum_{r=2}^{n-1} b_{r,o} c_{r,o},$$

and for $N_{\mathbb{C}}$:

$$\begin{aligned}
Q_{\mathbb{C}}(X) = & \left(t_1 c_{1,o} + t_2 b_{n,o}\right)\left(a_1 c_{1,1} + d_1 b_{n,1}\right) \\
& + \left(t_1 c_{1,o} + t_2 b_{n,o}\right) \sum_{r=2}^{n-1} \left(b_{r,o} c_{r,o} + b_{r,1} c_{r,1}\right) \\
& + \left(-t_1 c_{1,1} + t_2 b_{n,1}\right)\left(c_{1,o} a_1 - d_1 b_{n,o}\right) \\
& + \left(-t_1 c_{1,1} + t_2 b_{n,1}\right) \sum_{r=2}^{n-1} \left(-c_{r,1} b_{r,o} + c_{r,o} b_{r,1}\right).
\end{aligned}$$

3.2.3. *The space* $N_{(9,6)}$. This space has dimension 30 and $m_1 = m_3 = 9$, $m_2 = m_4 = 6$. The ambient is the tangent space of the symmetric space EIII of dimension 32. We adopt the following notation for the ambient space $\mathbb{R}^{32}$, which we identify with $\mathbb{H}^8$:

$$(A, B) = \left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_5 & b_6 \\ b_7 & b_8 \end{bmatrix}\right), \quad a_r, b_s \in \mathbb{H}.$$

We set the inner product on $\mathbb{H}^8$ as

$$\langle (A, B), (C, D) \rangle = \sum_{s=1}^{4} \langle a_s, c_s \rangle + \sum_{k=5}^{8} \langle b_k, d_k \rangle,$$

where $\langle a_s, c_s\rangle$ is the inner product of quaternions.

We take

$$E = \left(\begin{bmatrix} t_1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & t_6 \\ 0 & 0 \end{bmatrix} \right), \quad \Omega = \left(\begin{bmatrix} t_6 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & (-t_1) \\ 0 & 0 \end{bmatrix} \right),$$

where, as before, $t_1 = \cos\left(\frac{\pi}{8}\right)$, $t_6 = \sin\left(\frac{\pi}{8}\right)$. Clearly $\|E\| = 1$ and the normal space to $N_{(9,6)}$ at E is the subspace $T_E\left(N_{(9,6)}\right)^{\perp} = RE \oplus R\Omega$. In turn the tangent space at E is

$$T_E\left(N_{(9,6)}\right) = \left\{ \begin{bmatrix} \alpha & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_5 & \beta \\ b_7 & b_8 \end{bmatrix} \right\}, \tag{3.6}$$

with $a_r, b_s \in \mathbb{H}$ and α, β pure quaternions.

To present the polynomial, we require the following refined notation:

$$\begin{aligned} a_s &= u_{s,0} + iu_{s,1} + ju_{s,2} + ku_{s,3}, \quad s = 2,3,4, \\ b_r &= v_{r,0} + iv_{r,1} + jv_{r,2} + kv_{r,3}, \quad r = 5,7,8, \\ \alpha &= i\alpha_1 + j\alpha_2 + k\alpha_3, \\ \beta &= i\beta_1 + j\beta_2 + k\beta_3. \end{aligned} \tag{3.7}$$

Then the expression of the polynomial for $X \in T_E\left(N_{(9,6)}\right)$ is

$$\begin{aligned} P_{(9,6)}(X) = {} & (t_1 v_{5,0} + t_6 u_{2,0}) \left[\langle \alpha, b_5\rangle + \langle a_2, \beta\rangle + \langle a_3, b_7\rangle + \langle a_4, b_8\rangle\right] \\ & + (-t_1 v_{5,1} + t_6 u_{2,1}) \left[\langle \alpha, ib_5\rangle + \langle a_2, i\beta\rangle - \langle a_3, ib_7\rangle - \langle a_4, ib_8\rangle\right] \\ & + (-t_1 v_{5,2} + t_6 u_{2,2}) \left[\langle \alpha, jb_5\rangle + \langle a_2, j\beta\rangle - \langle a_3, jb_7\rangle - \langle a_4, jb_8\rangle\right] \\ & + (-t_1 v_{5,3} + t_6 u_{2,3}) \left[\langle \alpha, kb_5\rangle + \langle a_2, k\beta\rangle - \langle a_3, kb_7\rangle - \langle a_4, kb_8\rangle\right] \\ & + (t_1 v_{8,0} - t_6 u_{3,0}) \left[\langle \alpha, b_8\rangle + \langle a_2, b_7\rangle - \langle a_3, \beta\rangle - \langle a_4, b_5\rangle\right] \\ & + (-t_1 v_{7,1} + t_6 u_{4,1}) \left[\langle \alpha, b_7 i\rangle + \langle a_2, b_8 i\rangle + \langle a_3, b_5 i\rangle + \langle a_4, \beta i\rangle\right] \\ & + (-t_1 v_{7,2} + t_6 u_{4,2}) \left[\langle \alpha, b_7 j\rangle + \langle a_2, b_8 j\rangle + \langle a_3, b_5 j\rangle + \langle a_4, \beta j\rangle\right] \\ & + (-t_1 v_{7,3} + t_6 u_{4,3}) \left[\langle \alpha, b_7 k\rangle + \langle a_2, b_8 k\rangle + \langle a_3, b_5 k\rangle + \langle a_4, \beta k\rangle\right] \\ & + (-t_1 v_{7,0} - t_6 u_{4,0}) \left[- \langle \alpha, b_7\rangle + \langle a_2, b_8\rangle + \langle a_3, b_5\rangle - \langle a_4, \beta\rangle\right]. \end{aligned}$$

3.3. **Spaces with $g = 6$.** These are $M_{\mathbb{B}}$ and $M_{\mathbb{S}}$. The complex simple Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$, of type G_2, has only two real forms, namely the compact one $\mathfrak{g}_2$ and the split (or normal) real form $\mathfrak{g}$. The real algebra $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. That is, the subalgebra $\mathfrak{k}$ and the complementary subspace $\mathfrak{p}$ satisfy

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k},$$

and $\mathfrak{k} \oplus i\mathfrak{p} = \mathfrak{g}_2$ is the compact real form. As in [8] we identify $\mathfrak{p}$ with $\mathfrak{p}_u := i\mathfrak{p}$ by the map

$$iX \mapsto X, \tag{3.8}$$

which in turn identifies $\mathfrak{g}_2$ and $\mathfrak{g}$. Furthermore we have $\mathfrak{k} \simeq \mathfrak{so}(4)$.

As in [5] and [6], it is possible to choose a *convenient orthonormal basis* for $\mathfrak{g}_2$, $\{H_j : 1 \leq j \leq 14\}$, such that

$$\begin{gathered}\operatorname{Span}_{\mathbb{R}}\{H_3, H_4, H_5, H_6, H_7, H_8\} = \mathfrak{k} \simeq \mathfrak{so}(4),\\ \operatorname{Span}_{\mathbb{R}}\{H_1, H_2, H_9, H_{10}, H_{11}, H_{12}, H_{13}, H_{14}\} = \mathfrak{p},\\ \mathfrak{a} = \operatorname{Span}_{\mathbb{R}}\{H_1, H_2\}.\end{gathered}$$

$\mathfrak{a}$ is a *Cartan subalgebra* of $\mathfrak{g}_2$ (and hence a maximal abelian subspace of $\mathfrak{p}$). Since the Cartan subalgebra $\mathfrak{a}$ is contained in $\mathfrak{p}$, the *restricted roots* coincide with the roots of $\mathfrak{g}_2^{\mathbb{C}}$. We take the point $E = H_1$, which happens to be a *regular* element in $\mathfrak{a}$; then the orbits of E by the compact groups G_2 and $\mathrm{SO}(4)$ are both principal orbits.

$$\begin{gathered}M_{\mathbb{B}} = G_2/T^2 = G_2(E) \subset \mathfrak{g}_2,\\ M_{\mathbb{S}} = \mathrm{SO}(4)/(Z_2 \times Z_2) = \mathrm{SO}(4)(E) \subset \mathfrak{p} \subset \mathfrak{g}_2,\\ M_{\mathbb{B}} \subset \mathbb{S}(\mathfrak{g}_2) = \mathbb{S}^{13} M_{\mathbb{S}} \subset \mathbb{S}(\mathfrak{p}) = \mathbb{S}^7.\end{gathered}$$

We have

$$\begin{gathered}T_E(M_{\mathbb{B}}) = [\mathfrak{g}_2, E] = \operatorname{Span}_{\mathbb{R}}\{[H_j, E] : 3 \leq j \leq 14\},\\ T_E(M_{\mathbb{S}}) = [\mathfrak{k}, E] = \operatorname{Span}_{\mathbb{R}}\{[H_j, E] : 3 \leq j \leq 8\},\\ T_E^{\perp}(M_{\mathbb{B}}) = T_E^{\perp}(M_{\mathbb{S}}) = \operatorname{Span}_{\mathbb{R}}\{H_1, H_2\} = \mathfrak{a}.\end{gathered}$$

On $X = \sum_{j=3}^{14} r_j [H_j, E] \in T_E(M_{\mathbb{B}})$, the polynomial for $\widehat{X}_E[M_{\mathbb{B}}]$ is of the form

$$\begin{aligned}P_{\mathbb{B}}(X) = {} & r_3r_5r_7 + r_3r_6r_8 + r_3r_{11}r_{13} + r_3r_{12}r_{14}\\ & + r_4r_{12}r_{13} + r_7r_9r_{11} + r_8r_9r_{12}\\ & + (-r_4r_6r_7 - r_5r_9r_{13} - r_6r_{10}r_{13} - r_6r_9r_{14} - r_7r_{10}r_{12})\\ & + \frac{2}{3}\sqrt{3}(-r_3r_6r_7 - r_3r_{12}r_{13} - r_6r_9r_{13} + r_7r_9r_{12})\\ & + 3(r_4r_5r_8 + r_5r_{10}r_{14} + r_8r_{10}r_{11} - r_4r_{11}r_{14}),\end{aligned} \tag{3.9}$$

and the polynomial defining $\widehat{X}_E[M_{\mathbb{S}}]$ is obtained by restricting $P_{\mathbb{B}}(X)$ to $T_E(M_{\mathbb{S}})$ (i.e., vanishing r_j, $9 \leq j \leq 14$). We get

$$P_{\mathbb{S}}(X) = r_3r_5r_7 + r_3r_6r_8 + (-r_4r_6r_7) + \frac{2}{\sqrt{3}}(-r_3r_6r_7) + 3(r_4r_5r_8). \tag{3.10}$$

4. Pro-sets

As in Section 2 we use M to indicate any of the hypersurfaces in Table 1. The polynomials of normal sections of our isoparametric hypersurfaces M are defined in $T_E(M)$ where we have (in all cases) an orthogonal system of coordinates $\{x_1, \ldots, x_m\}$ and the polynomials are written is terms of these variables. As we mentioned above, the polynomials $P(X)$ defining $\widehat{X}_E[M]$ have degree 3 and the variables in each monomial have degree 1.

We want to indicate the presence of certain subsets of each set of variables which are (when they exist) particularly important to our objective of proving Theorem 1.1.

Definition 4.1. We shall say that a subset $A \subset \{x_1, \ldots, x_m\}$ is a *pro-set* for the polynomial $P(X)$ if each of its monomials has one and only one variable in the subset A.

In the next section we describe pro-sets for each $P(X)$ where they exist. Each pro-set A defines, obviously, a corresponding companion "subspace" $V(A) \subset (T_E(M) - \{0\})$ by vanishing the variables included in A:

$$V(A) = \{X \neq 0 \in T_E(M) : x_j(X) = 0,\ \forall x_j \in A\}. \tag{4.1}$$

Note that we are excluding $\{0\}$ and we call them "subspaces" of $(T_E(M) - \{0\})$. It is obvious that $V(A) \subset \Lambda \widehat{X}_E[M]$.

5. Description of pro-sets

We shall indicate *only two* pro-sets (when they exist) for each space even if there are others. As mentioned above, we have a companion "subspace" for each of them.

5.1. **In the spaces with $g = 3$.** By (3.3) for each $F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ we have two obvious pro-sets, namely $A_k = \{x_k\}$, $k = 1, 2$. Notice that A_k contains one real variable for $F = \mathbb{R}$, two for $F = \mathbb{C}$, four for $F = \mathbb{H}$, and eight for $F = \mathbb{O}$. We have the associated "subspaces" which are denoted by $V_k(M_F)$ for $k = 1, 2$. Clearly $\dim V_k(M_F) = 2, 4, 8, 16$, $k = 1, 2$, for $F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, respectively.

Let us denote by $\mathbb{S}(V_k(M_F))$ the unit sphere in $V_k(M_F)$. We notice that

$$\mathbb{S}(V_1(M_F)) \cap \mathbb{S}(V_2(M_F)) \supset \{X \in T_E(M_F) : \|x_3\|^2 = 1\} \simeq \mathbb{S}(F) \neq \emptyset,$$

and $\dim(\mathbb{S}(F)) = 0, 1, 3, 7$ ($F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$).

We must observe also that

$$V_1(M_F) + V_2(M_F) = (T_E(M_F) - \{0\}), \qquad F = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}. \tag{5.1}$$

5.2. **In the spaces with $g = 4$.**

5.2.1. *Spaces $W_{\mathbb{R}}$ and $W_{\mathbb{C}}$.* For $W_{\mathbb{R}} = \mathrm{SO}(5)/T^2$ we have the polynomial $P_{\mathbb{R}}(X)$ with variables $\{x_3, \ldots, x_{10}\}$. We find the pro-sets

$$A_1(W_{\mathbb{R}}) = \{x_7, x_8\}, \qquad A_2(W_{\mathbb{R}}) = \{x_9, x_{10}\},$$

and the associated "subspaces" $V_1(W_{\mathbb{R}})$ and $V_2(W_{\mathbb{R}})$. Here the dimension of $V_1(W_{\mathbb{R}})$ and $V_2(W_{\mathbb{R}})$ is 6. Then $\dim(V_1(W_{\mathbb{R}}) \cap V_2(W_{\mathbb{R}})) = 4$. Therefore $\mathbb{S}(V_1(W_{\mathbb{R}})) \cap \mathbb{S}(V_2(W_{\mathbb{R}})) \simeq \mathbb{S}^3$.

Similarly for $W_{\mathbb{C}}$ (recalling that $z_j = x_j + iy_j$) we find pro-sets

$$A_1(W_{\mathbb{C}}) = \{y_1, y_2, x_7, y_7, x_8, y_8\},$$
$$A_2(W_{\mathbb{C}}) = \{x_3, y_3, x_5, y_5, x_9, y_9, x_{10}, y_{10}\},$$

and the corresponding "subspaces" are $V_1(W_{\mathbb{C}})$ and $V_2(W_{\mathbb{C}})$. Then the dimension of $V_1(W_{\mathbb{C}})$ is 12 and that of $V_2(W_{\mathbb{C}})$ is 10. Again $\dim(V_1(W_{\mathbb{C}}) \cap V_2(W_{\mathbb{C}})) = 4$ and $\mathbb{S}(V_1(W_{\mathbb{C}})) \cap \mathbb{S}(V_2(W_{\mathbb{C}})) \simeq \mathbb{S}^3$.

We must observe also that, in both cases,

$$V_1(W_F) + V_2(W_F) = (T_E(M_F) - \{0\}), \qquad F = \mathbb{R}, \mathbb{C}. \tag{5.2}$$

5.2.2. *Spaces $N_{\mathbb{R}}$, $N_{\mathbb{C}}$, $N_{\mathbb{H}}$.* Note that here $n \geq 3$.

- For $N_{\mathbb{R}}$. Recalling (3.5) we see that for $N_{\mathbb{R}}$ we have $\alpha = \beta = 0$ and the whole set of variables is $\{c_{r,o}, b_{s,o},\ r = 1, \ldots, n-1,\ s = 2, \ldots, n\}$. Two pro-sets for $Q_{\mathbb{R}}(X)$ are

$$A_1(N_{\mathbb{R}}) = \{c_{1,o}, b_{n,o}\},$$
$$A_2(N_{\mathbb{R}}) = \{b_{r,o}\}_{2 \leq r \leq n-1},$$

and associated to them we have $V_1(N_{\mathbb{R}})$ and $V_2(N_{\mathbb{R}})$. Let us observe that $\dim V_1(N_{\mathbb{R}}) = 2n-4$ and $\dim V_2(N_{\mathbb{R}}) = n$. Also notice that

$$\mathbb{S}(V_1(N_{\mathbb{R}})) \cap \mathbb{S}(V_2(N_{\mathbb{R}})) \supset \left\{X \in T_E(N_{\mathbb{R}}) : \textstyle\sum_{s=2}^{n-1} c_{s,o}^2 = 1\right\} \simeq \mathbb{S}^{n-3}, \tag{5.3}$$

$$V_1(N_{\mathbb{R}}) + V_2(N_{\mathbb{R}}) = (T_E(N_{\mathbb{R}}) - \{0\}). \tag{5.4}$$

- For $N_{\mathbb{C}}$. We have the set of variables

$$\alpha = a_1 i, \quad \delta = d_1 i,$$
$$u_s = b_{s,o} + b_{s,1} i, \quad v_s = c_{s,o} + c_{s,1} i \quad s = 1, \ldots, n,$$

and two pro-sets for $Q_{\mathbb{C}}(X)$ are

$$A_1(N_{\mathbb{C}}) = \{c_{1,1}, b_{n,1}, v_r\}_{2 \leq r \leq n-1},$$
$$A_2(N_{\mathbb{C}}) = \{a_1, d_1, u_r\}_{2 \leq r \leq n-1}.$$

The associated "subspaces" are $V_1(N_{\mathbb{C}})$ and $V_2(N_{\mathbb{C}})$. We have

$$\mathbb{S}(V_1(N_{\mathbb{C}})) \cap \mathbb{S}(V_2(N_{\mathbb{C}})) \supset \left\{X \in T_E(N_{\mathbb{C}}) : |c_{1,o}|^2 + |b_{n,o}|^2 = 1\right\} \simeq \mathbb{S}^1,$$
$$V_1(N_{\mathbb{C}}) + V_2(N_{\mathbb{C}}) = (T_E(N_{\mathbb{C}}) - \{0\}). \tag{5.5}$$

- For $N_{\mathbb{H}}$. Looking at $Q_{\mathbb{H}}(X)$ and (3.5) we find

$$\begin{aligned} A_1(N_{\mathbb{H}}) &= \{\alpha, \delta, v_r\}_{2 \leq r \leq n-1}, \\ A_2(N_{\mathbb{H}}) &= \{\alpha, \delta, u_r\}_{2 \leq r \leq n-1}. \end{aligned} \tag{5.6}$$

They are pro-sets, but we notice that here we have a situation different from previous cases, that is

$$A_1(N_{\mathbb{H}}) \cap A_2(N_{\mathbb{H}}) = \{\alpha, \delta\}. \tag{5.7}$$

The corresponding "subspaces" are $V_1(N_{\mathbb{H}})$ and $V_2(N_{\mathbb{H}})$ and we observe that

$$\mathbb{S}(V_1(N_{\mathbb{H}})) \cap \mathbb{S}(V_2(N_{\mathbb{H}})) \supset \left\{X \in T_E(N_{\mathbb{H}}) : |v_1|^2 + |u_n|^2 = 1\right\} \simeq \mathbb{S}^7. \tag{5.8}$$

We have here another difference with the previous cases. Namely,

$$V_1(N_{\mathbb{H}}) + V_2(N_{\mathbb{H}}) \subsetneqq (T_E(M_F) - \{0\}). \tag{5.9}$$

This situation is responsible for the need of an *ad hoc* proof for this space.

- For $N_{(9,6)}$. The polynomial $P_{(9,6)}(X)$, when expanded in its real variables, has 252 monomials and a patient search into them shows that there are no pro-sets among its 30 variables.

5.3. **In the spaces with $g = 6$.** In $M_{\mathbb{B}}$, whose polynomial is (3.9), we have the pro-sets

$$A_1(M_{\mathbb{B}}) = \{r_3, r_4, r_9, r_{10}\}, \quad A_2(M_{\mathbb{B}}) = \{r_5, r_6, r_{11}, r_{12}\}$$

and corresponding "subspaces" $V_1(M_{\mathbb{B}})$ and $V_2(M_{\mathbb{B}})$. Clearly,

$$\mathbb{S}(V_1(M_{\mathbb{B}})) \cap \mathbb{S}(V_2(M_{\mathbb{B}})) \supset \left\{X \in T_E(M_{\mathbb{B}}) : r_7^2 + r_8^2 + r_{13}^2 + r_{14}^2 = 1\right\} \simeq \mathbb{S}^3.$$

Furthermore,

$$V_1(M_{\mathbb{B}}) + V_2(M_{\mathbb{B}}) = (T_E(M_{\mathbb{B}}) - \{0\}).$$

Similarly for $M_{\mathbb{S}}$:

$$A_1(M_{\mathbb{S}}) = \{r_3, r_4\}, \quad A_2(M_{\mathbb{S}}) = \{r_5, r_6\},$$

with "subspaces" $V_1(M_{\mathbb{B}})$ and $V_2(M_{\mathbb{B}})$. We have here

$$\mathbb{S}(V_1(M_{\mathbb{S}})) \cap \mathbb{S}(V_2(M_{\mathbb{S}})) \supset \left\{X \in T_E(M_{\mathbb{S}}) : r_7^2 + r_8^2 = 1\right\} \simeq \mathbb{S}^1,$$

and also

$$V_1(M_{\mathbb{S}}) + V_2(M_{\mathbb{S}}) = (T_E(M_{\mathbb{S}}) - \{0\}).$$

6. Proof of the theorem

This section contains the proof of Theorem 1.1.

6.1. **General case.** We shall do first the proof for the spaces in Table 1 different from $N_{\mathbb{H}}$ and $N_{(9,6)}$. We use the generic notation M for our manifold and let $\{x_1, \ldots, x_m\}$ be the orthogonal coordinates in $T_E(M)$ in which the polynomial $P_M(X)$ is written. We have determined two pro-sets $A_j(M)$ $(j = 1, 2)$ and corresponding "subspaces" $V_j(M)$. In all cases considered (those in Table 1 except $N_{\mathbb{H}}$ and $N_{(9,6)}$) we have

$$\begin{gathered} A_1(M) \cap A_2(M) = \emptyset \\ V_1(M) + V_2(M) = (T_E(M) - \{0\}) \\ \mathbb{S}(V_1(M)) \cap \mathbb{S}(V_2(M)) \neq \emptyset. \end{gathered} \tag{6.1}$$

Let us take now an *arbitrary point X in* $\Lambda\widehat{X}[M]$. We write it in terms of the coordinates as $X = (x_1, \ldots, x_m) \neq 0$; it satisfies $P_M(X) = 0$. Now, with the coordinates of X, we construct two new points in $T_E(M)$, namely

$$\begin{aligned} &Y\text{: coodinates of } X \text{ that are in } A_1(M)\text{, others 0;} \\ &Z\text{: coodinates of } X \text{ that are } \textit{not} \text{ in } A_1(M)\text{, others 0.} \end{aligned} \tag{6.2}$$

We have now three alternatives, namely

$$\begin{aligned} &((1)) \quad Y \neq 0 \text{ and } Z \neq 0, \\ &((2)) \quad Y = 0 \Longrightarrow Z = X \neq 0, \\ &((3)) \quad Z = 0 \Longrightarrow Y = X \neq 0. \end{aligned} \tag{6.3}$$

(The alternative $Y = 0 = Z$ is ruled out since $X \in \Lambda\widehat{X}[M]$.)

Let us assume first that we have the situation ((1)) in (6.3). We must observe that by definition and (6.1) we have $Z \in V_1(M)$ and $Y \in V_2(M)$, and also $\langle Y, Z\rangle = 0$. Let us take now the points

$$X(t) = (tY) + Z \in T_E(M), \qquad \forall t \in [0,1].$$

Of course $X(1) = X \in \Lambda\widehat{X}[M]$ and $X(0) = Z \in V_1(M) \subset \Lambda\widehat{X}[M]$. Also, by assumption ((1)) (6.3), $X(t) \neq 0$, $\forall t \in (0,1]$.

Since $A_1(M)$ is a pro-set, in *every monomial of* $P_M(X)$ *there is one and only one* variable in $A_1(M)$ and we see that

$$P_M(X(t)) = tP_M(X) = 0, \qquad \forall t \in (0,1].$$

Then we have that

$$X(t) \in \Lambda\widehat{X}[M], \qquad \forall t \in [0,1],$$

and therefore we have *a continuous curve* $X(t) \in \Lambda\widehat{X}_E[M]$, $\forall t \in [0,1]$, which joins the starting point $X \in \Lambda\widehat{X}_E[M]$ to the point $Z \in V_1(M) \subset \Lambda\widehat{X}[M]$.

So we have proved that any $X \in \Lambda\widehat{X}_E[M]$ *for which* ((1)) *of* (6.3) *holds* can be joined, by a continuous curve contained in $\Lambda\widehat{X}_E[M]$, to a point in $V_1(M) \subset \Lambda\widehat{X}[M]$.

We have to consider now the cases ((2)) and ((3)) in (6.3).

Assume ((2)). If $X = Z \in V_1(M)$ then it obviously can be joined (by a continuous curve contained in $V_1(M)$) to any other point in $V_1(M) \subset \Lambda\widehat{X}[M]$.

Assume ((3)). We have $X = Y \in V_2(M)$, then (as was shown for all the hypersurfaces $M \neq N_{\mathbb{H}}$, $N_{(9,6)}$) we have that $\mathbb{S}(V_1(M)) \cap \mathbb{S}(V_2(M)) \supset \mathbb{S}^p$ for some $p \geq 0$. Then we have two sets, connected by arcs, namely $V_1(M)$ and $V_2(M)$, with at least a point in common. Therefore any point $X \in V_2(M)$ can be joined (by a continuous curve in $\Lambda\widehat{X}_E[M]$) to any other in $V_1(M)$. This shows that $\Lambda\widehat{X}_E[M]$ *is connected by arcs*, and in turn so are $\widehat{X}_E[M]$ and $X[M]$.

We present now a somewhat different proof for $N_{\mathbb{H}}$.

6.2. **Proof for $N_{\mathbb{H}}$.** The reasons for taking this case separately are (5.7) and (5.9).

We take an arbitrary point X in $\Lambda\widehat{X}[N_{\mathbb{H}}]$; then

$$Q_{\mathbb{H}}(X) = 0, \qquad X \neq 0, \tag{6.4}$$

and write it in coordinates as

$$X = ((\alpha, u_2, \dots, u_n), (v_1, \dots, v_{n-1}, \delta)) \in T_E(N_{\mathbb{H}}),$$
$$u_j, v_j \in \mathbb{H}, \quad \alpha, \delta \text{ pure quaternions.}$$

Now (with the coordinates of X) we construct two new points in $T_E(M)$ as in (6.2) but with $A_1(N_{\mathbb{H}})$ (5.6) instead of $A_1(M)$. Then Z and Y are respectively of the form

$$Z = ((0, u_2, \dots, u_{n-1}, u_n), (v_1, 0 \dots, 0, 0)) \in V_1(N_{\mathbb{H}}) \subset \Lambda\widehat{X}[N_{\mathbb{H}}],$$
$$Y = ((\alpha, 0, \dots, 0, 0), (0, v_2 \dots, v_{n-1}, \delta)).$$

We may write X as $X = Y + Z$ and have again the three alternatives ((1)), ((2)) and ((3)) in (6.3).

Let us assume first that ((1)) holds. Then $Z \in V_1(N_{\mathbb{H}}) \subset \Lambda\widehat{X}_E[N_{\mathbb{H}}]$ but Y is *neither in* $V_1(N_{\mathbb{H}})$ *nor* $V_2(N_{\mathbb{H}})$ (*so it may not even be in* $\Lambda\widehat{X}_E[N_{\mathbb{H}}]$). However $\langle Y, Z\rangle = 0$.

Let us take again

$$X(t) = (tY) + Z, \qquad \forall t \in [0,1],$$

which in this case takes the form

$$X(t) = ((t\alpha, u_2 \ldots, u_n), (v_1, tv_2, \ldots, tv_{n-1}, t\delta)), \qquad t \in [0,1].$$

$X(1) = X \in \Lambda\widehat{X}_E[N_{\mathbb{H}}]$ and $X(0) = Z \in V_1(N_{\mathbb{H}}) \subset \Lambda\widehat{X}_E[N_{\mathbb{H}}]$.

Again, since $A_1(N_{\mathbb{H}})$ (5.6) is a pro-set, we see that

$$Q_{\mathbb{H}}(X(t)) = tQ_{\mathbb{H}}(X) = 0, \qquad \forall t \in (0,1].$$

Then $X(t) \in \Lambda\widehat{X}_E[N_{\mathbb{H}}]$, $\forall t \in [0,1]$, and we we have *a continuous curve*

$$X(t) \in \Lambda\widehat{X}_E[N_{\mathbb{H}}], \qquad \forall t \in [0,1]$$

which joins the starting point $X \in \Lambda\widehat{X}_E[N_{\mathbb{H}}]$ to the point $Z \in V_1(N_{\mathbb{H}}) \subset \Lambda\widehat{X}_E[N_{\mathbb{H}}]$. Also $Z \in V_1(N_{\mathbb{H}})$ can be joined to any other point in $V_1(N_{\mathbb{H}})$ (by a continuous curve contained there). So we have proved that any $X \in \Lambda\widehat{X}_E[N_{\mathbb{H}}]$ *such that* $Y \neq 0 \neq Z$ can be joined, by a continuous curve contained in $\Lambda\widehat{X}_E[N_{\mathbb{H}}]$, to any point in $V_1(N_{\mathbb{H}})$.

Now we have to study the other alternatives in (6.3).

Assume ((2)), i.e., $Y = 0$, $Z = X \neq 0$. Then there is nothing to prove since we already have $X \in V_1(N_{\mathbb{H}})$ and so it can be joined, by a curve in $V_1(N_{\mathbb{H}})$, to any other point there.

Assume ((3)), i.e., $Z = 0$. Then we have $Y = X \in \Lambda\widehat{X}_E[N_{\mathbb{H}}]$. In this situation the above procedure (multiplying by t) leads to 0. So we have to use a different approach.

Let us recall that we are assuming now

$$Y = X = ((\alpha, 0, \ldots, 0, 0), (0, v_2 \ldots, v_{n-1}, \delta)) \in \Lambda\widehat{X}_E[N_{\mathbb{H}}] \tag{6.5}$$

and

$$\|Y\|^2 = \|\alpha\|^2 + \|\delta\|^2 + \sum_{s=2}^{n-1} \|v_s\|^2.$$

Under assumption ((3)) we have a new alternative, namely

$$\begin{array}{ll} (3.1) & \|\alpha\|^2 + \|\delta\|^2 = 0, \\ (3.2) & \|\alpha\|^2 + \|\delta\|^2 \neq 0. \end{array} \tag{6.6}$$

If we have (3.1) then clearly $Y = X \in V_2(N_{\mathbb{H}})$ and, since (5.8) holds, we can join $Y = X$ to any point in $V_1(N_{\mathbb{H}})$.

We *may assume from now on that* (3.2) *holds.* Then we have a new alternative:

$$\begin{aligned} &(3.2.1) \quad \sum_{s=2}^{n-1} \|v_s\|^2 \neq 0, \\ &(3.2.2) \quad \sum_{s=2}^{n-1} \|v_s\|^2 = 0, \end{aligned} \tag{6.7}$$

and study, separately, both situations.

Let us assume first (3.2.1). Since $Y = X$, we have $Q_{\mathbb{H}}(Y) = Q_{\mathbb{H}}(X) = 0$ and by (6.5) and (3.5), for our Y, we have

$$b_{r,s} = 0, \qquad 2 \le r \le n \text{ and } 0 \le s \le 3.$$

Then we may eliminate, from the polynomial $Q_{\mathbb{H}}$, the terms containing these variables. By doing this we get

$$\begin{aligned} 0 = Q_{\mathbb{H}}(Y) = &\left(t_1 c_{1,o}\right)\left(a_1 c_{1,1} + a_2 c_{1,2} + a_3 c_{1,3}\right) \\ &+ \left(-t_1 c_{1,1}\right)\left(c_{1,o} a_1 - c_{1,3} a_2 + c_{1,2} a_3\right) \\ &+ \left(-t_1 c_{1,2}\right)\left(c_{1,3} a_1 + c_{1,o} a_2 - c_{1,1} a_3\right) \\ &+ \left(-t_1 c_{1,3}\right)\left(-c_{1,2} a_1 + c_{1,1} a_2 + c_{1,o} a_3\right). \end{aligned} \tag{6.8}$$

Now we take

$$Y(t) = \left(\left((t\alpha), 0, \ldots, 0, 0\right), \left(0, v_2 \ldots, v_{n-1}, (t\delta)\right)\right), \qquad \forall t \in [0,1].$$

Then, considering (6.8), we see that

$$Q_{\mathbb{H}}(Y(t)) = t\left(Q_{\mathbb{H}}(Y)\right) = 0, \qquad \forall t \in [0,1].$$

Then, in the same way as before, we can join $Y = X \in \Lambda\widehat{X}_E[N_{\mathbb{H}}]$, by a continuous curve contained in $\Lambda\widehat{X}_E[N_{\mathbb{H}}]$, to a point of the form

$$H = \left((0, 0, \ldots, 0, 0), (0, v_2 \ldots, v_{n-1}, 0)\right).$$

This H is not zero, due to (3.2.1) in (6.7), and $H \in V_2(N_{\mathbb{H}})$ but is not contained in $V_1(N_{\mathbb{H}}) \cap V_2(N_{\mathbb{H}})$. However (by (5.8)) we can, in turn, join H to any point in $V_1(N_{\mathbb{H}})$ by a continuous curve contained in $\Lambda\widehat{X}_E[N_{\mathbb{H}}]$.

It remains to consider the case (3.2.2) in (6.7) (we still have $Y = X \in \Lambda\widehat{X}_E[N_{\mathbb{H}}]$). If (3.2.2) holds then Y is of the form

$$Y = X = \left((\alpha, 0, \ldots, 0, 0), (0, 0 \ldots, 0, \delta)\right) \in \Lambda\widehat{X}_E[N_{\mathbb{H}}]. \tag{6.9}$$

We must show that also in this case we can find a continuous curve in $\Lambda\widehat{X}_E[N_{\mathbb{H}}]$ joining Y to one point in $V_1(N_{\mathbb{H}})$.

Let us consider an extra point C of the form

$$C = \left((0, 0, \ldots, 0, u_n), (v_1, 0, \ldots, 0, 0)\right) \in V_1(N_{\mathbb{H}}) \cap V_2(N_{\mathbb{H}})$$

defined as follows. Recalling the notation (3.5), we may take u_n and v_1 *real.* That is,

$$u_n = b_{n,o}, \qquad v_1 = c_{1,o}, \qquad b_{n,q} = 0 = c_{1,q}, \quad 2 \le q \le 3.$$

Furthermore, calling $\lambda^2 = \|\alpha\|^2 + \|\delta\|^2 = \|Y\|^2$ (we may take $\lambda > 0$ since we are in (3.2.2)) we take

$$|v_1|^2 + |u_n|^2 = c_{1,o}^2 + b_{n,o}^2 = \lambda^2.$$

Then $\langle C, Y \rangle = 0$ and by definition

$$\frac{1}{\lambda} C \in \mathbb{S}\left(V_1\left(N_{\mathbb{H}}\right)\right) \cap \mathbb{S}\left(V_2\left(N_{\mathbb{H}}\right)\right).$$

Let us consider now in $\mathbb{S}\left(T_E\left(N_{\mathbb{H}}\right)\right)$ the curve

$$\Omega(\theta) = \cos(\theta)\frac{C}{\lambda} + \sin(\theta)\frac{Y}{\lambda}, \quad \theta \in \left[0, \frac{\pi}{2}\right]$$

$$\lambda\Omega(\theta) = \left(\left(\sin(\theta)\alpha, 0, \ldots, 0, \cos(\theta) b_{n,o}\right), \left(\cos(\theta) c_{1,o}, 0, \ldots, 0, \sin(\theta)\delta\right)\right).$$

Now, a glance at $Q_{\mathbb{H}}$ shows that we have

$$Q_{\mathbb{H}}\left(\lambda\Omega(\theta)\right) = 0, \quad \forall\theta \in \left[0, \frac{\pi}{2}\right],$$

and so any point of the form (6.9) can be joined to a point in the intersection $V_1\left(N_{\mathbb{H}}\right) \cap V_2\left(N_{\mathbb{H}}\right)$ by a continuous curve in $\Lambda\widehat{X}\left[N_{\mathbb{H}}\right]$. This finally proves that $\Lambda\widehat{X}\left[N_{\mathbb{H}}\right]$ is connected by arcs.

6.3. **The case $N_{(9,6)}$.** We can compute the shape operator A_Ω on $T_E\left(N_{(9,6)}\right)$ and obtain the eigenspaces. There are two of dimension 9 and two of dimension 6. It is now convenient to set the following notation: we write a quaternion $q = q_0 + iq_1 + jq_2 + kq_3$ as

$$q = q_0 + Iq, \qquad Iq = iq_1 + jq_2 + kq_3. \tag{6.10}$$

The eigenspaces are:

$$Q_1 = \left\{X \in T_E\left(N_{(9,6)}\right) : -u_{4,0} = v_{7,0}, -u_{2,0} = v_{5,0}, u_{3,0} = v_{8,0},\right.$$
$$\left. Ia_4 = Ib_7, Ia_2 = Ib_5, \text{others} = 0\right\},$$

$$Q_2 = \left\{X \in T_E\left(N_{(9,6)}\right) : u_{4,0} = v_{7,0}, u_{2,0} = v_{5,0}, -u_{3,0} = v_{8,0},\right.$$
$$\left. -Ia_4 = Ib_7, -Ia_2 = Ib_5, \text{others} = 0\right\}$$

$$W_1 = \left\{X \in T_E\left(N_{(9,6)}\right) : \alpha, Ia_3, \text{others} = 0\right\}, \tag{6.11}$$

$$W_2 = \left\{X \in T_E\left(N_{(9,6)}\right) : \beta, Ib_8, \text{others} = 0\right\}. \tag{6.12}$$

Clearly we have $\dim Q_j = 9$ and $\dim W_j = 6$ for $j = 1, 2$.

Our interest in these subspaces comes from the fact that they vanish the polynomial $P_{(9,6)}(X)$. This is a general fact [9, Proposition 4.1] but can be checked directly with $P_{(9,6)}(X)$ and is obvious for the subspaces W_1 and W_2. We consider their direct sums, which are:

space	dimension
$Q_1 \oplus Q_2$	18
$W_1 \oplus W_2$	12
$Q_r \oplus W_s$, $1 \leq r, s \leq 2$	15

because they also vanish the polynomial $P_{(9,6)}(X)$ [9, Corollary 4.2] (again this can be verified by direct computation). We shall use $\Lambda(W_1 \oplus W_2)$, $\Lambda(Q_1 \oplus Q_2)$ and $\Lambda(Q_r \oplus W_s)$ to indicate the set of non-zero vectors in these subspaces.

The subspace $Q_1 \oplus Q_2$ consists of the nine 2-dimensional planes (the other variables zero)

$$\begin{aligned}&(v_{5,k}, u_{2,k}), \quad 0 \le k \le 3\\ &(v_{7,h}, u_{4,h}), \quad 0 \le h \le 3\\ &(v_{8,0}, u_{3,0}).\end{aligned} \tag{6.13}$$

As indicated in the Appendix, the polynomial $P_{(9,6)}(X)$ splits as a sum (8.1). Let us note that $\Omega_{(9,6)}(X) = P_{(9,6)}(X)\big|_{(Q_1 \oplus Q_2)}$. Note also that we may consider $\Omega_{(9,6)}(X)$ *defined in the whole space* $T_E(N_{(9,6)})$ since the other 12 variables do not appear in this polynomial. But since $P_{(9,6)}(X)$ vanishes on $X \in \Lambda(Q_1 \oplus Q_2)$ we have the following important fact:

$$\Omega_{(9,6)} \equiv 0 \quad \text{on } T_E\left(N_{(9,6)}\right).$$

Therefore the polynomial $P_{(9,6)}(X)$ reduces to $\Theta_{(9,6)}(X)$ on the tangent space $T_E\left(N_{(9,6)}\right)$, that is,

$$P_{(9,6)}(X) = \Theta_{(9,6)}(X).$$

Let us consider now the 2-dimensional plane

$$(v_{8,0}, u_{3,0}) \tag{6.14}$$

and take new orthogonal coordinates in it. We have a line that vanishes the factor $(t_1 v_{8,0} - t_6 u_{3,0})$, that is

$$(t_1 v_{8,0} - t_6 u_{3,0}) = 0 \iff v_{8,0} = \frac{t_6}{t_1} u_{3,0} \iff (v_{8,0}, u_{3,0}) = u_{3,0}\left(\frac{t_6}{t_1}, 1\right),$$

and by taking the orthogonal vector $\left(1, -\frac{t_6}{t_1}\right)$ we may set new orthogonal coordinates in the plane (6.14) as follows:

$$\begin{bmatrix} v_{8,0} \\ u_{3,0} \end{bmatrix} = \begin{bmatrix} \frac{t_6}{t_1} & 1 \\ 1 & -\frac{t_6}{t_1} \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} =: \begin{bmatrix} \frac{1}{t_1} t_6 y + x \\ y - \frac{1}{t_1} t_6 x \end{bmatrix}.$$

By replacing the new variables in the factor $(t_1 v_{8,0} - t_6 u_{3,0})$ (which is the only place in $\Theta_{(9,6)}(X)$ where the variables $(v_{8,0}, u_{3,0})$ appear) we obtain

$$\begin{aligned}(t_1 v_{8,0} - t_6 u_{3,0}) &= \left(t_1\left(\frac{1}{t_1} t_6 y + x\right) - t_6\left(y - \frac{1}{t_1} t_6 x\right)\right) =: \frac{1}{t_1}\left(t_1^2 + t_6^2\right) x\\ &= \frac{1}{t_1} x,\end{aligned}$$

and we may replace this into $\Theta_{(9,6)}(X)$ getting

$$\begin{aligned}\Theta_{(9,6)}(X) = {} & (t_1 v_{5,0} + t_6 u_{2,0})\,\Phi_1 + (-t_1 v_{5,1} + t_6 u_{2,1})\,\Phi_2 \\ & + (-t_1 v_{5,2} + t_6 u_{2,2})\,\Phi_3 + (-t_1 v_{5,3} + t_6 u_{2,3})\,\Phi_4 \\ & + \frac{1}{t_1} x \left[\langle \alpha, b_8 \rangle - \langle a_3, \beta \rangle\right] \\ & + (-t_1 v_{7,1} + t_6 u_{4,1})\,\Phi_6 + (-t_1 v_{7,2} + t_6 u_{4,2})\,\Phi_7 \\ & + (-t_1 v_{7,3} + t_6 u_{4,3})\,\Phi_8 + (-t_1 v_{7,0} - t_6 u_{4,0})\,\Phi_9.\end{aligned} \tag{6.15}$$

It is important to observe that the polynomial $\Theta_{(9,6)}$ does not depend on the variable y. However this variable must be considered.

6.4. **Proof for $N_{(9,6)}$.** We have in (3.6) and (3.7) the variables that we considered in the expression of $P_{(9,6)}(X)$. Now we may use the new set of variables

$$\{x, y, \alpha, \beta, a_2, Ia_3, a_4, b_5, b_7, Ib_8\}, \tag{6.16}$$

where we use (6.10), the new variables (x, y) and the old variables retain its meaning in (3.7).

We divide the new set of variables into the disjoint sets

$$\begin{gathered}(x, y, 0, 0, a_2, 0, a_4, b_5, b_7, 0) \in Q_1 \oplus Q_2, \\ (0, \alpha, \beta, 0, Ia_3, 0, 0, 0, Ib_8) \in W_1 \oplus W_2, \\ Q_1 \oplus Q_2 \oplus W_1 \oplus W_2 = T_E\left(N_{(9,6)}\right).\end{gathered}$$

Let us take now an arbitrary point $X_0 \in \Lambda\widehat{X}\left[N_{(9,6)}\right]$ (then $X_0 \neq 0$), which we may write, using the new variables, as

$$X_0 = (x, y, \alpha, \beta, a_2, Ia_3, a_4, b_5, b_7, Ib_8) : \Theta_{(9,6)}(X_0) = 0.$$

As before, we may write $X_0 = Y + Z$, where

$$\begin{aligned}Y &= (x, y, 0, 0, a_2, 0, a_4, b_5, b_7, 0) \in Q_1 \oplus Q_2, \\ Z &= (0, 0, \alpha, \beta, 0, Ia_3, 0, 0, 0, Ib_8) \in W_1 \oplus W_2,\end{aligned}$$

and we have again the alternative (6.3).

We assume first ((1)) of (6.3), that is, $Y \neq 0 \neq Z$. Under this assumption we divide our considerations into two possible cases, namely $x = 0$ and $x \neq 0$.

First case, $x = 0$. If the point $X_0 \in \Lambda\widehat{X}\left[N_{(9,6)}\right]$ has the form

$$X_0 = (0, y, \alpha, \beta, a_2, Ia_3, a_4, b_5, b_7, Ib_8), \quad \Theta_{(9,6)}(X_0) = 0;$$

then $\Theta_{(9,6)}(X_0)$ reduces to

$$\begin{aligned}\Theta_{(9,6)}(X_0) = {} & (t_1 v_{5,0} + t_6 u_{2,0})\,\Phi_1 + (-t_1 v_{5,1} + t_6 u_{2,1})\,\Phi_2 \\ & + (-t_1 v_{5,2} + t_6 u_{2,2})\,\Phi_3 + (-t_1 v_{5,3} + t_6 u_{2,3})\,\Phi_4 \\ & + (-t_1 v_{7,1} + t_6 u_{4,1})\,\Phi_6 + (-t_1 v_{7,2} + t_6 u_{4,2})\,\Phi_7 \\ & + (-t_1 v_{7,3} + t_6 u_{4,3})\,\Phi_8 + (-t_1 v_{7,0} - t_6 u_{4,0})\,\Phi_9,\end{aligned}$$

and we consider the points

$$X(s) = (0, y, s\alpha, s\beta, a_2, sIa_3, a_4, b_5, b_7, sIb_8), \quad s \in [0,1].$$

Since all the factors Φ_j *are linear in the variables* $\{\alpha, \beta, Ia_3, Ib_8\}$ we see that, for every $s \in (0,1]$, we have the equality

$$\Theta_{(9,6)}(X(s)) = s\Theta_{(9,6)}(X_0) = 0$$

and hence (since $Y \in \Lambda(Q_1 \oplus Q_2) \subset \Lambda\widehat{X}\left[N_{(9,6)}\right]$) we have that

$$X(s) \in \Lambda\widehat{X}\left[N_{(9,6)}\right], \quad \forall s \in [0,1].$$

Second case, $x \neq 0$. In this case the point $X_0 \in \widehat{X}\left[N_{(9,6)}\right]$ has the form

$$X_0 = (x, y, \alpha, \beta, a_2, Ia_3, a_4, b_5, b_7, Ib_8), \quad x \neq 0, \quad \Theta_{(9,6)}(X_0) = 0.$$

We take now the points

$$X(s) = \left(\left(s^2\right)x, y, \alpha, \beta, sa_2, Ia_3, sa_4, sb_5, sb_7, Ib_8\right), \quad s \in [0,1],$$

and we see that each one of the nine terms of $\Theta_{(9,6)}(X(s))$ in (6.15) has a factor s^2 (because each term that does not contain x has two factors, a parenthesis and a bracket and each one of these is linear in the variables multiplied by s; on the other hand, the variables in the factor companion of x are not multiplied by s).

Hence we have

$$\Theta_{(9,6)}(X(s)) = \left(s^2\right)\Theta_{(9,6)}(X_0) = 0, \quad s \in (0,1],$$

which again yields

$$X(s) \in \Lambda\widehat{X}\left[N_{(9,6)}\right], \quad \forall s \in (0,1],$$

and in turn (since $Z = X(0) \in \Lambda(W_1 \oplus W_2) \subset \Lambda\widehat{X}\left[N_{(9,6)}\right]$) we have again

$$X(s) \in \Lambda\widehat{X}\left[N_{(9,6)}\right], \quad \forall s \in [0,1].$$

Now we have to consider the other two possibilities, namely ((2)) and ((3)) of (6.3). Then X_0 is a point in either $\Lambda(W_1 \oplus W_2)$ or $\Lambda(Q_1 \oplus Q_2)$. But now by means of

$$\Lambda(Q_r \oplus W_s), \quad 1 \leq r, s \leq 2,$$

we can go between any point in $\Lambda(Q_1 \oplus Q_2)$ and any other in $\Lambda(W_1 \oplus W_2)$ by a continuous curve in $\Lambda\widehat{X}\left[N_{(9,6)}\right]$. This completes the proof of Theorem 1.1. □

7. Consequence for $\Xi(M)$

Given a point p in the isoparametric submanifold M we have the algebraic set $\widehat{X}_p[M]$ and by considering (as in [9]) this set for each point in M we obtain a subset of the unit tangent bundle of M which we have denoted by $\Xi(M)$. The topology of $\Xi(M)$ is the one induced from the unit tangent bundle $\mathbb{S}(T(M))$ of M.

The objective of this section is to show that for the submanifolds M in Table 1 the set $\Xi(M)$ is also connected by arcs.

Theorem 7.1. *For all the homogeneous isoparametric hypersurfaces* $M^n \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ *(those in Table 1), the set* $\Xi(M) \subset \mathbb{S}(T(M))$ *is connected by arcs.*

Proof. Let M be such a submanifold. Then the tangent bundle splits as a direct sum

$$T(M) = D_1 \oplus \cdots \oplus D_g$$

of the simultaneous eigenspaces of the shape operators (which commute because the normal bundle is flat). The distributions D_j are autoparallel and hence integrable with totally geodesic leaves which are round spheres.

To prove that $\Xi(M)$ is connected by arcs we use Theorem D in [4], which says that any two points, say p and q, in M can be joined by a piecewise differentiable curve in M whose differentiable pieces are tangent to one of the D_j. In Theorem D we take $I = \{1, \ldots, g\}$; then the ψ_i $(i \in I)$ generate $\widetilde{W}$, which is the hypothesis of that theorem (see [12] for details).

We take two arbitrary points p and q in M. The piecewise differentiable curve γ in M joining points p and q in M given by Theorem D in [4], can be taken to be

$$\gamma : [0, b] \longrightarrow M, \qquad \gamma(0) = p,\ \gamma(b) = q,$$

and the interval $[0, b]$ has a partition

$$0 < s_1 < s_2 < \cdots < s_{h-1} < s_h = b$$

such that $\gamma|_{[s_j, s_{j+1}]}$ is a *geodesic in one of the spheres* integrating one of the distributions D_k for each $j = 0, \ldots, h-1$, and we may assume that the images of two consecutive subintervals $[s_j, s_{j+1}]$ belong to *different* spheres (otherwise we may take a single geodesic joining the initial point of the first piece and the final point of the second one).

At each point $\gamma(s_j)$, $j = 1, \ldots, h-1$, we have two vectors in $T_{\gamma(s_j)}(M)$, the *left* and *right* derivatives of γ at s_j which are orthogonal (since they belong to different D_j at $\gamma(s_j)$). Let us denote these two derivatives by

$$\gamma'(s_j(-)) \quad \text{and} \quad \gamma'(s_j(+)), \qquad j = 1, \ldots, h-1.$$

We have also the derivatives at the two extremes of the interval $[0, b]$, that is

$$\gamma'(0(+)) \quad \text{and} \quad \gamma'(b(-)).$$

Let us take now two arbitrary points $v_p \in \widehat{X}_p[M]$ and $w_q \in \widehat{X}_q[M]$ in $\Xi(M)$ for some p and q in M. Let γ be the curve described above (joining p and q) given by Theorem D.

Since $\widehat{X}_p[M]$ is arc-wise connected, we can join v_p to $\gamma'(0(+))$ by a continuous curve in $\widehat{X}_p[M]$. Outside the singular set, that is, if a point t is in $([0, b] - \{s_1, s_2, \ldots, s_{h-1}\})$, then $\gamma'(t) \in \widehat{X}_{\gamma(t)}[M]$. Now at each $\{s_1, s_2, \ldots, s_{h-1}\}$ we have the two orthogonal derivatives $\gamma'(s_j(-))$ and $\gamma'(s_j(+))$, $j = 1, \ldots, h-1$, and they satisfy

$$\begin{aligned} &\gamma'(s_j(-)) \in \mathbb{S}(D_r(\gamma(s_j))), \quad \text{for some } 1 \le r \le g, \\ &\gamma'(s_j(+)) \in \mathbb{S}(D_u(\gamma(s_j))), \quad \text{for some } 1 \le u \le g, \\ &u \ne r. \end{aligned}$$

Since, by [9, p. 45, Cor. 4.2], we have that

$$\mathbb{S}\left(D_r\left(\gamma\left(s_j\right)\right) \oplus D_u\left(\gamma\left(s_j\right)\right)\right) \subset \widehat{X}_{\gamma(s_j)}\left[M\right],$$

we can join $\gamma'(s_j(-))$ and $\gamma'(s_j(+))$ by a continuous curve in $\widehat{X}_{\gamma(s_j)}\left[M\right]$.

When we reach the final point $\gamma(b)$ we may join $\gamma'(b(-)) \in \widehat{X}_q[M]$ to $w_q \in \widehat{X}_q[M]$ by a continuous curve in $\widehat{X}_q[M]$.

Then we can join v_p to w_q by a continuous curve in $\Xi\left(M\right)$. □

8. Appendix

With the coordinates in (3.7) the expression of the polynomial $P_{(9,6)}\left(X\right)$ is given under (3.7). It has nine terms and each of them consists of two factors (one between parentheses and the other between brackets). We want to split each factor *between brackets* into two terms, placing in the first one the terms containing $u_{3,0}$ and $v_{8,0}$ and lumping the rest of them into Φ_j. We write the bracket from each term (indicated by the order in the polynomial) as follows.

The first bracket is

$$(b1) \quad \left[\langle\alpha, b_5\rangle + \langle a_2, \beta\rangle + \langle a_3, b_7\rangle + \langle a_4, b_8\rangle\right] = \left[u_{3,0}v_{7,0} + u_{4,0}v_{8,0}\right] + \Phi_1,$$

where, as indicated above,

$$\begin{aligned}\Phi_1 = \langle\alpha, b_5\rangle + \langle a_2, \beta\rangle &+ \left(u_{3,1}v_{7,1} + u_{3,2}v_{7,2} + u_{3,3}v_{7,3}\right)\\ &+ \left(u_{4,1}v_{8,1} + u_{4,2}v_{8,2} + u_{4,3}v_{8,3}\right).\end{aligned}$$

We continue similarly with the brackets in the other eight terms:

$$\begin{aligned}(b2) \quad &\left[\langle\alpha, ib_5\rangle + \langle a_2, i\beta\rangle - \langle a_3, ib_7\rangle - \langle a_4, ib_8\rangle\right]\\ &\qquad = \left[-\left(u_{3,0}\left(-v_{7,1}\right)\right) - \left(u_{4,1}\left(v_{8,0}\right)\right)\right] + \Phi_2,\end{aligned}$$

$$\begin{aligned}\Phi_2 = \langle\alpha, ib_5\rangle + \langle a_2, i\beta\rangle &- \left(u_{3,1}v_{7,0} - u_{3,2}v_{7,3} + u_{3,3}v_{7,2}\right)\\ &- \left(-u_{4,0}v_{8,1} + u_{4,3}v_{8,2} - u_{4,2}v_{8,3}\right);\end{aligned}$$

$$\begin{aligned}(b3) \quad &\left[\langle\alpha, jb_5\rangle + \langle a_2, j\beta\rangle - \langle a_3, jb_7\rangle - \langle a_4, jb_8\rangle\right]\\ &\qquad = \left[-\left(u_{3,0}\left(-v_{7,2}\right)\right) - \left(u_{4,2}\left(v_{8,0}\right)\right)\right] + \Phi_3,\end{aligned}$$

$$\begin{aligned}\Phi_3 = \langle\alpha, jb_5\rangle + \langle a_2, j\beta\rangle &- \left(u_{3,1}v_{7,3} + u_{3,2}v_{7,0} - u_{3,3}v_{7,1}\right)\\ &- \left(-u_{4,3}v_{8,1} - u_{4,0}v_{8,2} + u_{4,1}v_{8,3}\right);\end{aligned}$$

$$\begin{aligned}(b4) \quad &\left[\langle\alpha, kb_5\rangle + \langle a_2, k\beta\rangle - \langle a_3, kb_7\rangle - \langle a_4, kb_8\rangle\right]\\ &\qquad = \left[-\left(u_{3,0}\left(-v_{7,3}\right)\right) - \left(u_{4,3}\left(v_{8,0}\right)\right)\right] + \Phi_4,\end{aligned}$$

$$\begin{aligned}\Phi_4 = \langle\alpha, kb_5\rangle + \langle a_2, k\beta\rangle &- u_{3,1}\left(-v_{7,2}\right) + u_{3,2}\left(v_{7,1}\right) + u_{3,3}\left(v_{7,0}\right)\\ &- \left(u_{4,0}\left(-v_{8,3}\right) + u_{4,1}\left(-v_{8,2}\right) + u_{4,2}\left(v_{8,1}\right)\right).\end{aligned}$$

The fifth term

$$(b5) \quad U = \left(t_1v_{8,0} - t_6u_{3,0}\right)\left[\langle\alpha, b_8\rangle + \langle a_2, b_7\rangle - \langle a_3, \beta\rangle - \langle a_4, b_5\rangle\right]$$

does not contain $u_{3,0}$ and $v_{8,0}$ in the bracket so we split this one as

$$U=(t_1v_{8,0}-t_6u_{3,0})\left[\langle a_2,b_7\rangle-\langle a_4,b_5\rangle\right]+(t_1v_{8,0}-t_6u_{3,0})\left[\langle \alpha,b_8\rangle-\langle a_3,\beta\rangle\right].$$

We continue the splitting with the previous procedure:

(b6) $$[\langle \alpha,b_7i\rangle+\langle a_2,b_8i\rangle+\langle a_3,b_5i\rangle+\langle a_4,\beta i\rangle] = [+(u_{2,1}v_{8,0})+(u_{3,0}(-v_{5,1}))]+\Phi_6,$$

$$\Phi_6=\langle \alpha,b_7i\rangle+\langle a_4,\beta i\rangle+(u_{3,1}(v_{5,0})+u_{3,2}(v_{5,3})+u_{3,3}(-v_{5,2})) + (u_{2,0}(-v_{8,1})+u_{2,2}(v_{8,3})+u_{2,3}(-v_{8,2}));$$

(b7) $$[\langle \alpha,b_7j\rangle+\langle a_2,b_8j\rangle+\langle a_3,b_5j\rangle+\langle a_4,\beta j\rangle] = [+(u_{2,2}(v_{8,0}))+(u_{3,0}(-v_{5,2}))]+\Phi_7,$$

$$\Phi_7=\langle \alpha,b_7j\rangle+\langle a_4,\beta j\rangle+(u_{3,1}(-v_{5,3})+u_{3,2}(v_{5,0})+u_{3,3}(v_{5,1})) + ((u_{2,0}(-v_{8,2})+u_{2,1}(-v_{8,3})+u_{2,3}(v_{8,1})));$$

(b8) $$[\langle \alpha,b_7k\rangle+\langle a_2,b_8k\rangle+\langle a_3,b_5k\rangle+\langle a_4,\beta k\rangle] = [+(u_{2,3}(v_{8,0}))+(u_{3,0}(-v_{5,3}))]+\Phi_8,$$

$$\Phi_8=\langle \alpha,b_7k\rangle+\langle a_4,\beta k\rangle+(u_{3,1}(v_{5,2})+u_{3,2}(-v_{5,1})+u_{3,3}(v_{5,0})) + (u_{2,0}(-v_{8,3})+u_{2,1}(v_{8,2})+u_{2,2}(-v_{8,1}));$$

(b9) $$[-\langle \alpha,b_7\rangle+\langle a_2,b_8\rangle+\langle a_3,b_5\rangle-\langle a_4,\beta\rangle] = [+(u_{2,0}v_{8,0})+(u_{3,0}(v_{5,0}))]+\Phi_9,$$

$$\Phi_9=-\langle \alpha,b_7\rangle-\langle a_4,\beta\rangle+(u_{3,1}(v_{5,1})+u_{3,2}(v_{5,2})+u_{3,3}(v_{5,3})) + (u_{2,1}(v_{8,1})+u_{2,2}(v_{8,2})+u_{2,3}(v_{8,3})).$$

With this procedure we may write

$$P_{(9,6)}(X)=\Omega_{(9,6)}(X)+\Theta_{(9,6)}(X), \tag{8.1}$$

where

$$\begin{aligned}\Omega_{(9,6)}(X)=&(t_1v_{5,0}+t_6u_{2,0})[u_{3,0}v_{7,0}+u_{4,0}v_{8,0}]\\&+(-t_1v_{5,1}+t_6u_{2,1})[-(u_{3,0}(-v_{7,1}))-(u_{4,1}(v_{8,0}))]\\&+(-t_1v_{5,2}+t_6u_{2,2})[-(u_{3,0}(-v_{7,2}))-(u_{4,2}(v_{8,0}))]\\&+(-t_1v_{5,3}+t_6u_{2,3})[-(u_{3,0}(-v_{7,3}))-(u_{4,3}(v_{8,0}))]\\&+(t_1v_{8,0}-t_6u_{3,0})[\langle a_2,b_7\rangle-\langle a_4,b_5\rangle]\\&+(-t_1v_{7,1}+t_6u_{4,1})[(u_{2,1}v_{8,0})+(u_{3,0}(-v_{5,1}))]\\&+(-t_1v_{7,2}+t_6u_{4,2})[(u_{2,2}(v_{8,0}))+(u_{3,0}(-v_{5,2}))]\\&+(-t_1v_{7,3}+t_6u_{4,3})[(u_{2,3}(v_{8,0}))+(u_{3,0}(-v_{5,3}))]\\&+(-t_1v_{7,0}-t_6u_{4,0})[(u_{2,0}v_{8,0})+(u_{3,0}(v_{5,0}))].\end{aligned}$$

and

$$\begin{aligned}\Theta_{(9,6)}(X) = &(t_1 v_{5,0} + t_6 u_{2,0})\,\Phi_1 + (-t_1 v_{5,1} + t_6 u_{2,1})\,\Phi_2 \\ &+ (-t_1 v_{5,2} + t_6 u_{2,2})\,\Phi_3 + (-t_1 v_{5,3} + t_6 u_{2,3})\,\Phi_4 \\ &+ (t_1 v_{8,0} - t_6 u_{3,0})\,[\langle \alpha, b_8\rangle - \langle a_3, \beta\rangle] \\ &+ (-t_1 v_{7,1} + t_6 u_{4,1})\,\Phi_6 + (-t_1 v_{7,2} + t_6 u_{4,2})\,\Phi_7 \\ &+ (-t_1 v_{7,3} + t_6 u_{4,3})\,\Phi_8 + (-t_1 v_{7,0} - t_6 u_{4,0})\,\Phi_9.\end{aligned}$$

Acknowledgement

The referee's comments lead to an improvement in the presentation of the paper. The author is very grateful.

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Received: August 4, 2015
Accepted: April 11, 2016