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VANISHING, BASS NUMBERS, AND COMINIMAXNESS OF LOCAL COHOMOLOGY MODULES

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ABSTRACT. Let (R, m) be a commutative Noetherian regular local ring and I be a proper ideal of R. It is shown that $H^{d-1}_{\mathfrak{p}}(R) = 0$ for any prime ideal $\mathfrak{p}$ of R with $\dim(R/\mathfrak{p}) = 2$, whenever the set $\{n \in \mathbb{N} : R/\mathfrak{p}^{(n)}$ is Cohen–Macaulay$\}$ is infinite. Now, let (R, m) be a commutative Noetherian unique factorization local domain of dimension d, I an ideal of R, and M a finitely generated R-module. It is shown that the Bass numbers of the R-module $H^i_I(M)$ are finite, for all integers $i \geq 0$, whenever height$(I) = 1$ or $d \leq 3$.

1. Introduction

Throughout this paper, let R denote a commutative Noetherian local ring (with identity), I a proper ideal of R, and M an R-module. The local cohomology modules $H^i_I(M)$ arise as the derived functors of the left exact functor $\Gamma_I(-)$, where for an R-module M, $\Gamma_I(M)$ is the submodule of M consisting of all elements annihilated by some power of I, i.e., $\bigcup_{n=1}^{\infty}(0 :_M I^n)$. There is a natural isomorphism:

$$H^i_I(M) = \varinjlim_{n\geq 1} \operatorname{Ext}^i_R(R/I^n, M).$$

It is well-known that if $(R, \mathfrak{m})$ is a regular local ring of dimension $d > 0$ then the top local cohomology module $H^d_{\mathfrak{m}}(R)$ is not a finitely generated R-module. But for each $i \geq 0$ and each finitely generated module over an arbitrary Noetherian local ring $(R, \mathfrak{m})$ the R-module $H^i_{\mathfrak{m}}(M)$ is Artinian and hence the R-module $\operatorname{Hom}_R(R/\mathfrak{m}, H^i_{\mathfrak{m}}(M))$ is finitely generated. This led Grothendieck to conjecture in [7] that for any ideal I of a Noetherian ring R and any finitely generated R-module M, the module $\operatorname{Hom}_R(R/I, H^i_I(M))$ is finitely generated. This conjecture is not true in general and a number of counterexamples are given by several authors (for example see [8, 3, 6, 4]). In fact using [3, Theorem 3.9] it is easy to see that, for any Noetherian ring of dimension $d \geq 3$, there is an ideal I of R and a finitely generated R-module M, such that the module $\operatorname{Hom}_R(R/I, H^i_I(M))$ is not finitely generated. Hartshorne was the first to present a counterexample to Grothendieck's conjecture (see [8] for details and proof). However, he defined an R-module M

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to be I-cofinite if $\operatorname{Supp} M \subseteq V(I)$ and $\operatorname{Ext}^j_R(R/I, M)$ is finitely generated for all j. In this paper we prove the following about the vanishing of local cohomology modules. Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$, and $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim(R/\mathfrak{p}) = 2$. If the set

$$A := \{n \in \mathbb{N} : R/\mathfrak{p}^{(n)} \text{ is Cohen–Macaulay}\}$$

is infinite, then $H^{d-1}_{\mathfrak{p}}(R) = 0$. Also we consider the finiteness of the Bass numbers and cominimaxness of local cohomology modules over unique factorization domains. The main results of this paper in this direction are the following.

Theorem 1.1. *Let $(R, \mathfrak{m})$ be a unique factorization domain of dimension $d \geq 1$ and M a finitely generated R-module. Let I be an ideal of R, such that* $\operatorname{height}(I) = 1$. *Then the Bass numbers $\mu^j(\mathfrak{m}, H^i_I(M))$ are finite, for all $i \geq 0$ and all $j \geq 0$.*

Theorem 1.2. *Let $(R, \mathfrak{m})$ be a unique factorization domain of dimension $d \leq 3$, I an ideal of R and M a finitely generated R-module. Then the following statements hold:*

(i) *The Bass numbers of the R-module $H^i_I(M)$ are finite, for each $i \geq 0$.*
(ii) *The R-module $H^i_I(M)$ is I-cominimax, for each $i \geq 0$.*

For an R-module M, the *cohomological dimension of M with respect to I* is defined as

$$\operatorname{cd}(I, M) := \max\{i \in \mathbb{Z} : H^i_I(M) \neq 0\}.$$

For each R-module L, we denote by $\operatorname{Assh}_R L$ (resp. $\operatorname{mAss}_R L$) the set $\{\mathfrak{p} \in \operatorname{Ass}_R L : \dim R/\mathfrak{p} = \dim L\}$ (resp. the set of minimal primes of $\operatorname{Ass}_R L$). Also, for any ideal $\mathfrak{a}$ of R, we denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq \mathfrak{a}\}$ by $V(\mathfrak{a})$. Finally, for any ideal $\mathfrak{b}$ of R, *the radical of* $\mathfrak{b}$, denoted by $\operatorname{Rad}(\mathfrak{b})$, is defined to be the set $\{x \in R : x^n \in \mathfrak{b}$ for some $n \in \mathbb{N}\}$. We recall that the R-module M is said to be I-cominimax if support of M is contained in $V(I)$ and $\operatorname{Ext}^i_R(R/I, M)$ is minimax for all $i \geq 0$. The concept of I-cominimax modules was introduced in [2], as a generalization of the important notion of I-cofinite modules. Note that an R-module M is said to be minimax if there exists a finitely generated submodule N of M such that M/N is Artinian (see [13, 14]).

2. Main results

Theorem 2.1. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$, and $\mathfrak{p} \in$* $\operatorname{Spec}(R)$ *such that* $\dim(R/\mathfrak{p}) = 2$. *Then*

$$H^{d-1}_{\mathfrak{p}}(R) \cong \varinjlim_{n \geq 1} \operatorname{Ext}^{d-1}_R(R/\mathfrak{p}^{(n)}, R),$$

where for each $n \in \mathbb{N}$, $\mathfrak{p}^{(n)}$ denotes the n-th symbolic power of the prime ideal $\mathfrak{p}$.

Proof. Since for each $n \geq 1$ there is $s \in R \backslash \mathfrak{p}$ such that $s(\mathfrak{p}^{(n)}/\mathfrak{p}^n) = 0$, it follows that $\dim(\mathfrak{p}^{(n)}/\mathfrak{p}^n) \leq 1$, and so by the local duality theorem,

$$\operatorname{Ext}^{d-2}_R(\mathfrak{p}^{(n)}/\mathfrak{p}^n, R) = 0.$$

Also, since the ideal $\operatorname{Ann}_R(\mathfrak{p}^{(n)}/\mathfrak{p}^n)$ contains an R-regular sequence as $x_1, \ldots, x_{d-1}$, it follows from [11, §18 Lemma 2] that

$$\operatorname{Ext}_R^{d-1}(\mathfrak{p}^{(n)}/\mathfrak{p}^n, R) \cong \operatorname{Hom}_{R/(x_1,\ldots,x_{d-1})}(\mathfrak{p}^{(n)}/\mathfrak{p}^n, R/(x_1, \ldots, x_{d-1})).$$

So, as $\mathfrak{m} \notin \operatorname{Ass}_R(R/(x_1, \ldots, x_{d-1}))$ it follows that $\mathfrak{m} \notin \operatorname{Ass}_R(\operatorname{Ext}_R^{d-1}(\mathfrak{p}^{(n)}/\mathfrak{p}^n, R))$. Hence, $\Gamma_{\mathfrak{m}}(\operatorname{Ext}_R^{d-1}(\mathfrak{p}^{(n)}/\mathfrak{p}^n, R)) = 0$. Now the exact sequence

$$0 \to \mathfrak{p}^{(n)}/\mathfrak{p}^n \to R/\mathfrak{p}^n \to R/\mathfrak{p}^{(n)} \to 0$$

induces the exact sequence

$$\begin{aligned} \operatorname{Ext}_R^{d-2}(\mathfrak{p}^{(n)}/\mathfrak{p}^n, R) = 0 \to \operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^{(n)}, R) \\ \to \operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^n, R) \to \operatorname{Ext}_R^{d-1}(\mathfrak{p}^{(n)}/\mathfrak{p}^n, R). \end{aligned}$$

From this exact sequence we get an exact sequence

$$\begin{aligned} 0 \to \Gamma_{\mathfrak{m}}(\operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^{(n)}, R)) \\ \to \Gamma_{\mathfrak{m}}(\operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^n, R)) \to \Gamma_{\mathfrak{m}}(\operatorname{Ext}_R^{d-1}(\mathfrak{p}^{(n)}/\mathfrak{p}^n, R)) = 0. \end{aligned}$$

Therefore, by the local duality theorem, as

$$\operatorname{Supp} \operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^{(n)}, R) \subseteq V(\mathfrak{m}),$$

it follows that

$$\Gamma_{\mathfrak{m}}(\operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^{(n)}, R)) = \operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^{(n)}, R)$$

and so

$$\Gamma_{\mathfrak{m}}(\operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^n, R)) \cong \operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^{(n)}, R).$$

On the other hand, using the Lichtenbaum-Hartshorne vanishing theorem, it is easy to see that

$$\operatorname{Supp} H_{\mathfrak{p}}^{d-1}(R) \subseteq V(\mathfrak{m}),$$

and hence

$$\Gamma_{\mathfrak{m}}(H_{\mathfrak{p}}^{d-1}(R)) = H_{\mathfrak{p}}^{d-1}(R).$$

Therefore, as for each pair of positive integers n, k the natural commutative diagram

$$\begin{array}{ccc} R/\mathfrak{p}^{n+k} & \xrightarrow{f_{n+k}} & R/\mathfrak{p}^{(n+k)} \\ {\scriptstyle g_n}\downarrow & & \downarrow{\scriptstyle g'_n} \\ R/\mathfrak{p}^n & \xrightarrow[f_n]{} & R/\mathfrak{p}^{(n)} \end{array}$$

induces a commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^{(n)}, R) & \xrightarrow[\cong]{\widetilde{f_n}} & \Gamma_{\mathfrak{m}}(\operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^n, R)) \\ {\scriptstyle \widetilde{g'_n}}\downarrow & & \downarrow{\scriptstyle \widetilde{g_n}} \\ \operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^{(n+k)}, R) & \xrightarrow[\widetilde{f_{n+k}} \, \cong]{} & \Gamma_{\mathfrak{m}}(\operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^{n+k}, R)) \end{array}$$

it follows that

$$H_{\mathfrak{p}}^{d-1}(R) = \Gamma_{\mathfrak{m}}(H_{\mathfrak{p}}^{d-1}(R)) \cong \varinjlim_{n\geq 1} \Gamma_{\mathfrak{m}}(\operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^n, R)) \cong \varinjlim_{n\geq 1} \operatorname{Ext}_R^{d-1}(R/\mathfrak{p}^{(n)}, R),$$

as required. □

Corollary 2.2. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \geq 2$, and $\mathfrak{p} \in$ $\operatorname{Spec}(R)$ such that $\dim(R/\mathfrak{p}) = 2$. If the set*

$$A := \{n \in \mathbb{N} : R/\mathfrak{p}^{(n)} \text{ is Cohen–Macaulay}\}$$

is infinite, then $H_{\mathfrak{p}}^{d-1}(R) = 0$.

Proof. The assertion follows from Theorem 2.1, using the local duality theorem. □

Lemma 2.3. *Let R be a Noetherian local ring of dimension $d \geq 3$, $x \in \mathfrak{m}$ and I an ideal of R such that $I \subseteq Rx$. Then the R-homomorphism $H_I^j(M) \xrightarrow{.x} H_I^j(M)$ is an isomorphism, for each $j \geq 2$.*

Proof. See [1, Lemma 2.5]. □

Lemma 2.4. *Let R be a Noetherian ring, I an ideal of R, M a finitely generated R-module such that $\dim M \leq 2$. Then the R-module $H_I^i(M)$ is I-cofinite, for each $i \geq 0$. In particular, the Bass numbers of $H_I^i(M)$ are finite.*

Proof. See [1, Lemma 2.11]. □

Theorem 2.5. *Let $(R, \mathfrak{m})$ be a unique factorization domain of dimension $d \geq 1$ and M a finitely generated R-module. Let I be an ideal of R, such that $\operatorname{height}(I) = 1$. Then the Bass numbers $\mu^j(\mathfrak{m}, H_I^i(M))$ are finite, for all $i \geq 0$ and all $j \geq 0$.*

Proof. As the R-module $H_I^0(M)$ is finitely generated, in view of [10, Corollary 3.9] it is enough to prove that $\operatorname{Ext}_R^j(R/\mathfrak{m}, H_I^i(M)) = 0$, for $j = 0, 1, 2, \ldots$ and $i = 2, 3, \ldots$. Now suppose that $i \geq 2$ and let

$$X := \{\mathfrak{p} \in \operatorname{mAss}_R R/I \mid \operatorname{height} \mathfrak{p} = 1\},$$
$$Y := \{\mathfrak{p} \in \operatorname{mAss}_R R/I \mid \operatorname{height} \mathfrak{p} \geq 2\}.$$

Then $X \neq \emptyset$. Let $J = \bigcap_{\mathfrak{p} \in X} \mathfrak{p}$. Since R is a UFD, it follows from [11, Exercise 20.3] that J is a principal ideal, so there is an element $x \in R$ such that $J = Rx$. Hence if $Y = \emptyset$ then $\sqrt{I} = J$, and so it follows from [5, Theorem 3.3.1] that $H_I^i(M) = 0$ for all $i > 1$. Therefore we may assume that $Y \neq \emptyset$. Then $\sqrt{I} = Rx \cap K$, where $K = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$. Therefore as $H_I^i(M) = H_{\sqrt{I}}^i(M)$ and $\sqrt{I} \subseteq Rx$, it follows from Lemma 2.3 that the R-homomorphism $H_I^i(M) \xrightarrow{.x} H_I^i(M)$ is an isomorphism, for each $i \geq 2$. Therefore, for each $j \geq 0$, the R-homomorphism

$$\operatorname{Ext}_R^j(R/\mathfrak{m}, H_I^i(M)) \xrightarrow{.x} \operatorname{Ext}_R^j(R/\mathfrak{m}, H_I^i(M))$$

is an isomorphism. Since $x \in \mathfrak{m}$, it follows that $\operatorname{Ext}_R^j(R/\mathfrak{m}, H_I^i(M)) = 0$, for each $j \geq 0$. Consequently, we have $\mu^j(\mathfrak{m}, H_I^i(M)) = 0$, for $j = 0, 1, 2, \ldots$ and $i = 2, 3, \ldots$, as required. □

Lemma 2.6. *Let $(R, \mathfrak{m})$ be a unique factorization domain of dimension $d \leq 3$, I an ideal of R, and M a finitely generated R-module. Then the Bass numbers of $H^i_I(M)$ are finite, for all $i \geq 0$.*

Proof. Let $\mathfrak{p} \in \operatorname{Spec}(R)$. Then the case $\dim(R/\mathfrak{p}) \leq 2$ follows from Lemma 2.4, by localization. So we may assume that $\mathfrak{p} = \mathfrak{m}$. If $\dim(R/I) = 1$ the assertion follows from [6, Corollary 2]. The cases $\dim(R/I) = 0$ and $\dim(R/I) = 3$ are clear. Let $\dim(R/I) = 2$. Then we have $\operatorname{height}(I) = 1$ and so in view of Theorem 2.5 the Bass numbers of $H^i_I(M)$ are finite, for all $i \geq 0$. □

The following result is a partial generalization of a result given in [9].

Theorem 2.7. *Let R be a unique factorization domain of dimension $d \leq 3$, I an ideal of R, and M a finitely generated R-module. Then the Bass numbers of $H^i_I(M)$ are finite, for all $i \geq 0$.*

Proof. The assertion follows immediately from Lemma 2.6, by localization. □

Theorem 2.8. *Let $(R, \mathfrak{m})$ be a unique factorization domain of dimension $d \leq 3$, I an ideal of R, and M a finitely generated R-module. Then the R-module $H^i_I(M)$ is I-cominimax, for all $i \geq 0$.*

Proof. By Lemma 2.6 we may assume that $\dim M = 3$. Now by Grothendieck's vanishing theorem $H^i_I(M) = 0$, for all $i \geq 4$. On the other hand $H^0_I(M)$ is finitely generated and so is I-cominimax. Moreover, in view of [12, Proposition 5.1], $H^3_I(M)$ is I-cofinite and so is I-cominimax. Therefore in view of [4, Proposition 2.14], it is enough to prove that the R-module $H^2_I(M)$ is I-cominimax. To do this, if $\dim(R/I) \leq 1$, it follows from [4, Corollary 2.7] that $H^2_I(M)$ is I-cofinite and hence is I-cominimax. Also if $\dim(R/I) = 3$, then $I = 0$, and hence $H^2_I(M) = 0$. Finally, let $\dim(R/I) = 2$. Then we have $\operatorname{height}(I) = 1$. Let

$$X := \{\mathfrak{p} \in \operatorname{mAss}_R R/I \mid \operatorname{height} \mathfrak{p} = 1\},$$
$$Y := \{\mathfrak{p} \in \operatorname{mAss}_R R/I \mid \operatorname{height} \mathfrak{p} \geq 2\}.$$

Then $X \neq \emptyset$. Let $J = \bigcap_{\mathfrak{p} \in X} \mathfrak{p}$. Since R is a UFD, it follows from [11, Exercise 20.3] that J is a principal ideal, so there is an element $x \in R$ such that $J = Rx$. Hence if $Y = \emptyset$ then $\sqrt{I} = J$, and so it follows from [5, Theorem 3.3.1] that $H^2_I(M) = 0$. Therefore we may assume that $Y \neq \emptyset$. Then $\sqrt{I} = Rx \cap K$, where $K = \bigcap_{\mathfrak{p} \in Y} \mathfrak{p}$. Then $\operatorname{height} K = 2$, and so $\dim(R/K) = 1$. Consequently in view of [4, Corollary 2.7], the R-module $H^2_K(M)$ is K-cofinite, and hence is K-cominimax. We claim that $\operatorname{height}(J + K) = 3$. To see this, suppose there exists a prime ideal $\mathfrak{q}$ of R such that $J + K \subseteq \mathfrak{q}$ and $\operatorname{height} \mathfrak{q} = 2$. Then there exist $\mathfrak{p}_1 \in X$ and $\mathfrak{p}_2 \in Y$ such that $\mathfrak{p}_1 + \mathfrak{p}_2 \subseteq \mathfrak{q}$. As $\operatorname{height} \mathfrak{p}_2 \geq 2$, it follows that $\mathfrak{q} = \mathfrak{p}_2$, and so $\mathfrak{p}_1 \subset \mathfrak{p}_2$, a contradiction. Since $\operatorname{height}(J + K) = 3$ it follows from [5, Theorem 7.1.3] that the R-modules $H^2_{J+K}(M) \cong H^2_{\mathfrak{m}}(M)$ and $H^3_{J+K}(M) \cong H^3_{\mathfrak{m}}(M)$ are Artinian. Now, since $\sqrt{I} = J \cap K$, the Mayer–Vietoris sequence (see e.g. [5, Theorem 3.2.3]) yields the exact sequence

$$H^2_{J+K}(M) \longrightarrow H^2_K(M) \longrightarrow H^2_I(M) \longrightarrow H^3_{J+K}(M),$$

that implies that the R-module $H^2_I(M)$ is K-cominimax. In view of Lemma 2.3, there exists an exact sequence

$$0 \longrightarrow H^2_I(M) \xrightarrow{x} H^2_I(M) \longrightarrow 0,$$

that implies $\mathrm{Ext}^i_R(R/J, H^2_I(M)) = 0$, for each $i \geq 0$. On the other hand, it follows from [2, Theorem 2.7] that the R-module $\mathrm{Ext}^i_R(R/J+K, H^2_I(M))$ is minimax for all $i \geq 0$. Consequently, the exact sequence

$$0 \longrightarrow R/\sqrt{I} \longrightarrow R/J \oplus R/K \longrightarrow R/J+K \longrightarrow 0$$

induces a long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_R(R/J+K, H^2_I(M)) \longrightarrow \mathrm{Hom}_R(R/K, H^2_I(M)) \\ \longrightarrow \mathrm{Hom}_R(R/\sqrt{I}, H^2_I(M)) \longrightarrow \mathrm{Ext}^1_R(R/J+K, H^2_I(M)) \\ \longrightarrow \mathrm{Ext}^1_R(R/K, H^2_I(M)) \longrightarrow \mathrm{Ext}^1_R(R/\sqrt{I}, H^2_I(M)) \longrightarrow \cdots \end{aligned}$$

that implies that the R-module $H^2_I(M)$ is $\sqrt{I}$-cominimax. Now, as $\mathrm{Supp}(R/I) = \mathrm{Supp}(R/\sqrt{I})$, it follows from [2, Theorem 2.7] that the R-module $H^2_I(M)$ is I-cominimax. This completes the proof. □

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