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SOFT IDEALS IN ORDERED SEMIGROUPS

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Abstract. The notions of soft left and soft right ideals, soft quasi-ideal and soft bi-ideal in ordered semigroups are introduced. We show here that in ordered groupoids the soft right and soft left ideals are soft quasi-ideals, and in ordered semigroups the soft quasi-ideals are soft bi-ideals. Moreover, we prove that in regular ordered semigroups the soft quasi-ideals and the soft bi-ideals coincide. We finally show that in an ordered semigroup the soft quasi-ideals are just intersections of soft right and soft left ideals.

1. Introduction

The theory of soft sets is introduced by Molodtsov in [14]. The basic aim of this theory is to introduce a new tool to discuss uncertainty. Maji et al. [13] gave the operations of soft sets and their properties. Since then, based on these operations, the theory of soft sets has developed in many directions and found its applications in a wide variety of fields. In soft set theory, we have enough number of parameters to deal with uncertainty. This quality makes it prominent among its predecessor theories such as probability theory, interval mathematics, fuzzy sets and rough sets. In other words, a soft set is a set-valued function from a set of parameters to the power set of an initial universe set. In order to establish the algebraic structures of soft sets, Aktas and Cagman [1] gave a definition of soft groups that is actually a parametrized family of subgroups of a group. That is, a soft set is called a soft group over a group if and only if all the values of the set-valued function are subgroups of the group. On the basis of this definition, some other authors ([2],[4]-[7],[11],[12]) have studied the algebraic properties of soft sets. Cagman et al. [3] studied a new type of soft groups on a soft set, which are different from the definition of soft groups in [1]. The new group is called soft intersection group (abbreviated as *soft int-group*). This new approach is based on the inclusion relation and intersection of sets. It brings the soft set theory, set theory and the group theory together. Some supplementary properties of soft int-groups and normal soft int-groups, analogues to classical group theory and fuzzy group theory, are introduced in [9, 15, 16]. Recently, Ideal theory in semigroups based on soft int-semigroup is investigated in [17]. E. Hamouda [5] discussed soft ideals and soft filters of soft ordered groupoids.

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In this paper, the notions of soft left and soft right ideals, soft quasi-ideal and soft bi-ideal in ordered semigroups are introduced. We show here that in ordered groupoids the soft right and soft left ideals are soft quasi-ideals, and in ordered semigroups the soft quasi-ideals are soft bi-ideals. Moreover, we prove that in regular ordered semigroups the soft quasi-ideals and the soft bi-ideals coincide. We finally show that in an ordered semigroup the soft quasi-ideals are just intersections of soft right and soft left ideals.

2. Preliminaries

We denote by $(S, \cdot, \leq)$ an ordered semigroup, that is, a semigroup $(S, \cdot)$ with a simple order $\leq$ which satisfies the following condition:

$$\text{for } x, y, z \in S,\ x \leq y \text{ implies that } xz \leq yz \text{ and } zx \leq zy.$$

Let U be an initial universe set and let E be a set of parameters. Let $P(U)$ denote the power set of U and $A, B, C, \cdots \subseteq E$.

Definition 2.1 ([9])**.** A soft set (α, A) over U is defined to be the set of ordered pairs $(\alpha, A) = \{(x, \alpha(x)) : x \in E, \alpha(x) \in P(U)\}$, where $\alpha : E \longrightarrow P(U)$ such that $\alpha(x) = \emptyset$ if $x \notin A$.

Example 2.2 ([14])**.**

(1) A fuzzy set $\mu : U \longrightarrow [0, 1]$ may be considered as a special case of the soft set. Let us consider the family of t-level sets for the function μ,

$$\alpha(t) = \{x \in U : \mu(x) \geq t\}, \qquad t \in [0, 1].$$

If we know the family α, we can find the functions $\mu(x)$ by means of the following formula:

$$\mu(x) = \bigvee_{x \in \alpha(t)} t.$$

Thus, every fuzzy set μ may be considered as the soft set $(\alpha, [0, 1])$.

(2) Let (X, τ) be a topological space. Then, the family of open neighborhoods $T(x)$ of the point x, where $T(x) = \{V \in \tau : x \in V\}$, may be considered as the soft set $(T(x), \tau)$.

Definition 2.3 ([17])**.** Let (α, A) and (β, A) be two soft sets. (α, A) is a soft subset of (β, A), denoted by $(\alpha, A) \Subset (\beta, A)$, if $\alpha(x) \subseteq \beta(x)$ for all $x \in E$ and (α, A); (β, A) are called soft equal, denoted by $(\alpha, A) = (\beta, A)$, if and only if $\alpha(x) = \beta(x)$ for all $x \in E$.

Definition 2.4 ([17])**.** Let (α, A) and (β, A) be two soft sets. Union $(\alpha, A) \Cup (\beta, A)$ and intersection $(\alpha, A) \Cap (\beta, A)$ are defined by

$$(\alpha \Cup \beta)(x) = \alpha(x) \cup \beta(x),$$
$$(\alpha \Cap \beta)(x) = \alpha(x) \cap \beta(x),$$

respectively.

Definition 2.5 ([17])**.** Let S be a semigroup and (α, S) be a soft set. (α, S) is called an int-soft semigroup of S over U if it satisfies

$$\alpha(xy) \supseteq \alpha(x) \cap \alpha(y) \qquad \text{for all } x, y \in G.$$

Example 2.6 ([8])**.** Let $S = \{a, b, c, d\}$ be a semigroup with the following Cayley table:

*	a	b	c	d
a	a	a	a	a
b	a	b	a	a
c	c	c	c	c
d	c	c	c	d

Let (α, S) be a soft set over $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ defined as follows:

$$\alpha : S \longrightarrow P(U), \quad x \longmapsto \begin{cases} U, & \text{if } x = a, \\ \{2, 4, 6, 8, 10\}, & \text{if } x = b, \\ \{3, 6, 9\}, & \text{if } x = c, \\ \{2, 6, 10\}, & \text{if } x = d. \end{cases}$$

Then (α, S) is not an int-soft semigroup over U since

$$\alpha(d \star b) = \alpha(c) = \{3, 6, 9\} \not\supseteq \{2, 6, 10\} = \alpha(d) \cap \alpha(b).$$

The soft set (β, S) over U given by

$$\beta : S \longrightarrow P(U), \quad x \longmapsto \begin{cases} U, & \text{if } x = a, \\ \{3, 6, 9\}, & \text{if } x = c, \\ \{6\}, & \text{if } x \in \{b, d\} \end{cases}$$

is an int-soft semigroup over U.

Throughout this paper, S denotes an arbitrary semigroup and the set of all int-soft semigroups with parameter set S over U will be denoted by $S_S(U)$, unless otherwise stated.

For $a \in S$, define $A_a = \{(x, y) \in S \times S : a \leq xy\}$. For two soft sets (α, S) and (β, S), we define the soft product of (α, S) and (β, S) as the soft set $(\alpha \circ \beta, S)$ over U defined by

$$(\alpha \circ \beta)(a) = \begin{cases} \bigcup_{(x,y) \in A_a} \{\alpha(x) \cap \beta(y)\}, & \text{if } A_a \neq \emptyset, \\ \emptyset, & \text{otherwise.} \end{cases}$$

For an ordered semigroup S, let $(\theta, S) \in S_S(U)$ be the soft set over U defined by $\theta(x) = U$ for all $x \in S$. It is clear that the soft set (θ, S) is the greatest element of the ordered set $(S_S(U), \Subset)$.

3. Main results

Proposition 3.1. *If $(S, \cdot, \leq)$ is an ordered groupoid and (α_1, S), (α_2, S), (β_1, S), (β_2, S) are soft sets over U such that $(\alpha_1, S) \Subset (\beta_1, S)$ and $(\alpha_2, S) \Subset (\beta_2, S)$, then $(\alpha_1 \circ \alpha_2, S) \Subset (\beta_1 \circ \beta_2, S)$.*

Proof. Let $a \in S$. If $A_a = \emptyset$, then $(\alpha_1 \circ \alpha_2)(a) = \emptyset \subseteq (\beta_1 \circ \beta_2)(a)$. Let $A_a \neq \emptyset$, then we have

$$(\alpha_1 \circ \alpha_2)(a) = \bigcup_{(x,y) \in A_a} \{\alpha_1(x) \cap \beta_2(y)\},$$

$$(\beta_1 \circ \beta_2)(a) = \bigcup_{(x,y) \in A_a} \{\beta_1(x) \cap \beta_2(y)\}.$$

Let $(x, y) \in A_a$. Since $x, y \in S$, we have $\alpha_1(x) \subseteq \beta_1(x)$ and $\alpha_2(y) \subseteq \beta_2(y)$, then

$$\bigcup_{(x,y) \in A_a} \{\alpha_1(x) \cap \beta_2(y)\} \subseteq \bigcup_{(x,y) \in A_a} \{\beta_1(x) \cap \beta_2(y)\}.$$

Thus $(\alpha_1 \circ \alpha_2)(a) \subseteq (\beta_1 \circ \beta_2)(a)$. Therefore, $(\alpha_1 \circ \alpha_2, S) \Subset (\beta_1 \circ \beta_2, S)$. □

By Proposition 3.1, the set $S_S(U)$ of all soft sets over U endowed with the multiplication $\circ$ and the order $\Subset$ is an ordered groupoid.

Definition 3.2. Let $(S, \cdot, \leq)$ be an ordered semigroup. A soft set (α, S) is called a soft left ideal (resp. soft right ideal) over U if

(1) $\alpha(xy) \supseteq \alpha(x)$ (resp. $\alpha(xy) \supseteq \alpha(y)$) for every $x, y \in S$,
(2) $x \leq y \Longrightarrow \alpha(x) \supseteq \alpha(y)$.

Definition 3.3. Let $(S, \cdot, \leq)$ be an ordered semigroup. A soft set (α, S) is called a soft quasi-ideal over U if

(1) $(\alpha \circ \theta, S) \Cap (\theta \circ \alpha, S) \Subset (\alpha, S)$,
(2) $x \leq y \Longrightarrow \alpha(x) \supseteq \alpha(y)$.

Definition 3.4. Let $(S, \cdot, \leq)$ be an ordered semigroup. A soft set (α, S) is called a soft bi-ideal of S if the following assertions are satisfied:

(1) $\alpha(xyz) \supseteq \alpha(x) \cap \alpha(z)$ for all $x, y, z \in S$,
(2) $x \leq y \Longrightarrow \alpha(x) \supseteq \alpha(y)$.

Theorem 3.5. *If $(S, \cdot, \leq)$ is an ordered semigroup, then the soft right and the soft left ideals over U are soft quasi-ideals over U.*

Proof. Let (α, S) be a soft right ideal over U and $x \in S$. If $A_x = \emptyset$, then we have $(\alpha \circ \theta)(x) = (\theta \circ \alpha)(x) = \emptyset \subseteq \alpha(x)$. If $A_x \neq \emptyset$, then

$$(\alpha \circ \theta)(x) = \bigcup_{(y,z) \in A_x} \{\alpha(y) \cap \theta(z)\}.$$

On the other hand, $\alpha(x) \supseteq \alpha(y) \cap \theta(z)$ for every $(y, z) \in A_x$. Indeed, if $(y, z) \in A_x$ then $x \leq yz$ and $\alpha(x) \supseteq \alpha(yz) \supseteq \alpha(y) = \alpha(y) \cap \theta(z)$. This implies that

$$\alpha(x) \supseteq \bigcup_{(y,z) \in A_x} \{\alpha(y) \cap \theta(z)\} = (\alpha \circ \theta)(x).$$

Hence we have

$$\begin{aligned}((\alpha \circ \theta) \Cap (\theta \circ \alpha))(x) &= (\alpha \circ \theta)(x) \cap (\theta \circ \alpha)(x) \\ &\subseteq \bigcup_{(y,z)\in A_x} \{\alpha(y) \cap \theta(z)\} \\ &= (\alpha \circ \theta)(x) \subseteq \alpha(x).\end{aligned}$$

Therefore (α, S) is a soft quasi-ideal over U. A similar argument for soft left ideals completes the proof. □

In the following example, we show that the converse of the above theorem does not hold in general.

Example 3.6. Let $S = \{0, a, b, c\}$ be the ordered semigroup defined by the multiplication and the order below:

*	0	a	b	c
0	0	0	0	0
a	0	a	b	0
b	0	0	0	0
c	0	c	0	0

$$\leq := \{(0,0), (a,a), (b,b), (c,c), (o,c)\}.$$

Let (α, S) be a soft set over $U = \{1, 2, 3, 4, 5, 6\}$ defined as follows:

$$\alpha : S \longrightarrow P(U), \quad x \longmapsto \begin{cases} \{2,4,6\}, & \text{if } x \in \{0, a\}, \\ \emptyset, & \text{if } x \in \{b, c\}. \end{cases}$$

Then (α, S) is an int-soft quasi-ideal over U [17] and is not an int-soft left (resp. right) ideal over U since

$$\alpha(a \star b) = \alpha(b) = \emptyset \not\supseteq \{2,4,6\} = \alpha(a) \quad (\text{resp. } \alpha(c \star a) = \alpha(c) = \emptyset \not\supseteq \alpha(a)).$$

Theorem 3.7. *If $(S, \cdot, \leq)$ is an ordered semigroup, then the soft quasi-ideals over U are soft bi-ideals over U.*

Proof. Assume that (α, S) is a soft quasi-ideal over U and $x, y, z \in S$; then we have

$$\alpha(xyz) \supseteq ((\alpha \circ \theta) \Cap (\theta \circ \alpha))(xyz) = (\alpha \circ \theta)(xyz) \cap (\theta \circ \alpha)(xyz).$$

Since $(x, yz) \in A_{(xyz)}$, we get

$$(\alpha \circ \theta)(xyz) = \bigcup_{(a,b)\in A_{(xyz)}} \{\alpha(a) \cap \theta(b)\} \supseteq \alpha(x) \cap \theta(yz) = \alpha(x).$$

Since $(xy, z) \in A_{(xyz)}$, we have

$$(\theta \circ \alpha)(xyz) = \bigcup_{(a,b)\in A_{(xyz)}} \{\theta(a) \cap \alpha(b)\} \supseteq \theta(xy) \cap \alpha(z) = \alpha(z).$$

Thus,

$$\alpha(xyz) \supseteq (\alpha \circ \theta)(xyz) \cap (\theta \circ \alpha)(xyz) \supseteq \alpha(x) \cap \alpha(z).$$

This proves that (α, S) is a soft bi-ideal over U. □

The converse of Theorem 3.7 is not generally true as shown by the following example.

Example 3.8. Consider the semigroup $S = \{0, a, b, c\}$ in Example 3.6 with the new order

$$\leq := \{(0, 0), (a, a), (b, b), (c, c), (b, c), (b, 0)\}.$$

Let (β, S) be a soft set over U defined by

$$\beta : S \longrightarrow P(U), \quad x \longmapsto \begin{cases} V_1 & \text{if } x = 0, \\ V_2 & \text{if } x = a, \\ V_3 & \text{if } x = b, \\ V_4 & \text{if } x = c, \end{cases}$$

where $V_1 \supset V_2 \supset V_3 \supset V_4$ are subsets of U. Then (β, S) is not soft quasi-ideal over U since

$$(\beta \circ \theta)(b) \cap (\theta \circ \beta)(b) = V_1 \not\subseteq \beta(b) = V_3,$$

but (β, S) is a soft bi-ideal over U.

An ordered semigroup S is called regular if for every $a \in S$ there exist $x \in S$ such that $a \leq axa$ [10].

Theorem 3.9. *In a regular ordered semigroup S, the soft quasi-ideals and the soft bi-ideals coincide.*

Proof. Let (α, S) be a soft bi-ideal over U and $x \in S$. We will show that

$$((\alpha \circ \theta) \Cap (\theta \circ \alpha))(x) \subseteq \alpha(x). \tag{1}$$

By Definition 2.4, we have

$$((\alpha \circ \theta) \Cap (\theta \circ \alpha))(x) = (\alpha \circ \theta)(x) \cap (\theta \circ \alpha))(x).$$

If $A_x = \emptyset$ then, as we have already seen in Theorem 3.5, condition (1) is satisfied. Now assume that $A_x \neq \emptyset$ and that $(\alpha \circ \theta)(x) \subseteq \alpha(x)$. Then we have

$$((\alpha \circ \theta) \Cap (\theta \circ \alpha))(x) = (\alpha \circ \theta)(x) \cap (\theta \circ \alpha)(x) \subseteq (\alpha \circ \theta)(x) \subseteq \alpha(x),$$

and condition (1) is satisfied.

Let $(\alpha \circ \theta)(x) \supset \alpha(x)$. Then by definition of soft product, there exists $(a, b) \in A_x$ such that

$$\alpha(a) \cap \theta(b) \supseteq \alpha(x) \Longleftrightarrow \alpha(a) \supset \alpha(x). \tag{2}$$

In order to complete the proof, all that we need is $(\theta \circ \alpha)(x) \subseteq \alpha(x)$. Then

$$(\alpha \circ \theta)(x) \cap (\theta \circ \alpha)(x) \subseteq (\theta \circ \alpha)(x) \subseteq \alpha(x),$$

so, condition (1) is satisfied. By definition of soft product, it is sufficient to show that

$$\theta(u) \cap \alpha(v) \subseteq \alpha(x) \qquad \forall\, (u, v) \in A_x.$$

So, let $(u, v) \in A_x,\ x \leq uv$. Since S is regular, there exists $t \in S$ such that $x \leq xtx$. Then $x \leq abtuv$. Since (α, S) is a soft bi-ideal over U, we have

$$\alpha(x) \supseteq \alpha(abtuv) \supseteq \alpha(z) \cap \alpha(v).$$

If $\alpha(z) \cap \alpha(v) = \alpha(z)$, then $\alpha(z) \subseteq \alpha(x)$, which is a contradiction by (2). Thus we get $\alpha(z) \cap \alpha(v) = \alpha(v)$ and hence $\alpha(x) \supseteq \alpha(v) = \theta(u) \cap \alpha(v)$. □

In the rest of this section, we show that the soft quasi-ideals of an ordered semigroup are just intersections of soft right and soft left ideals.

Lemma 3.10. *Let $(S, \cdot, \leq)$ be an ordered semigroup and (α, S) a soft set over S. Then the following holds:*

(i) $(\theta \circ \alpha)(xy) \supseteq \alpha(y)$ *for all* $x, y \in S$,
(ii) $(\theta \circ \alpha)(xy) \supseteq (\theta \circ \alpha)(y)$ *for all* $x, y \in S$.

Proof. (i) Let $x, y \in S$. Since $(x, y) \in A_{xy}$, we have

$$(\theta \circ \alpha)(xy) = \bigcup_{(a,b) \in A_{xy}} \{\theta(a) \cap \alpha(b)\} \supseteq \theta(x) \cap \alpha(y).$$

(ii) Let $x, y \in S$. If $A_y = \emptyset$, then $(\theta \circ \alpha)(y) = \emptyset$. Since $\theta \circ \alpha$ is a soft set over S, we have $(\theta \circ \alpha)(xy) \supseteq \emptyset = (\theta \circ \alpha)(y)$. Now, assume that $A_y \neq \emptyset$. Then

$$(\theta \circ \alpha)(y) = \bigcup_{(w,z) \in A_y} \{\theta(w) \cap \alpha(z)\}.$$

On the other hand,

$$(\theta \circ \alpha)(xy) \supseteq \theta(w) \cap \alpha(z) \quad \forall\, (w, z) \in A_y. \tag{3}$$

Indeed, let $(w, z) \in A_y$. Since $(x, y) \in A_{xy}$, we have

$$(\theta \circ \alpha)(xy) = \bigcup_{(s,t) \in A_{xy}} \{\theta(s) \cap \alpha(t)\}.$$

Since $(w, z) \in A_y$, we have $y \leq wz$, then $xy \leq xwz$, and $(xw, z) \in A_{xy}$. Hence we have

$$(\theta \circ \alpha)(xy) \supseteq \theta(xw) \cap \alpha(z) = \theta(w) \cap \alpha(z).$$

Using (3), we have

$$(\theta \circ \alpha)(xy) = \bigcup_{(w,z) \in A_y} \{\theta(w) \cap \alpha(z)\} = (\theta \circ \alpha)(x).$$

This completes the proof. □

In a similar way we prove the following lemma.

Lemma 3.11. *Let $(S, \cdot, \leq)$ be an ordered semigroup and (α, S) a soft set over S. Then the following holds:*

(i) $(\alpha \circ \theta)(xy) \supseteq \alpha(x)$ *for all* $x, y \in S$,
(ii) $(\alpha \circ \theta)(xy) \supseteq (\alpha \circ \theta)(x)$ *for all* $x, y \in S$.

Lemma 3.12. *If $(S, \cdot, \leq)$ is an ordered semigroup, (α, S) a soft set over S and $x \leq y$, then*

$$(\theta \circ \alpha)(x) \supseteq (\theta \circ \alpha)(y).$$

Proof. If $A_y = \emptyset$, then $(\theta \circ \alpha)(y) = \emptyset$. Hence $(\theta \circ \alpha)(y) = \emptyset \subseteq (\theta \circ \alpha)(x)$. Let $A_y \neq \emptyset$. Then

$$(\theta \circ \alpha)(y) = \bigcup_{(w,z)\in A_y} \{\theta(w) \cap \alpha(z)\} = \bigcup_{(w,z)\in A_y} \{\alpha(z)\}.$$

On the other hand,

$$(\theta \circ \alpha)(x) \supseteq \alpha(z) \quad \forall\, (w,z) \in A_y. \tag{4}$$

To show that, let $(wz) \in A_y$. Since $x \leq y \leq wz$, we have $(wz) \in A_x$. Then

$$(\theta \circ \alpha)(xy) = \bigcup_{(s,t)\in A_{xy}} \{\theta(s) \cap \alpha(t)\} \supseteq \alpha(z).$$

Thus, by (4), we conclude that

$$(\theta \circ \alpha)(x) \supseteq \bigcup_{(w,z)\in A_y} \{\alpha(z)\} = (\theta \circ \alpha)(y).$$ □

In a similar argument we prove the following lemma.

Lemma 3.13. *If $(S, \cdot, \leq)$ is an ordered semigroup, (α, S) a soft set over S and $x \leq y$, then*

$$(\alpha \circ \theta)(x) \supseteq (\alpha \circ \theta)(y).$$

Lemma 3.14. *Let $(S, \cdot, \leq)$ be an ordered semigroup and (α, S) a soft set over S. Then*

$$(\alpha \Cup (\theta \circ \alpha))(xy) \supseteq (\alpha \Cup (\theta \circ \alpha))(y) \qquad \forall x, y \in S.$$

Proof. Let $x, y \in S$. Since $(\theta \circ \alpha, S) \Subset (\alpha \Cup (\theta \circ \alpha), S)$, we have

$$(\alpha \Cup (\theta \circ \alpha))(xy) \supseteq (\theta \circ \alpha)(xy).$$

By Lemma 3.10, $(\theta \circ \alpha)(xy) \supseteq \alpha(y)$ and $(\theta \circ \alpha)(xy) \supseteq (\theta \circ \alpha)(y)$, so we have

$$(\theta \circ \alpha)(xy) \supseteq (\theta \circ \alpha)(y) \cup \alpha(y) = (\alpha \Cup (\theta \circ \alpha))(y).$$

Therefore $(\alpha \Cup (\theta \circ \alpha))(xy) \supseteq (\alpha \Cup (\theta \circ \alpha))(y)$. □

In a similar way we prove the following lemma.

Lemma 3.15. *Let $(S, \cdot, \leq)$ be an ordered semigroup and (α, S) a soft set over S. Then*

$$(\alpha \Cup (\alpha \circ \theta))(xy) \supseteq (\alpha \Cup (\alpha \circ \theta))(x) \qquad \forall x, y \in S.$$

Lemma 3.16. *Let $(S, \cdot, \leq)$ be an ordered semigroup and (α, S) a soft set over S. If $x \leq y$ implies $\alpha(x) \supseteq \alpha(y)$ for all $x, y \in S$, then the soft set $(\alpha \Cup (\theta \circ \alpha), S)$ is a soft left ideal over U.*

Proof. Let $x, y \in S$. Lemma 3.14 implies that $(\alpha \Cup (\theta \circ \alpha))(xy) \supseteq (\alpha \Cup (\theta \circ \alpha))(y)$. Now, let $x \leq y$. Then

$$(\alpha \Cup (\theta \circ \alpha))(x) \supseteq (\alpha \Cup (\theta \circ \alpha))(y).$$

Indeed, since (α, S) is a soft set over U and $x \leq y$, by Lemma 3.12 we get $(\theta\circ\alpha)(x) \supseteq (\theta \circ \alpha)(y))$ and, by hypothesis, $\alpha(x) \supseteq \alpha(y)$. Then

$$\begin{aligned}(\alpha \Cup (\theta \circ \alpha))(x) &= \alpha(x) \cup (\theta \circ \alpha)(x)\\ &\supseteq \alpha(y) \cup (\theta \circ \alpha)(y) = (\alpha \Cup (\theta \circ \alpha))(y).\end{aligned}$$

Therefore, $(\alpha \Cup (\theta \circ \alpha), S)$ is a soft left ideal over U. □

In a similar way we prove the following lemma.

Lemma 3.17. *Let $(S, \cdot, \leq)$ be an ordered semigroup and (α, S) a soft set over S. If $x \leq y$ implies $\alpha(x) \supseteq \alpha(y)$ for all $x, y \in S$, then the soft set $(\alpha \Cup (\alpha \circ \theta), S)$ is a soft right ideal over U.*

Theorem 3.18. *Let $(S, \cdot, \leq)$ be an ordered semigroup. A soft set (α, S) is a soft quasi-ideal over U if and only if there exist a soft right ideal (β, S) and a soft left ideal (γ, S) over U such that $(\alpha, S) = (\beta \cap \gamma, S)$.*

Proof. Assume that (α, S) is a soft quasi-ideal over U. From Lemmas 3.16 and 3.17, $(\alpha \Cup (\theta \circ \alpha), S)$ is a soft left ideal and $(\alpha \Cup (\alpha \circ \theta), S)$ is a soft right ideal over U. Moreover, we have

$$(\alpha, S) = (\alpha \Cup (\theta \circ \alpha) \Cap (\alpha \Cup (\alpha \circ \theta)), S).$$

In fact, by simple calculations, we have

$$(\alpha \Cup (\theta \circ \alpha) \Cap (\alpha \Cup (\alpha \circ \theta)), S) = (\alpha \Cup ((\theta \circ \alpha) \Cap \alpha) \Cup (\alpha \Cap (\alpha \circ \theta)) \Cup ((\theta \circ \alpha) \Cap (\alpha \circ \theta)), S).$$

Since (α, S) is a soft quasi-ideal over U, we have $(\alpha \circ \theta) \Cap (\theta \circ \alpha) \Subset \alpha$. In addition, $((\alpha \circ \theta) \Cap \alpha) \Subset \alpha$ and $(\alpha \Cap (\theta \circ \alpha)) \Subset \alpha$. Hence

$$(\alpha, S) = (\alpha \Cup (\theta \circ \alpha) \Cap (\alpha \Cup (\alpha \circ \theta)), S).$$

Conversely, let $x \in S$. Then

$$((\alpha \circ \theta) \Cap (\theta \circ \alpha))(x) \subseteq \alpha(x). \tag{5}$$

In fact, $((\alpha \circ \theta) \Cap (\theta \circ \alpha))(x) = (\alpha \circ \theta)(x) \cap (\theta \circ \alpha)(x)$. If $A_x = \emptyset$, then $(\alpha \circ \theta)(x) = \emptyset = (\theta \circ \alpha)(x)$. Thus condition (5) is satisfied.

If $A_x \neq \emptyset$, then

$$(\alpha \circ \theta)(x) = \bigcup_{(y,z) \in A_x} \{\alpha(y) \cap \theta(z)\} = \bigcup_{(y,z) \in A_x} \{\alpha(y)\}. \tag{6}$$

Also, we have

$$\alpha(y) \subseteq \gamma(x) \qquad \forall (y, z) \in A_x \tag{7}$$

Indeed, for $(y, z) \in A_x$ we have $x \leq yz$ and $\gamma(x) \supseteq \gamma(yz) \supseteq \gamma(y)$ because (γ, S) is a soft left ideal over U. By applying (7) to (6), we get

$$(\alpha \circ \theta)(x) = \bigcup_{(y,z) \in A_x} \{\alpha(y)\} \subseteq \gamma(x).$$

In a similar argument, we get $(\theta \circ \alpha)(x) \subseteq \beta(x)$. Hence

$$\begin{aligned}((\alpha \circ \theta) \Cap (\theta \circ \alpha))(x) &= (\alpha \circ \theta)(x) \cap (\theta \circ \alpha))(x) \\ &\subseteq \beta(x) \cap \gamma(x) = (\beta \Cap \gamma)(x) = \alpha(x),\end{aligned}$$

which completes the proof of (5). $\square$

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