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CERTAIN CURVES ON SOME CLASSES OF THREE-DIMENSIONAL ALMOST CONTACT METRIC MANIFOLDS

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Abstract. The object of the present paper is to characterize three-dimensional trans-Sasakian generalized Sasakian space forms admitting biharmonic almost contact curves with respect to generalized Tanaka Webster Okumura (gTWO) connections and to give illustrative examples. The mean curvature vector of almost contact curves has been analyzed on trans-Sasakian manifolds with gTWO connections. Some properties of slant curves on the same manifolds have been established. Finally curvature and torsion, with respect to gTWO connections, of C-parallel and C-proper slant curves in three-dimensional almost contact metric manifolds have been deduced.

1. Introduction

A beautiful notion of classical differential geometry of curves is that of curves of constant slope, also called cylindrical helix. This is a curve in the Euclidean space E^3 for which the tangent vector field has a constant angle with a fixed direction called the axis. Analogous to the above concept there are notions of slant curves and in particular Legendre curves in contact structure geometry. In the study of contact manifolds, Legendre curves play an important role, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair in [3]. Originally Legendre curves were defined on three-dimensional contact metric manifolds with the contact form η. Later the Legendre property has been extended to almost contact metric manifolds [54]. Legendre curves on almost contact metric manifolds are called almost contact curves [23]. After the work of Baikoussis and Blair [3], a good number of works have been done on Legendre curves or almost contact curves: [12, 24, 32, 41, 42, 43, 44, 50, 51, 52]. For the study of slant curves we refer to [8, 12, 21, 27]. Slant curves with C-parallel mean curvature vector fields have been

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studied in [31]. Again in [21] slant curves with C-parallel mean curvature vector field have been investigated.

An important class of almost contact metric manifolds is trans-Sasakian manifolds. Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by D. Chinea and C. Gonzalez [10], and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. Again in the Gray-Hervella classification of almost Hermite manifolds [20], there appears a class W_4 of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [40] if the product manifold $M \times R$ belongs to the class W_4. The class $C_6 \oplus C_5$ [34, 35] coincides with the class of trans-Sasakian structures of type (α, β). In [35], the local nature of the two subclasses C_5 and C_6 of trans-Sasakian structures is characterized completely. In [9], some curvature identities and sectional curvatures for C_5, C_6 and trans-Sasakian manifolds are obtained. It is known that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic, β-Kenmotsu, and α-Sasakian respectively [29].

The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by J. C. Marrero [34]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu manifold. So proper trans-Sasakian manifolds exist only for dimension three. The first author of the present paper has studied trans-Sassakian manifolds of dimension three [14].

The nature of a Riemannian manifold mostly depends on the manifold's curvature tensor R. It is well known that the sectional curvatures of a manifold determine curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as real-space-form and its curvature tensor is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}.$$

A Sasakian manifold with constant ϕ-sectional curvature is a Sasakian-space-form and it has a specific form of its curvature tensor. A similar notion also holds for Kenmotsu and cosymplectic space-forms. In order to generalize such space-forms in a common frame, P. Alegre, D. E. Blair, and A. Carriazo introduced in 2004 the notion of generalized Sasakian-space-forms [1]. But it is to be noted that generalized Sasakian-space-forms are not merely a generalization of such space-forms; it also contains a large class of almost contact manifolds. For example, it is known that any three-dimensional (α, β)-trans Sasakian manifold, with α, β depending on the Reeb vector field ξ, is a generalized Sasakian-space-form [2]. However, we can find generalized Sasakian-space-forms with non-constant functions and arbitrary dimensions. In [1], the authors cited several examples of generalized Sasakian-space-forms in terms of warped product spaces. In this regard, it should be mentioned that in 1989 Z. Olszak [38] studied generalized complex-space-forms and proved their existence. A generalized Sasakian-space-form is defined as follows [1].

Given an almost contact metric manifold $M(\phi, \xi, \eta, g)$, we say that M is generalized Sasakian-space-form if there exist three functions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$\begin{aligned}R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\},\end{aligned}$$

for any vector fields X, Y, Z on M. In such a case we denote the manifold as $M(f_1, f_2, f_3)$. Here we shall denote this manifold simply by M. If $f_1 = (c + 3)/4$, $f_2 = (c - 1)/4$, and $f_3 = (c - 1)/4$, then a generalized Sasakian-space-form with Sasakian structure becomes a Sasakian-space-form. The first author of the present paper has also studied generalized Sasakian space forms [15, 16, 22, 45, 46, 47, 48, 49].

In [1] and [2], the authors have shown the existence of trans-Sasakian generalized Sasakian space forms. By a trans-Sasakian generalized Sasakian space form we mean an almost contact manifold with trans-Sasakian metric and whose curvature tensor R is of the above form. In [1], the authors have given examples of trans-Sasakian generalized Sasakian space forms. For instance, if $N(c)$ is a complex space form, and if we consider the warped product $M = (-\frac{\pi}{2}, \frac{\pi}{2}) \times_f N$, with $f(t) = \cos t$, then M is a generalized Sasakian space form with

$$f_1 = \frac{c - 4\sin^2 t}{4\cos^2 t}, \quad f_2 = \frac{c}{4\cos^2 t}, \quad f_3 = \frac{c - 4\sin^2 t}{4\cos^2 t} - 1.$$

It is proved in [1, Theorem 4.8] that M is a trans-Sasakian manifold of type $(0, -\tan t)$. This proves the existence of trans-Sasakian generalized Sasakian space forms.

After the work of Jiang [30], there has been a growing interest in the theory of biharmonic maps which can be divided into three main research directions. On the one side, the differential geometric aspect has driven attention to the construction of examples and classification results. The other side is the analytic aspect from the point of view of PDEs: *biharmonic maps are solutions of a fourth order strongly elliptic semi linear PDE.* The theory of biharmonic functions is an old and rich subject; they have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. For more details we refer to [36]. Again for the study of biharmonic maps we may refer to [4, 5, 17, 33].

Biharmonic curves have been studied extensively in [18, 24, 26, 42]. Further references can be found there.

The study of pseudo-Hermitian geometry of curves with respect to the Tanaka Webster connection was initiated by J. T. Cho and collaborators [12, 32]. Pseudo-Hermitian geometry of curves were further studied in [13, 27, 25, 41]. The pseudo-Hermitian geometry of curves is studied with the Tanaka Webster connection, which is meaningful on contact manifolds. In contrast to the contact case, the Levi-form of almost contact metric manifolds may be degenerate. In addition Tanaka Webster connection is defined under contact condition. So to develop curve theory or submanifold theory in almost contact manifolds analogous to pseudo Hermitian connection we need an appropriate affine connection. Inoguchi and Lee introduced generalized Tanaka Webster Okumura connections [24] to curves in almost contact manifolds. Here the generalized Tanaka Webster Okumura connections will be called gTWO connections for short. In this paper we are interested in studying three-dimensional trans-Sasakian generalized Sasakian space forms admitting almost contact curves with gTWO connections. Attention has been given also to some slant curves with respect to gTWO connections on three-dimensional trans-Sasakian manifolds.

The present paper is organized as follows: After the introduction, required preliminaries are given in Section 2. Section 3 contains illustrative examples of almost contact curves in trans-Sasakian generalized Sasakian space forms. Section 4 is devoted to characterizing three-dimensional trans-Sasakian generalized Sasakian space forms admitting biharmonic almost contact curves with respect to generalized Tanaka Webster Okumura connections. In Section 5, we study the mean curvature vector of an almost contact curve with gTWO connections. Section 6 is concerned with some properties of slant curves on trans-Sasakian manifolds with gTWO connections. Curvature and torsion with respect to gTWO connections of slant curves with C-parallel mean curvature vector field on three-dimensional almost contact metric manifolds are calculated in Section 7. The last section gives curvature and torsion of C-proper slant curves in three-dimensional almost contact metric manifolds with respect to gTWO connections.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that (see [6])

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X),$$

for all $X, Y \in T(M)$ [3]. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y),$$

for $X, Y \in T(M)$.

An almost contact metric manifold is normal if $[\phi, \phi](X, Y) + 2d\eta(X, Y)\xi = 0$. Normal almost contact manifolds of dimension three have been studied in [39].

An almost contact metric structure (ϕ, ξ, η, g) on a manifold M is called trans-Sasakian structure [40] if $(M \times R, J, G)$ belongs to the class W_4 [20], where J is the almost complex structure on $M \times R$ defined by

$$J(X, f d/dt) = (\phi X - f\xi, \eta(X) d/dt),$$

for all vector fields X on M, a smooth function f on $M \times R$, and the product metric G on $M \times R$. This may be expressed by the condition (see [7])

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.1}$$

for smooth functions α and β on M. Here ∇ is the Levi-Civita connection on M. We call M a trans-Sasakian manifold of type (α, β). From (2.1) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{2.2}$$

A trans-Sasakian manifold is said to be

- a cosymplectic or co Kähler manifold if $\alpha = \beta = 0$,
- a quasi-Sasakian manifold if $\beta = 0$ and $\xi(\alpha) = 0$,
- an α-Sasakian manifold if α is a non-zero constant and $\beta = 0$,
- a β-Kenmotsu manifold if $\alpha = 0$ and β is a non-zero constant.

Therefore, a trans-Sasakian manifold generalizes a large class of almost contact manifolds.

For a generalized Sasakian space form, the Riemannian curvature tensor is given by

$$\begin{aligned} R(X,Y)Z = {} & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{2.3}$$

for any vector fields X, Y, Z on M.

The generalized Tanaka Webster Okumura connections [24] $\widetilde{\nabla}$ and the Levi-Civita connection ∇ are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + A(X,Y), \tag{2.4}$$

for all vector fields X, Y on M. Here

$$A(X,Y) = \alpha(g(X,\phi Y)\xi + \eta(Y)\phi X) + \beta(g(X,Y)\xi - \eta(Y)X) - l\eta(X)\phi Y, \tag{2.5}$$

where l is a real constant.

The torsion $\widetilde{\mathcal{T}}$ of the gTWO connections $\widetilde{\nabla}$ is given by

$$\begin{aligned} \widetilde{\mathcal{T}}(X,Y) = {} & \alpha(2g(X,\phi Y)\xi - \eta(X)\phi Y + \eta(Y)\phi X) \\ & + \eta(X)(\beta Y - l\phi Y) - \eta(Y)(\beta X - l\phi X). \end{aligned} \tag{2.6}$$

For $l = 0$, the connections become the connection introduced by Sasaki and Hatakeyama. When $l = 1$, it is the connection introduced by Cho [11]. For the Sasakian case and $l = 1$, the connection is the Okumura connection [37]. The connections become the generalized Tanaka Webster connection introduced by Tanno [53], when $l = -1$.

Let M be a 3-dimensional Riemannian manifold. Let $\gamma : I \to M$, with I an interval, be a curve in M which is parametrized by arc length, and let $\nabla_{\dot\gamma}$ denote the covariant differentiation along γ with respect to the Levi-Civita connection on M. It is said that γ is a Frenet curve if one of the following three cases holds:

(a) γ is of osculating order 1, i.e., $\nabla_t t = 0$ (geodesic), $t = \dot\gamma$. Here, the dot denotes differentiation with respect to arc parameter.

(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $t(= \dot\gamma)$, n and a non-negative function k (curvature) along γ such that $\nabla_t t = kn$, $\nabla_t n = -kt$.

(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $t(=\dot{\gamma})$, n, b and two non-negative functions k (curvature) and τ (torsion) along γ such that

$$\begin{aligned}\nabla_t t &= kn,\\ \nabla_t n &= -kt + \tau b,\\ \nabla_t b &= -\tau n.\end{aligned}$$

With respect to the Levi-Civita connection, a Frenet curve of osculating order 3 for which k is a positive constant and $\tau = 0$ is called a circle in M; a Frenet curve of osculating order 3 is called a helix in M if k and τ both are positive constants, and the curve is called a generalized helix if k/τ is a constant.

Let $\widetilde{\nabla}_{\dot{\gamma}}$ denote the covariant differentiation along γ with respect to gTWO connections on M. We shall say that γ is a Frenet curve with respect to gTWO connections if one of the following three cases holds:

(a) γ is of osculating order 1, i.e., $\widetilde{\nabla}_t t = 0$ (geodesic).
(b) γ is of osculating order 2, i.e., there exist two orthonormal vector fields $t(=\dot{\gamma})$, n and a non-negative function $\tilde{k}$ (curvature) along γ such that $\widetilde{\nabla}_t t = \tilde{k}n$, $\widetilde{\nabla}_t n = -\tilde{k}t$.
(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $t(=\dot{\gamma})$, n, b and two non-negative functions $\tilde{k}$ (curvature) and $\tilde{\tau}$ (torsion) along γ such that

$$\widetilde{\nabla}_t t = \tilde{k}n, \tag{2.7}$$

$$\widetilde{\nabla}_t n = -\tilde{k}t + \tilde{\tau}b, \tag{2.8}$$

$$\widetilde{\nabla}_t b = -\tilde{\tau}n. \tag{2.9}$$

With respect to the gTWO connection, a Frenet curve of osculating order 3 for which $\tilde{k}$ is a positive constant and $\tilde{\tau} = 0$ is called a circle in M; a Frenet curve of osculating order 3 is called a helix in M if $\tilde{k}$ and $\tilde{\tau}$ both are positive constants, and the curve is called a generalized helix with respect to gTWO connections if $\tilde{k}/\tilde{\tau}$ is a constant.

A Frenet curve γ in an almost contact metric manifold is said to be almost contact curve if it is an integral curve of the distribution $\mathcal{D} = \ker\eta$. Formally, it is also said that a Frenet curve γ in an almost contact metric manifold is an almost contact curve if and only if $\eta(\dot{\gamma}) = 0$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$. For more details we refer to [3, 24, 54].

It is to be mentioned that in [24], curves satisfying the above properties on almost contact manifolds have been termed *almost contact curves*, while Welyczko [54] has termed such curves on almost contact manifolds *Legendre curves*. Henceforth by Legendre curves on almost contact manifolds we shall mean almost contact curves.

If we consider the binormal vector field b along ξ, then by (2.2) and (2.4) $\widetilde{\nabla}_t b = 0$ holds for an almost contact curve with respect to gTWO connections. Hence in that case we can say that the torsion $\tilde{\tau}$ of an almost contact curve on a three-dimensional almost contact metric manifold with respect to gTWO connections is given by

$$\tilde{\tau} = 0. \tag{2.10}$$

A Frenet curve is called a slant curve if it makes a constant angle with the Reeb vector field ξ [12]. If a curve γ on an almost contact metric manifold is a slant curve then $\eta(\dot{\gamma}) = \cos\theta$ and $g(\dot{\gamma}, \dot{\gamma}) = 1$, where θ is a constant and is called slant angle. In particular if the angle is $\frac{\pi}{2}$, the curve becomes an almost contact curve.

The example of slant curves in an almost contact metric manifold is given in [21, Example 4, p. 98].

3. Examples of almost contact curves on three-dimensional trans-Sasakian generalized Sasakian space forms

Example 3.1. Let us consider $\mathbb{R}^2$ with the metric $G = dx^2 + dy^2$. We can consider $(\mathbb{R}^2, G)$ as a Kählerian manifold $\mathbb{C}^1$ with $G = (dx + idy) \odot (dx - idy)$. Then, according to [1], $M = \mathbb{R} \times_f \mathbb{C}^1$ is a generalized Sasakian space form with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where $f(z) = e^{2z}$, the warped product metric $g = e^{2z}(dx^2 + dy^2) + dz^2$. Obviously M is a subset of R^3.

We can consider $M = \{(x, y, z) \in R^3, z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \quad e_2 = e^{-z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. The metric g defined above can also be written in the following format as an inner product, instead of a quadratic form:

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = e_2$, $\phi(e_2) = -e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1, e_2] = 0, \quad [e_2, e_3] = e_2, \quad [e_1, e_3] = e_1.$$

By Koszul's formula,

$$\begin{array}{lll}
\nabla_{e_1} e_3 = e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = -e_3, \\
\nabla_{e_2} e_3 = e_2, & \nabla_{e_2} e_2 = -e_3, & \nabla_{e_2} e_1 = 0, \\
\nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0.
\end{array}$$

From above it can be observed that M is a trans-Sasakian manifold of type $(0,1)$. Hence, according to [1], M is a trans-Sasakian generalized Sasakian space form with

$$f_1 = -\frac{(f')^2}{f^2}, \quad f_2 = 0, \quad f_3 = -\frac{(f')^2}{f^2} + \frac{f''}{f},$$

where $f(z) = e^{2z}$.

Using (2.4) and (2.5), it can be easily calculated that

$$\widetilde{\nabla}_{e_3} e_1 = -le_2, \qquad \widetilde{\nabla}_{e_3} e_2 = le_1.$$

The other $\widetilde{\nabla}_{e_i} e_j$ are 0. Consider a curve $\gamma : I \to M$ defined by $\gamma(s) = (\gamma_1(s), \gamma_2(s), \gamma_3(s)) = \left(\sqrt{\frac{2}{3}}s, \sqrt{\frac{1}{3}}s, 1\right)$. Hence $\dot{\gamma}_1 = \sqrt{\frac{2}{3}}$, $\dot{\gamma}_2 = \sqrt{\frac{1}{3}}$, and $\dot{\gamma}_3 = 0$. Now

$$\begin{aligned} \eta(\dot{\gamma}) &= g(\dot{\gamma}, e_3) = g(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3, e_3) = 0, \\ g(\dot{\gamma}, \dot{\gamma}) &= g(\dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3, \dot{\gamma}_1 e_1 + \dot{\gamma}_2 e_2 + \dot{\gamma}_3 e_3) \\ &= \dot{\gamma}_1^2 + \dot{\gamma}_2^2 + \dot{\gamma}_3^2 \\ &= \dot{\gamma}_1^2 + \dot{\gamma}_2^2 \\ &= \frac{2}{3} + \frac{1}{3} \\ &= 1. \end{aligned}$$

Hence the curve is an almost contact curve. For this curve, $\widetilde{\nabla}_{\dot{\gamma}} \dot{\gamma} = 0$. So the curve is a geodesic with respect to gTWO connections.

Example 3.2. Explicit formulas for proper biharmonic curves γ satisfying $\eta(\dot{\gamma}) = 0$ with respect to the Levi-Civita connection are given in [19]. The example of non-geodesic biharmonic almost contact curve with respect to generalized Tanaka Webster Okumura connections can be found in [24]. In the following we restate it just for illustration.

Let us consider the unit three sphere $S^3 \subset R^4(x_1, x_2, x_3, x_4)$ in Euclidean 4-space, centered at the origin. A model helix in S^3 is given by

$$\gamma(s) = (\cos\phi \cos(as), \sin\phi \sin(as), \sin\phi \cos(bs), \sin\phi \sin(bs)), \tag{3.1}$$

where a, b and ϕ are constants satisfying $a^2 \cos\phi + b^2 \sin^2\phi = 1$. Here s is the arc length parameter. The helix has curvature k and torsion τ as follows:

$$k = \sqrt{(a^2-1)(1-b^2)}, \quad \tau = ab.$$

The helix lies in the flat torus is S^3 defined by

$$x_1^2 + x_2^2 = \cos^2\phi, \quad x_3^2 + x_4^2 = \sin^2\phi.$$

This flat torus has constant mean curvature $\cot 2\phi$. Every proper helix, i.e., a curve with constant curvature $\kappa \neq 0$ and constant torsion $\tau \neq 0$, is congruent to one of the helices. Thus choosing the parameters a and b so that $ab = 1$ and $2(1-t) = (a^2-1)(1-b^2)$ in (3.1) we get the parametrization of an almost contact curve in S^3. The curve is biharmonic with respect to gTWO connections [24].

Now S^3 is a trans-Sasakian generalized Sasakian space form with $f_1 = 1$, $f_2 = f_3 = 0$ and $\alpha = 1$, $\beta = 0$.

These examples will be used for the purpose of illustration in the next section.

4. Characterization of three-dimensional trans-Sasakian generalized Sasakian space forms admitting biharmonic almost contact curves with respect to gTWO connections

Definition 4.1. An almost contact curve γ on a three-dimensional trans-Sasakian manifold is called biharmonic with respect to gTWO connections $\widetilde{\nabla}$ if it satisfies the equation

$$\widetilde{\nabla}_t^3 t + \widetilde{\nabla}_t \widetilde{\mathcal{T}}(\widetilde{\nabla}_t t, t) + \tilde{R}(\widetilde{\nabla}_t t, t)t = 0, \tag{4.1}$$

where $\dot{\gamma} = t$, $\widetilde{\mathcal{T}}$ is the torsion of the connections, and $\tilde{R}$ is the curvature of the gTWO connections (see [24]).

Let us consider a biharmonic almost contact curve with respect to gTWO connections on a three-dimensional trans-Sasakian manifold. We take $\{T, N, B\}$ as a Frenet frame on a tubular neighborhood of the image of γ, where $N := -\phi T$ and $B := \xi$ is considered to fix an orientation. In view of (2.4), we get

$$\begin{aligned} \tilde{R}(X,Y)Z = R(X,Y)Z &+ \nabla_X A(Y,Z) - \nabla_Y A(X,Z) \\ &+ A(X, \nabla_Y Z) + A(X, A(Y,Z)) - A(Y, \nabla_X Z) \\ &- A(Y, A(X,Z)) - A([X,Y],Z), \end{aligned}$$

for any vector fields X, Y, Z on M. Here $\tilde{R}$ is the curvature of the gTWO connections and R is that of the Levi-Civita connection ∇. From the above equation we get, by putting $X = N$, $Y = Z = T$,

$$\begin{aligned} \tilde{R}(N,T)T = R(N,T)T &+ \nabla_X A(T,T) - \nabla_T A(N,T) \\ &+ A(N, \nabla_T T) + A(N, A(T,T)) - A(T, \nabla_N T) \\ &- A(T, A(N,T)) - A([N,T],T), \end{aligned} \tag{4.2}$$

In view of (2.5), we have

$$\begin{aligned} A(T,T) &= \beta\xi \\ \nabla_N A(T,T) &= \beta'\xi \\ A(N,T) &= -\alpha\xi \\ \nabla_T A(N,T) &= -\alpha'\xi \\ A(N, \nabla_T T) &= \kappa\beta\xi \\ A(N, A(T,T)) &= \alpha\beta T + \beta^2\phi T \\ A(T, \nabla_N T) &= -\beta\eta(\nabla_T T) + \alpha\eta(\nabla_N T)\phi T + g(\alpha N + \beta T, \nabla_T T)\xi \\ A(T, A(N,T)) &= -\alpha^2\phi N + \alpha\beta T \end{aligned} \tag{4.3}$$

Again $g(T,T) = 1$ implies, by covariant differentiation with respect to N, that

$$g(\nabla_N T, T) = 0. \tag{4.4}$$

Similarly $g(T,\xi)=0$ gives $\eta(\nabla_N T)=\alpha$ and $g(N,\xi)=0$ produces $\eta(\nabla_T N)=-\alpha$. Using equalities (4.3) in (4.2) we obtain

$$\tilde{R}(N,T)T = R(N,T)T + (\alpha^2+\beta^2)\phi T + (d\alpha(T) + d\beta(N) + \tilde{\kappa}\beta)\xi. \tag{4.5}$$

By (2.3), we have

$$R(N,T)T = -(f_1+3f_2)\phi T. \tag{4.6}$$

In view of (4.5) and (4.6) it follows that

$$\tilde{R}(N,T)T = -(f_1+3f_2)\phi T + (\alpha^2+\beta^2)\phi T + (d\alpha(T) + d\beta(N) + \tilde{\kappa}\beta)\xi.$$

Now by definition $N=-\phi T$ and $B=\xi$. So

$$\tilde{R}(N,T)T = (f_1+3f_2-\alpha^2-\beta^2)N - (d\alpha(T) + d\beta(N) + \tilde{\kappa}\beta)B. \tag{4.7}$$

Now

$$\begin{aligned} \tilde{R}(\tilde{\nabla}_T T, T)T &= \tilde{R}(\tilde{\kappa}N, T)T \\ &= \tilde{\kappa}\tilde{R}(N,T)T \\ &= \tilde{\kappa}((f_1+3f_2-\alpha^2-\beta^2)N - (d\alpha(T)+d\beta(N)+\tilde{\kappa}\beta)B). \end{aligned} \tag{4.8}$$

By the use of (2.7)–(2.9) and (2.10),

$$\tilde{\nabla}^3_T T = (\tilde{\kappa}'' - \tilde{\kappa}^3)N - 3\tilde{\kappa}\tilde{\kappa}' T. \tag{4.9}$$

In view of (2.6),

$$\tilde{\nabla}_T \tilde{\mathcal{T}}(\tilde{\nabla}_T T, T) = 2\alpha'\tilde{\kappa}'\xi. \tag{4.10}$$

By virtue of (4.8), (4.9) and (4.10) it follows that

$$\begin{aligned} &\tilde{\nabla}^3_T T + \tilde{\nabla}_T \tilde{T}(\tilde{\nabla}_T T, T) + \tilde{R}(\tilde{\nabla}_T T, T)T \\ &\qquad = -3\tilde{\kappa}\tilde{\kappa}' T + (\tilde{\kappa}'' - \tilde{\kappa}^3 + \tilde{\kappa}(f_1+3f_2-\alpha^2-\beta^2))N \\ &\qquad\quad + (2\alpha'\tilde{\kappa}' - d\alpha(T) - d\beta(N) - \tilde{\kappa}\beta)B. \end{aligned} \tag{4.11}$$

If the almost contact curve is biharmonic the above equation yields

(i) $\tilde{\kappa}\tilde{\kappa}'=0$,
(ii) $\tilde{\kappa}'' - \tilde{k}^3 + \tilde{\kappa}(f_1+3f_2-\alpha^2-\beta^2)=0$,
(iii) $2\alpha'\tilde{\kappa}' - d\alpha(T) - d\beta(N) - \tilde{\kappa}\beta = 0$.

Consider the curve is not a geodesic. Then, by virtue of (i) and (iii), $\beta=0$ if α,β are constants. Hence we are in a position to state the following theorem.

Theorem 4.2. *If a three-dimensional trans-Sasakian generalized Sasakian space form with constant structure function α,β admits a non-geodesic biharmonic almost contact curve with respect to $gTWO$ connections, then it is an α-Sasakian manifold.*

Remark 4.3. In Example 3.2, the manifold is α-Sasakian with $\alpha=1$. The curve is proper biharmonic. So the example agrees with the above theorem.

Theorem 4.2 can alternatively be stated as follows.

Theorem 4.4. *A three-dimensional non α-Sasakian trans-Sasakian generalized Sasakian space form does not admit non-geodesic biharmonic almost contact curves with respect to gTWO connections, i.e., an almost contact curve on a non-α-Sasakian trans-Sasakian generalized Sasakian space form is biharmonic if and only if it is a geodesic.*

Remark 4.5. Example 3.1 agrees with Theorem 4.4.

Again, from (i) and (ii),

$$f_1 + 3f_2 = \tilde{\kappa}^2 + \alpha^2 + \beta^2. \tag{4.12}$$

From [1], it is known that $f_1 + 3f_2$ is the ϕ-sectional curvature of a generalized Sasakian space form. Thus the above equation leads us to state the following corollary.

Corollary 4.6. *If a three-dimensional cosymplectic generalized Sasakian space form admits a non-geodesic biharmonic almost contact curve with respect to gTWO connections, then the curvature of the curve is equal to the positive square root of the ϕ-sectional curvature of the space form.*

Since by definition $\tilde{\kappa} > 0$, in view of (4.12) we also have the following corollary.

Corollary 4.7. *A three-dimensional cosymplectic generalized Sasakian space form with negative ϕ-sectional curvature does not admit a non-geodesic biharmonic almost contact curve with respect to gTWO connections.*

5. Mean curvature vector of almost contact curves in three-dimensional trans-Sasakian manifolds with gTWO connections

Definition 5.1. A curve γ in an almost contact metric manifold is said to have

- parallel mean curvature vector $\widetilde{H}$ with respect to gTWO connections if $\widetilde{\nabla}_T T = 0$, where $T = \dot{\gamma}$;
- proper mean curvature vector $\widetilde{H}$ with respect to gTWO connections if $\widetilde{\triangle}\widetilde{H} = \lambda\widetilde{H}$, for a real number λ. Here we shall consider $\widetilde{\triangle}\widetilde{H} = -\widetilde{\nabla}_T\widetilde{\nabla}_T\widetilde{\nabla}_T$ as a convention.

Let $\widetilde{H}$ be the mean curvature vector of an almost contact curve with respect to gTWO connections. Now $\widetilde{H} = \widetilde{\nabla}_T T$. In view of (2.7) and (2.8), it follows that

$$\widetilde{H} = \widetilde{\nabla}_T T = \tilde{k}N = \nabla_T T + \beta\xi = H + \beta\xi, \tag{5.1}$$

where H is the mean curvature vector with respect to the Levi-Civita connection. From above we have the following proposition.

Proposition 5.2. *An almost contact curve on a three-dimensional trans-Sasakian manifold is minimal with respect to gTWO connections if and only if it is geodesic with respect to gTWO connections.*

Corollary 5.3. *For a non-β-Kenmotsu trans-Sasakian manifold an almost contact curve is minimal with respect to gTWO connections if and only if it is so with respect to the Levi-Civita connection.*

In view of (2.7)–(2.9) and (2.10) it follows that

$$\widetilde{\nabla}_T \widetilde{H} = -\tilde{\kappa}^2 T + \tilde{\kappa}' N.$$

By observing the components from the above equation we can state the following proposition.

Proposition 5.4. *An almost contact curve on a three-dimensional trans-Sasakian manifold has parallel mean curvature vector if and only if it is a geodesic with respect to gTWO connections.*

Consider

$$\widetilde{\triangle} \widetilde{H} = \lambda \widetilde{H},$$

for a real number λ. Hence using (2.7)–(2.9) and (2.10), we have

$$3\tilde{\kappa}\tilde{\kappa}' T + (\tilde{\kappa} - \tilde{\kappa}'' - \lambda) N = 0.$$

From above we have, by observing the components, that

$$\tilde{\kappa} = 0.$$

Thus we have the following proposition.

Proposition 5.5. *The mean curvature vector of an almost contact curve in a three-dimensional trans Sasakian manifold is proper if and only if the curve is a geodesic with respect to gTWO connections.*

In view of Propositions 5.2, 5.4, and 5.5, we obtain the following theorem.

Theorem 5.6. *In a three-dimensional trans-Sasakian manifold the following conditions are equivalent:*

- *An almost contact curve in a three-dimensional trans-Sasakian manifold is minimal with respect to gTWO connections.*
- *The mean curvature vector with respect to gTWO connections of an almost contact curve in a trans-Sasakian manifold is parallel.*
- *The mean curvature vector of an almost contact curve in a three-dimensional trans-Sasakian manifold is proper with respect to gTWO connections.*
- *The almost contact curve on a three-dimensional trans-Sasakian manifold is a geodesic with respect to gTWO connections.*

Next, we consider mean curvature vector fields of an almost contact curve in a trans-Sasakian manifold with gTWO connections in the normal bundle. Consider

$$\widetilde{\triangle}^{\perp} \widetilde{H} = \lambda \widetilde{H}.$$

By the use of (2.7)–(2.9) and (2.10), it follows from above that

$$\tilde{\kappa}'' + \lambda \tilde{\kappa} = 0.$$

For λ a non-zero constant, the above equation yields

$$\tilde{\kappa}(s) = \cos(\pm\sqrt{\lambda}s + c),$$

where c is a constant.

Theorem 5.7. *If the mean curvature vector with respect to gTWO connections of an almost contact curve in a three-dimensional trans-Sasakian manifold is proper in the normal bundle, then $\tilde{\kappa}(s) = \cos(\pm\sqrt{\lambda}s + c)$. The converse is also true.*

6. Slant curves in trans-Sasakian manifolds with gTWO connections

Let us consider a slant curve γ on a trans-Sasakian manifold with gTWO connections. Here $\gamma'(s) = T(s)$ is given by

$$\cos\theta(s) = g(T(s), \xi), \tag{6.1}$$

where θ is the constant slant angle. By covariant differentiation with respect to $\widetilde{\nabla}$ we get from (6.1)

$$-\sin\theta\theta' = g(\widetilde{\nabla}_T T, \xi) - g(T, \tilde{\nabla}_T \xi).$$

Considering θ a constant and using (2.7), (2.4), and (2.5) we get from above

$$\tilde{\kappa}\eta(N) = 0. \tag{6.2}$$

In [28, p. 155] the following notion has been introduced: *A non-geodesic curve is called slant helix if the principal normal lines of γ makes a constant angle with a fixed direction.* So by virtue of (6.2) we state the following theorem.

Theorem 6.1. *A slant curve on a three-dimensional trans-Sasakian manifold is either a geodesic with respect to gTWO connections or a slant helix with the direction of ξ as fixed direction.*

Again from (2.4) and (2.5)

$$\widetilde{\nabla}_T T = \nabla_T T + (\alpha - l)\cos\theta\phi T - \beta\cos\theta T + \beta\xi. \tag{6.3}$$

So

$$\tilde{\kappa}N = \kappa N + (\alpha - l)\cos\theta\phi T - \beta\cos\theta T + \beta\xi. \tag{6.4}$$

Taking inner product with ξ we get from (6.4) that $\beta\sin^2\theta = 0$. Now we are in a position to state the following theorem.

Theorem 6.2. *If a three-dimensional trans-Sasakian manifold admits slant curves with respect to gTWO connections, then it becomes an α-Sasakian manifold.*

7. Curvature and torsion of C-parallel slant curves in three-dimensional almost contact metric manifolds with gTWO connections

Definition 7.1. A curve γ in an almost contact metric manifold is defined to be C-parallel with respect to gTWO connections [31] if

$$\widetilde{\nabla}_T \widetilde{H} = \lambda\xi, \tag{7.1}$$

where λ is a differentiable function along γ.

If $\{E_1, E_2, E_3\}$ is a Frenet frame then (7.1) implies

$$-\tilde{\kappa}E_1 + \tilde{\kappa}'E_2 + \tilde{\kappa}\tilde{\tau}E_3 = \lambda\xi, \tag{7.2}$$

from which it follows that

$$\eta(E_1) = -\frac{1}{\lambda}\tilde{\kappa}^2, \tag{7.3}$$

$$\eta(E_2) = \frac{1}{\lambda}\tilde{\kappa}', \tag{7.4}$$

$$\eta(E_3) = \frac{1}{\lambda}\tilde{\kappa}\tilde{\tau}. \tag{7.5}$$

But for slant curves with slant angle θ,

$$\eta(E_1) = \cos\theta. \tag{7.6}$$

Hence from (7.3)

$$\tilde{\kappa}^2 = -\lambda\cos\theta. \tag{7.7}$$

By virtue of (2.4), (2.5), (7.6), and $(\widetilde{\nabla}_{E_1}g)(E_1, \xi) = 0$, it follows after simplification that

$$\eta(E_2) = 0. \tag{7.8}$$

From (7.4) and (7.8) we obtain $\tilde{\kappa}' = 0$. So we have the following theorem.

Theorem 7.2. *The curvature $\tilde{\kappa}$ with respect to the gTWO connection of a C-parallel slant curve in a three-dimensional almost contact metric manifold is a constant.*

By virtue of (7.2) it follows that $\xi \in \operatorname{span}\{E_1, E_2, E_3\}$. So we can write

$$\xi = \cos\theta E_1 + \sin\theta(\cos\psi E_2 + \sin\psi E_3), \tag{7.9}$$

where ψ is the angle function between E_2 and the orthogonal projection of ξ onto $\operatorname{span}\{E_2, E_3\}$. Taking inner product with E_2 and E_3 respectively, and using (7.9), (7.7) and (7.8) we find

$$\cos\psi = 0, \quad \sin\psi = -\frac{\tilde{\tau}\cot\theta}{\tilde{\kappa}}. \tag{7.10}$$

Hence from (7.10)

$$\tilde{\tau}^2 = \tilde{\kappa}^2\tan^2\theta. \tag{7.11}$$

For the Reeb flow $\theta = 0$ and for an almost contact curve $\theta = \frac{\pi}{2}$. So by virtue of (7.11) and Theorem 7.2 we state the following theorems.

Theorem 7.3. *The Reeb flow on a three-dimensional almost contact metric manifold with gTWO connections is a circle.*

Theorem 7.4. *A three-dimensional almost contact metric manifold with gTWO connections does not admit a C-parallel almost contact curve with finite torsion.*

8. Curvature and torsion of C-proper slant curves in three-dimensional trans-Sasakian manifolds with gTWO connections

Definition 8.1. A curve γ in an almost contact metric manifold M is defined to be C-proper if

$$\widetilde{\triangle}H = \lambda\xi, \tag{8.1}$$

where $\widetilde{H}$ is the mean curvature vector field, $\widetilde{\triangle}$ is the Laplacian with respect to gTWO connections, and λ is a non-zero differentiable function along γ (see [31]).

Here we take $\widetilde{\triangle}\widetilde{H} = -\widetilde{\nabla}_T\widetilde{\nabla}_T\widetilde{\nabla}_T T$ as a convention. Consider a Frenet curve $\gamma : I \to M$ and $\{E_1, E_2, E_3\}$ as a Frenet frame. Then by the use of (2.7)–(2.9) we have

$$\widetilde{\triangle}H = -3\tilde{\kappa}\tilde{\kappa}E_1 + (\tilde{\kappa}'' - \tilde{\kappa}^3 - \tilde{\kappa}\tilde{\tau}^2)E_2 - (2\tilde{\kappa}'\tilde{\tau} + \tilde{\kappa}\tilde{\tau}')E_3.$$

Hence (8.1) implies

$$-3\tilde{\kappa}\tilde{\kappa}'E_1 + (\tilde{\kappa}'' - \tilde{\kappa}^3 - \tilde{\kappa}\tilde{\tau}^2)E_2 + (2\tilde{\kappa}'\tilde{\tau} + \tilde{\kappa}\tilde{\tau}')E_3 + \lambda\xi = 0. \tag{8.2}$$

Taking inner product with E_1 we get from above

$$\eta(E_1) = \frac{1}{\lambda}3\tilde{\kappa}\tilde{\kappa}'. \tag{8.3}$$

Similarly

$$\eta(E_2) = -\frac{1}{\lambda}(\tilde{\kappa}'' - \tilde{\kappa}^3 - \tilde{\kappa}\tilde{\tau}^2). \tag{8.4}$$

$$\eta(E_3) = -\frac{1}{\lambda}(2\tilde{\kappa}'\tilde{\tau} + \tilde{\kappa}\tilde{\tau}').$$

But for a slant curve with slant angle θ,

$$\eta(E_1) = \cos\theta. \tag{8.5}$$

Hence from (8.3)

$$\tilde{\kappa}\tilde{\kappa}' = \frac{1}{3}\lambda\cos\theta.$$

By virtue of (2.4), (2.5), (8.5), and $(\widetilde{\nabla}_{E_1}g)(E_1,\xi) = 0$ it follows after simplification that

$$\eta(E_2) = 0. \tag{8.6}$$

In view of (8.4) and (8.6) we obtain

$$\tilde{\kappa}'' = \tilde{\kappa}(\tilde{\kappa}^2 + \tilde{\tau}^2). \tag{8.7}$$

Again by (8.3) and (8.6), the equation $(\widetilde{\nabla}_{E_1}g)(E_2,\xi) = 0$ gives

$$\tilde{\tau}\eta(E_3) = \tilde{\kappa}\cos\theta. \tag{8.8}$$

As in the previous section,

$$\xi = \cos\theta E_1 + \sin\theta(\cos\psi E_2 + \sin\psi E_3), \tag{8.9}$$

where ψ is the angle function between E_2 and the orthogonal projection of ξ onto span$\{E_2, E_3\}$.

Taking inner product with E_2 and E_3 and using (8.8) in the above equation, we get

$$\cos\psi = 0 \tag{8.10}$$

$$\sin\psi = \frac{\tilde{\kappa}}{\tilde{\tau}}\cot\theta. \tag{8.11}$$

Thus (8.10) and (8.11) yield

$$\tilde{\tau}^2 = \tilde{\kappa}^2\cot^2\theta. \tag{8.12}$$

From (8.7) and (8.12) we have

$$\tilde{\kappa}'' = \mu\tilde{k}^3, \tag{8.13}$$

where $\mu = \operatorname{cosec}^2\theta$ is a constant. Multiplying both sides of the equation (8.13) by $2\frac{d\tilde{\kappa}}{ds}$ and integrating we get

$$\left(\frac{d\tilde{k}}{ds}\right)^2 = \frac{1}{2}\mu\tilde{\kappa}^4 + \frac{c}{2}, \tag{8.14}$$

where c is a constant of integration. The above equation implies

$$\sqrt{\frac{2c}{\mu}}\int\frac{d\tilde{\kappa}}{\sqrt{\tilde{\kappa}^4+\frac{c}{\mu}}} = s + d, \tag{8.15}$$

where d is a constant. Let

$$I = \int\frac{d\tilde{\kappa}}{\sqrt{\tilde{\kappa}^4+\frac{c}{\mu}}}. \tag{8.16}$$

Put

$$\tilde{\kappa} = \left(\frac{c}{\mu}\right)^{\frac{1}{4}}\sqrt{\tan x}. \tag{8.17}$$

Then

$$I = \left(\frac{c}{\mu}\right)^{-\frac{1}{2}}\int\sqrt{\tan x}\,dx \tag{8.18}$$

After integrating, we have from above

$$I = \left(\frac{c}{\mu}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{2}}\left\{\sin^{-1}(\sin x - \cos x) - \log\left|\sin x + \cos x + \sqrt{\sin 2x}\right|\right\}. \tag{8.19}$$

Using the substitution (8.17),

$$\begin{aligned} I = \left(\frac{c}{\mu}\right)^{-\frac{1}{2}}\frac{1}{\sqrt{2}}\Bigg\{&\sin^{-1}(\sin(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2) - \cos(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2)) \\ &- \log\left|\sin(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2) + \cos(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2) + \sqrt{\sin 2(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2)}\right|\Bigg\}. \end{aligned} \tag{8.20}$$

By (8.15) and (8.20), we get that the solution of (8.13) is

$$s + d = \sin^{-1}(\sin(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2) - \cos(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2)) \\ - \log\left|\sin(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2) + \cos(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2) + \sqrt{\sin 2(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2)}\right|, \tag{8.21}$$

where d is a constant of integration. The above discussion helps us to state the following theorem.

Theorem 8.2. *The curvature with respect to gTWO connections of a C-proper slant curve of a three-dimensional almost contact metric manifold is given by*

$$s + d = \sin^{-1}(\sin(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2) - \cos(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2)) \\ - \log\left|\sin(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2) + \cos(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2) + \sqrt{\sin 2(\tan^{-1}\sqrt{\frac{\mu}{c}}\tilde{\kappa}^2)}\right|,$$

and the torsion is given by $\tilde{\tau}^2 = \tilde{\kappa}^2 \cot\theta$.

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