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# GENERALIZATIONS OF CLINE'S FORMULA FOR THREE GENERALIZED INVERSES

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Abstract. It is shown that an element $a$ in a ring is Drazin invertible if and only if so is $a^n$; the Drazin inverse of $a$ is given by that of $a^n$, and vice versa. Using this result, we prove that, in the presence of $aba = aca$, for any natural numbers $n$ and $m$, $(ac)^n$ is Drazin invertible in a ring if and only if so is $(ba)^m$; the Drazin inverse of $(ac)^n$ is expressed by that of $(ba)^m$, and vice versa. Also, analogous results for the pseudo Drazin inverse and the generalized Drazin inverse are established on Banach algebras.

## 1. Introduction

Throughout this paper, $A$ will denote a complex Banach algebra with identity 1 and $R$ will denote an associative ring with identity 1. $J(R)$ (resp. $J(A)$) denotes the *Jacobson radical* of $R$ (resp. $A$). The *commutant* and *double commutant* of an element $a \in R$ are defined as usual by

$$\text{comm}(a) = \{x \in R,\ ax = xa\}$$

and

$$\text{comm}^2(a) = \{x \in R,\ xy = yx \text{ for all } y \in \text{comm}(a)\},$$

respectively. We say that $b \in R$ is an *outer generalized inverse* of an element $a \in R$ provided that $bab = b$. The element $b \in R$ is not unique in general. In order to force its uniqueness, further conditions have to be imposed.

In 1958, Drazin [4] introduced the following generalized inverse. An element $a \in R$ is called *Drazin invertible* provided that there is a common solution to the equations

$$b \in \text{comm}(a),\ bab = b \quad \text{and} \quad a^k ba = a^k \text{ for some } k \geq 0. \tag{1.1}$$

If such a solution exists, then it is unique and is called a *Drazin inverse* of $a$, denoted as usual by $b = a^D$. The minimal $k$ for which (1.1) holds is called the *Drazin index* $i(a)$ of $a$. Moreover, $a^D \in \text{comm}^2(a)$ (see [4, Theorem 1]). If $i(a) \leq 1$, then $a$

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is *group invertible* and $b$ is called a *group inverse* of $a$. Some applications of the group and Drazin inverses can be found in [5, 7, 8, 9, 10, 13, 16].

In 2002, Koliha and Patrício [12] introduced the notion of generalized Drazin inverse. The *generalized Drazin inverse* is the unique common solution to the equations

$$b \in \mathrm{comm}^2(a),\ bab = b \quad \text{and} \quad aba - a \text{ is quasinilpotent.} \tag{1.2}$$

Here an element $s \in R$ is said to be *quasinilpotent* [6] if $1 + st$ is invertible for all $t \in \mathrm{comm}(s)$. If such a solution to (1.2) exists, then it is denoted as usual by $b = a^{gD}$ and we shall call the element $a \in R$ *generalized Drazin invertible.* According to [11, Theorem 4.4], the condition $b \in \mathrm{comm}^2(a)$ in (1.2) can be weakened as $b \in \mathrm{comm}(a)$ in the Banach algebra case.

An intermediate between the Drazin inverse and the generalized Drazin inverse is the pseudo Drazin inverse, which was introduced by Wang and Chen [17] in 2012. The *pseudo Drazin inverse* is the unique common solution to the equations

$$b \in \mathrm{comm}^2(a),\ bab = b \quad \text{and} \quad a^k ba - a^k \in J(R) \tag{1.3}$$

for some $k \geq 0$. The minimal such $k$ is called the *pseudo Drazin index* $i(a)$ of $a$. If such a solution to (1.3) exists, then it is denoted as usual by $b = a^{pD}$ and we shall call the element $a \in R$ *pseudo Drazin invertible.* Also, in a Banach algebra, the condition $b \in \mathrm{comm}^2(a)$ in (1.3) can be weakened to $b \in \mathrm{comm}(a)$ (see [17, Remark 5.1]).

In 1965, Cline [2] showed that if $ab$ is Drazin invertible then so is $ba$ and $(ba)^D = b((ab)^D)^2a$. This equation is the so-called Cline's formula. Cline's formulas for the generalized Drazin inverse and the pseudo Drazin inverse were recently proved in [15] and [17], respectively. As extensions of Jacobson's Lemma, in 2013 Corach, Duggal and Harte [3] firstly investigated common properties of $ac - 1$ and $ba - 1$ under the assumption

$$aba = aca,$$

where $a, b, c \in R$. Recently, we extended Cline's formula for the Drazin inverse, the pseudo Drazin inverse and the generalized Drazin inverse to the case when $aba = aca$ (see [14, 18]).

In this paper, we show that the Drazin invertibility of an element $a \in R$ is equivalent to that of $a^n$; the Drazin inverse of $a$ is given by that of $a^n$, and vice versa. Using this result, we prove that, in the presence of $aba = aca$, for any natural numbers $n$ and $m$, the Drazin invertibility of $(ac)^n$ in a ring is equivalent to that of $(ba)^m$; the Drazin inverse of $(ac)^n$ is expressed by that of $(ba)^m$, and vice versa. Also, we establish analogous results for the pseudo Drazin inverse and the generalized Drazin inverse on Banach algebras.

## 2. Main results

In [1, Theorem 2.3], Berkani and Sarih showed that an element $a$ in an algebra with unit is Drazin invertible if and only if $a^n$ is Drazin invertible. In the following, we give a different proof which holds in the frame of rings.

**Theorem 2.1.** *Let $a \in R$ and $n \in \mathbb{N}$. Then $a$ is Drazin invertible if and only if $a^n$ is Drazin invertible. In this case, we have*

$$(a^n)^D = (a^D)^n, \tag{2.1}$$

$$a^D = (a^n)^D a^{n-1} \tag{2.2}$$

*and*

$$\frac{i(a)}{n} \leq i(a^n) < \frac{i(a)}{n} + 1. \tag{2.3}$$

*Proof.* Since (2.1) and (2.3) have been proved by Drazin (see [4, Theorem 2]), it suffices to show (2.2). Suppose that $a^n$ is Drazin invertible and let $b = (a^n)^D$. Next, we show that $a^D = ba^{n-1}$.

(i) Since $a^n a = aa^n$, $a(ba^{n-1}) = baa^{n-1} = (ba^{n-1})a$.

(ii) We have $(ba^{n-1})a(ba^{n-1}) = (ba^n b)a^{n-1} = ba^{n-1}$.

(iii) Let $i(a^n) = k$ and $l = nk$. Since $a^{n-1}a^n = a^n a^{n-1}$, we have

$$\begin{aligned} a^{l+1}(ba^{n-1}) &= a^{nk+1}(ba^{n-1}) \\ &= a^{nk+1}a^{n-1}b \\ &= (a^n)^{k+1}b \\ &= (a^n)^k = a^l. \end{aligned}$$

This completes the proof. □

Let us remark that the Drazin index of $a^n$ is uniquely determined by that of $a$, but not vice versa.

**Lemma 2.2.** ([18, Theorem 2.7]) *Suppose that $a, b, c \in R$ satisfy $aba = aca$. Then $ac$ is Drazin invertible if and only if $ba$ is Drazin invertible. In this case, we have*

(1) $|i(ac) - i(ba)| \leq 1$;

(2) $(ba)^D = b((ac)^D)^2 a$ *and* $(ac)^D = a((ba)^D)^2 c$.

In the following theorem, in the presence of $aba = aca$, we give explicit expressions for the Drazin inverses of $(ac)^n$ and $(ba)^m$, both in terms of each other.

**Theorem 2.3.** *Suppose that $a, b, c \in R$ satisfy $aba = aca$ and let $n, m \in \mathbb{N}$.*

(1) *If $(ac)^n$ is Drazin invertible, then $(ba)^m$ is Drazin invertible. In this case,*

$$((ba)^m)^D = b((ac)^n)^D a(ba)^{n-m-1} \quad \text{if } n \geq m+1,$$

$$((ba)^m)^D = b[((ac)^n)^D]^{m+2-n} a(ba)^{(n-1)(m+1-n)} \quad \text{if } n < m+1.$$

(2) *If $(ba)^n$ is Drazin invertible, then $(ac)^m$ is Drazin invertible. In this case,*

$$((ac)^m)^D = a((ba)^n)^D c(ac)^{n-m-1} \quad \text{if } n \geq m+1,$$

$$((ac)^m)^D = a[((ba)^n)^D]^{m+2-n} c(ac)^{(n-1)(m+1-n)} \quad \text{if } n < m+1.$$

*Proof.* (1) If $n \geq m+1$, by Theorem 2.1 and Lemma 2.2, we have

$$\begin{aligned}
((ba)^m)^D &= ((ba)^D)^m \\
&= [b((ac)^D)^2a]^m \\
&= \underbrace{[b((ac)^D)^2a][b((ac)^D)^2a]\cdots[b((ac)^D)^2a]}_{m \text{ times}} \\
&= [b((ac)^D)^2a][bac((ac)^D)^3a]\underbrace{[b((ac)^D)^2a][b((ac)^D)^2a]\cdots[b((ac)^D)^2a]}_{(m-2) \text{ times}} \\
&= [b((ac)^D)^2a][cac((ac)^D)^3a]\underbrace{[b((ac)^D)^2a][b((ac)^D)^2a]\cdots[b((ac)^D)^2a]}_{(m-2) \text{ times}} \\
&= [b((ac)^D)^2][(ac)^Da]\underbrace{[b((ac)^D)^2a][b((ac)^D)^2a]\cdots[b((ac)^D)^2a]}_{(m-2) \text{ times}} \\
&= \cdots \\
&= b((ac)^D)^{m+1}a \\
&= b((ac)^D)^n(ac)^{n-m-1}a \\
&= b((ac)^n)^Da(ba)^{n-m-1}.
\end{aligned}$$

If $n < m+1$, by Theorem 2.1 and Lemma 2.2, we have

$$\begin{aligned}
((ba)^m)^D &= ((ba)^D)^m \\
&= [b((ac)^D)^2a]^m \\
&= \underbrace{[b((ac)^D)^2a][b((ac)^D)^2a]\cdots[b((ac)^D)^2a]}_{m \text{ times}} \\
&= b((ac)^D)^{m+1}a \\
&= b((ac)^D)^n((ac)^D)^{m+1-n}a \\
&= b((ac)^n)^D[((ac)^n)^D(ac)^{n-1}]^{m+1-n}a \\
&= b[((ac)^n)^D]^{m+2-n}(ac)^{(n-1)(m+1-n)}a \\
&= b[((ac)^n)^D]^{m+2-n}a(ba)^{(n-1)(m+1-n)}.
\end{aligned}$$

(2) The proof is similar to that of (1). □

The following result concerns explicit expressions for the pseudo Drazin inverses of $a$ and $a^n$, both in terms of each other.

**Theorem 2.4.**

(1) *Let $a \in R$ and $n \in \mathbb{N}$. If $a^n$ is pseudo Drazin invertible, then $a$ is pseudo Drazin invertible and $a^{pD} = (a^n)^{pD}a^{n-1}$.*
(2) *Let $a \in A$ and $n \in \mathbb{N}$. If $a$ is pseudo Drazin invertible, then $a^n$ is pseudo Drazin invertible and $(a^n)^{pD} = (a^{pD})^n$.*

*Moreover,*

$$\frac{i(a)}{n} \le i(a^n) < \frac{i(a)}{n} + 1.$$

*Proof.* (1) Suppose that $a^n$ is pseudo Drazin invertible and let $b = (a^n)^{pD}$. Next, we show that $a^{pD} = ba^{n-1}$.

(i) We have $(ba^{n-1})a(ba^{n-1}) = (ba^n b)a^{n-1} = ba^{n-1}$.

(ii) Let $c \in \mathrm{comm}(a)$. Then $ca^n = a^n c$, and since $b \in \mathrm{comm}^2(a^n)$, $bc = cb$. Therefore $c(ba^{n-1}) = b(ca^{n-1}) = (ba^{n-1})c$. Consequently, $ba^{n-1} \in \mathrm{comm}^2(a)$.

(iii) Let $i(a^n) = k$ and $l = nk$. Since $a^{n-1}a^n = a^n a^{n-1}$, we have

$$\begin{aligned} a^{l+1}(ba^{n-1}) - a^l &= a^{nk+1}(ba^{n-1}) - a^{nk} \\ &= a^{nk+1}a^{n-1}b - a^{nk} \\ &= (a^n)^{k+1}b - (a^n)^k \in J(R). \end{aligned}$$

From the above argument, one can also infer that $\frac{i(a)}{n} \le i(a^n)$. To prove $i(a^n) < \frac{i(a)}{n} + 1$, we also let $i(a^n) = k$. Then we need to show that $i(a) > nk - n$. Otherwise, $i(a) \le nk - n$ would mean that

$$\begin{aligned} (a^n)^k(a^n)^{pD} - (a^n)^{k-1} &= a^{nk-n+1}a^{n-1}(a^n)^{pD} - a^{nk-n} \\ &= a^{nk-n+1}(a^n)^{pD}a^{n-1} - a^{nk-n} \\ &= a^{nk-n+1}a^{pD} - a^{nk-n} \in J(R), \end{aligned}$$

which contradicts the fact that $i(a^n) = k$.

(2) Suppose that $a$ is pseudo Drazin invertible and let $b = a^{pD}$. Next, we show that $(a^n)^{pD} = b^n$. Evidently, we have (i) $b^n a^n b^n = b^n$, and (ii) $b^n a^n = a^n b^n$. For (iii), let $i(a) = m$ and $q$ be an integer satisfying $q \ge \frac{m}{n}$. Since $ab = ba$ and $bab = b$, $a^{m+n}b^n - a^m = a^m ba - a^m \in J(A)$. Then

$$\begin{aligned} (a^n)^{q+1}(b^n) - (a^n)^q &= a^{nq-m}a^{m+n}b^n - a^{nq-m}a^m \\ &= a^{nq-m}(a^{m+n}b^n - a^m) \in J(A). \end{aligned}$$

This completes the proof. □

Similarly, the pseudo Drazin index of $a^n$ is uniquely determined by that of $a$, but not vice versa.

**Lemma 2.5.** ([14, Theorem 2.4]) *Suppose that $a, b, c \in R$ satisfy $aba = aca$. Then $ac$ is pseudo Drazin invertible if and only if $ba$ is pseudo Drazin invertible. In this case, we have*

(1) $|i(ac) - i(ba)| \le 1$;
(2) $(ba)^{pD} = b((ac)^{pD})^2 a$ *and* $(ac)^{pD} = a((ba)^{pD})^2 c$.

In the Banach algebra case and with the presence of $aba = aca$, we give in the following theorem explicit expressions for the pseudo Drazin inverses of $(ac)^n$ and $(ba)^m$, both in terms of each other.

**Theorem 2.6.** *Suppose that $a, b, c \in A$ satisfy $aba = aca$ and let $n, m \in \mathbb{N}$.*

(1) *If $(ac)^n$ is pseudo Drazin invertible, then $(ba)^m$ is pseudo Drazin invertible. In this case,*

$$((ba)^m)^{pD} = b((ac)^n)^{pD} a(ba)^{n-m-1} \quad \text{if } n \geq m+1,$$

$$((ba)^m)^{pD} = b[((ac)^n)^{pD}]^{m+2-n} a(ba)^{(n-1)(m+1-n)} \quad \text{if } n < m+1.$$

(2) *If $(ba)^n$ is pseudo Drazin invertible, then $(ac)^m$ is pseudo Drazin invertible. In this case,*

$$((ac)^m)^{pD} = a((ba)^n)^{pD} c(ac)^{n-m-1} \quad \text{if } n \geq m+1,$$

$$((ac)^m)^{pD} = a[((ba)^n)^{pD}]^{m+2-n} c(ac)^{(n-1)(m+1-n)} \quad \text{if } n < m+1.$$

*Proof.* Apply the proof of Theorem 2.3 to the pseudo Drazin inverse, using Theorem 2.4 and Lemma 2.5. □

In the following theorem we give explicit expression for the generalized Drazin inverse of $a$ in terms of $a^n$.

**Theorem 2.7.**

(1) *Let $a \in R$ and $n \in \mathbb{N}$. If $a^n$ is generalized Drazin invertible, then $a$ is generalized Drazin invertible and $a^{gD} = (a^n)^{gD} a^{n-1}$.*

(2) *Let $a \in A$ and $n \in \mathbb{N}$. If $a$ is generalized Drazin invertible, then $a^n$ is generalized Drazin invertible and $(a^n)^{gD} = (a^{gD})^n$.*

*Proof.* For part (2), see [11, Theorem 5.4(i)]. (1) Suppose that $a^n$ is generalized Drazin invertible and let $b = (a^n)^{gD}$. Next, we show that $a^{gD} = ba^{n-1}$. As in the proof of Theorem 2.4(1), we get (i) $(ba^{n-1})a(ba^{n-1}) = ba^{n-1}$, and (ii) $ba^{n-1} \in \mathrm{comm}^2(a)$. (iii) Since $b$ is a generalized Drazin inverse of $a^n$, $p = 1 - a^n b$ is an idempotent and commutes with $a$, and hence $(pa)^n = pa^n$ is quasinilpotent. Let $c \in \mathrm{comm}(pa)$. Then $c^n \in \mathrm{comm}((pa)^n)$ and $1 - (pa)^n c^n = (1 - pac)(1 + pac + (pac)^2 + \cdots + (pac)^{n-1})$ is invertible, and hence $1 - pac$ is invertible. Therefore, $a - a(ba^{n-1})a = (1 - a^n b)a = pa$ is quasinilpotent. □

**Lemma 2.8.** ([14, Theorem 2.3]) *Suppose that $a, b, c \in R$ satisfy $aba = aca$. Then $ac$ is generalized Drazin invertible if and only if $ba$ is generalized Drazin invertible. In this case, we have $(ba)^{gD} = b((ac)^{gD})^2 a$ and $(ac)^{gD} = a((ba)^{gD})^2 c$.*

In the following theorem, in the Banach algebra case and under the hypothesis $aba = aca$, we give explicit expressions for the generalized Drazin inverses of $(ac)^n$ and $(ba)^m$, both in terms of each other.

**Theorem 2.9.** *Suppose that $a, b, c \in A$ satisfy $aba = aca$ and let $n, m \in \mathbb{N}$.*

(1) *If $(ac)^n$ is generalized Drazin invertible, then $(ba)^m$ is generalized Drazin invertible. In this case,*

$$((ba)^m)^{gD} = b((ac)^n)^{gD} a(ba)^{n-m-1} \quad \text{if } n \geq m+1,$$

$$((ba)^m)^{gD} = b[((ac)^n)^{gD}]^{m+2-n} a(ba)^{(n-1)(m+1-n)} \quad \text{if } n < m+1.$$

(2) *If* $(ba)^n$ *is generalized Drazin invertible, then* $(ac)^m$ *is generalized Drazin invertible. In this case,*

$$((ac)^m)^{gD} = a((ba)^n)^{gD} c(ac)^{n-m-1} \quad \text{if } n \geq m+1,$$

$$((ac)^m)^{gD} = a[((ba)^n)^{gD}]^{m+2-n} c(ac)^{(n-1)(m+1-n)} \quad \text{if } n < m+1.$$

*Proof.* Apply the proof of Theorem 2.3 to the generalized Drazin inverse, using Theorem 2.7 and Lemma 2.8. □

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